Yuming Qin

Integral and Discrete Inequalities and Their Applications

Volume I: Linear Inequalities

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To my Parents Zhenrong Qin and Xilan Xia
and to my wife and son Yu Yin and Jia Qin

Preface

Integral and discrete inequalities are very important tools in classical analysis. This book focuses on one- and multidimensional linear integral and discrete Gronwall–Bellman-type inequalities. It provides a useful collection and systematic presentation of known and new results, as well as many applications to differential (ODE and PDE), difference and integral equations, and is therefore an ideal source for familiarising students with this tool. It is also useful for researchers working on these topics.

It is Part I of a two-volume work on inequalities. We start with an introduction to different types of linear one-dimensional inequalities:

Chapter 1 focuses on continuous integral inequalities.
Chapter 2 features discrete (difference) inequalities.
Chapter 3 introduces discontinuous integral inequalities.
Chapter 4 then studies applications of these inequalities. The second half of this book, Chaps. 5–8, considers corresponding multidimensional linear inequalities and applications.

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Shanghai, China Yuming Qin

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Volume I: Linear Integral and Difference Inequalities

Chapter 1
Linear One-Dimensional Continuous Integral Inequalities

1.1 Linear One-Dimensional Continuous Classical Gronwall-Bellman Inequalities

It is well-known that the classical integral inequalities, which furnish explicit bounds for an unknown function, have played a fundamental role in establishing the basis of the theory of differential and integral equations. Just for this reason, more and more researchers have found many useful inequalities in order to achieve their different desired goals. In this chapter, we shall collect some basic linear one-dimensional continuous integral inequalities which have found many important applications in integral equations.

The following is the very famous Gronwall-Bellman inequality [239] which plays a crucial role in analysis, especially in the study of existence, uniqueness and stability and estimates of solutions to differential equations (see, e.g., [61–63, 66]).

Integral inequalities of the Gronwall-Bellman type are frequently used in various contexts. Over the years several such inequalities have been developed and used considerably to study the various problems in the theory of differential and integral equations, see [42, 495] and the references therein.

Theorem 1.1.1 (The Gronwall Inequality [239]) *Let $u(t)$ be a continuous function defined on the interval $I = [\alpha, \alpha + h]$ and for all $t \in I$,*

$$0 \le u(t) \le \int_\alpha^t [bu(s) + a]ds, \tag{1.1.1}$$

where a and b are non-negative constants. Then for all $t \in I$,

$$0 \le u(t) \le ahe^{bh}. \tag{1.1.2}$$

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Proof By analogy with the process of integrating a linear differential equation of first order, we take $u = z\exp[b(t-\alpha)]$. Let the maximum of z on I occur at $t = t_1$. For this value of t, (1.1.1) implies

$$0 \le z_{\max}\exp[b(t_1-\alpha)] \le \int_{\alpha}^{t_1}[bz(s)\exp[b(s-\alpha)] + a]ds$$

whence, by the mean value theorem, we conclude

$$\begin{aligned} 0 \le z_{\max}\exp[b(t_1-\alpha)] &\le z_{\max}\int_{\alpha}^{t_1} b\exp[b(s-\alpha)]ds + \int_{\alpha}^{t_1} ads \\ &\le z_{\max}[\exp[b(t_1-\alpha)] - 1] + a(t_1-\alpha) \end{aligned}$$

or

$$0 \le z_{\max} \le a(t_1-\alpha) \le ah$$

which readily implies (1.1.2). □

Remark 1.1.1 It is worth pointing out that such an inequality (1.1.1) can be traced back at least to Peano [519], which explicitly dealt with the special case of the above theorem with $a = 0$, and some general results on the differential inequalities and maximal and minimal solution of differential equations were also obtained.

Theorem 1.1.2 (The Classical Bellman Inequality [61]) *Let $y(t)$ and $g(t)$ be non-negative, continuous functions on $0 \le t \le T$ satisfying for all $0 \le t \le T$,*

$$y(t) \le \eta + \int_0^t g(s)y(s)ds, \tag{1.1.3}$$

where η is a non-negative constant. Then for all $0 \le t \le T$,

$$y(t) \le \eta\exp\left(\int_0^t g(s)ds\right). \tag{1.1.4}$$

Proof Put

$$v(t) = \eta + \int_0^t g(s)y(s)ds. \tag{1.1.5}$$

Then it follows from (1.1.3) and (1.1.5) that for all $0 \le t \le T$,

$$v'(t) = g(t)y(t) \le g(t)v(t). \tag{1.1.6}$$

Multiplying (1.1.6) by $\exp\left(-\int_0^t g(s)ds\right)$, we get

$$\frac{d}{dt}\left(v(t)\exp\left(-\int_0^t g(s)ds\right)\right) \le 0$$

which gives us (1.1.4). $\square$

Corollary 1.1.1 *Let $u(t)$ and $b(t)$ be non-negative continuous functions for all $t \ge \alpha$, and let, for all $\alpha \le t \le T$,*

$$u(t) \le ae^{-\gamma(t-\alpha)} + \int_\alpha^t e^{-\gamma(t-s)}b(s)u(s)ds, \tag{1.1.7}$$

where $a \ge 0$ and γ are constants. Then for all $\alpha \le t \le T$,

$$u(t) \le a\exp\left(-\gamma(t-\alpha) + \int_\alpha^t b(s)ds\right). \tag{1.1.8}$$

Proof Setting $w(t) = e^{\gamma t}u(t)$, we obtain from (1.1.7) for all $t \ge \alpha$,

$$w(t) \le ae^{\gamma\alpha} + \int_\alpha^t b(s)\omega(s)ds. \tag{1.1.9}$$

By Theorem 1.1.2, we derive that (1.1.9) implies $w(t) \le ae^{\gamma\alpha}\exp\left(\int_\alpha^t b(s)ds\right)$, which gives us (1.1.8). $\square$

Remark 1.1.2 In 1919, Gronwall [239] showed the case of $g(t) = constant \ge 0$. Later on in 1943, Bellman [61] extended this result to the form of Theorem 1.1.2. Since this type of inequalities is a very powerful and useful tool in analysis, more and more improvements and generalizations of the classical Gronwall-Bellman inequality have been made.

Remark 1.1.3 Clearly Bellman's inequality includes Gronwall's inequality due to the fact $\int_\alpha^t ads \le ah$ for $t \in I = [\alpha, \alpha + h]$. Since Bellman's inequality was found, it has exerted a great deal of influence till recently, and the study of such a kind of inequalities has become a hot topic in various important applications of differential and integral equations.

We know that Theorem 1.1.2 provides bounds on solution of (1.1.3) in terms of the solution of a related linear integral equation

$$v(t) = \eta + \int_0^t g(s)v(s)ds \tag{1.1.10}$$

and is one of the basic tools in the theory of differential equations. On the basis of various motivations, it has been extended and used considerably in various context. For instance, in the Picard-Cauchy type of iteration for establishing existence and uniqueness of solutions, this inequality and its various variants play a significant role. Inequalities of this type (1.1.3) are also encountered frequently in the perturbation and stability theory of differential equations.

Since the establishment of the above inequality, many various generalizations have been made. These generalizations include linear generalizations, nonlinear generalizations, singular generalizations, uniform generalizations, and other generalizations involving operators in partially ordered linear spaces, etc.

Among the early users of the above inequality in the theory of ordinary differential equations was Reid [555] (which is on the two-sided estimates, see Lemma 1.1.1 and Remarks 1.1.5–1.1.6), who employed a slightly more general form of Theorem 1.1.2 to study the properties of solutions of infinite systems of linear ordinary differential equations.

Lemma 1.1.1 *Let $b(t)$ and $f(t)$ be continuous functions for all $t \geq \alpha$, let $v(t)$ be a differentiable function for all $t \geq \alpha$, and suppose*

$$v'(t) \leq b(t)v(t) + f(t), \quad t \geq \alpha; \quad v(\alpha) \leq v_0. \tag{1.1.11}$$

Then for all $t \geq \alpha$,

$$v(t) \leq v_0 \exp\Big(\int_\alpha^t b(s)ds\Big) + \int_\alpha^t f(s) \exp\Big(\int_s^t b(\tau)d\tau\Big)ds. \tag{1.1.12}$$

Proof Condition (1.1.11) implies that

$$\Big[v'(s) - b(s)v(s)\Big] \exp\Big(\int_s^t b(\tau)d\tau\Big) \leq f(s) \exp\Big(\int_s^t b(\tau)d\tau\Big), \ s \geq \alpha,$$

or

$$\frac{d}{ds}\Big[v(s) \exp\Big(\int_s^t b(\tau)d\tau\Big)\Big] \leq f(s) \exp\Big(\int_s^t b(\tau)d\tau\Big).$$

Integration over s from α to t gives

$$v(t) - v(\alpha) \exp\Big(\int_\alpha^t b(\tau)d\tau\Big) \leq \int_\alpha^t f(s) \exp\Big(\int_s^t b(\tau)d\tau\Big)ds,$$

which implies (1.1.12) since $v(\alpha) \leq v_0$. $\square$

Remark 1.1.4 Note that the right-hand side of (1.1.12) coincides with the unique solution of the equation

$$v'(t) = b(t)v(t) + f(t), \quad t \geq \alpha, \tag{1.1.13}$$

which satisfies

$$v(\alpha) = v_0. \tag{1.1.14}$$

Equation (1.1.13) is called the comparison differential equation of the inequality (1.1.11). The comparison of initial value problem (1.1.13)–(1.1.14) is obtained by replacing "$\leq$" by "$=$" in (1.1.11).

Remark 1.1.5 Lemma 1.1.1 remains valid if "$\leq$" is replaced by "$\geq$" in both (1.1.11) and (1.1.12).

Remark 1.1.6 If the function $b(t)$ and $f(t)$ are continuous for all $t \leq \alpha$,

$$v'(t) \leq b(t)v(t) + f(t), \tag{1.1.15}$$

then for all $t \leq \alpha$,

$$v(t) \geq v(\alpha)\exp\left(\int_\alpha^t b(s)ds\right) + \int_\alpha^t f(s)\exp\left(\int_\alpha^t b(\tau)d\tau\right)ds. \tag{1.1.16}$$

Moreover, this result remains valid if " $\leq$" in (1.1.12) is replaced by " $\geq$", and " $\geq$" in (1.1.16) is replaced by " $\leq$".

Theorem 1.1.3 (Bellman-Reid [555]) *Let $u(t)$ and $b(t)$ be non-negative continuous functions in $J = [\alpha, \beta]$, and suppose that for all $t \in J$,*

$$u(t) \leq a + \int_{t_0}^t b(s)u(s)|ds|,$$

where $t_0 \in J$ and $a \geq 0$ is a constant. Then for all $t \in J$,

$$u(t) \leq a\exp\left(\int_{t_0}^t b(s)|ds|\right). \tag{1.1.17}$$

Proof By Lemma 1.1.1 and Remark 1.1.6, it is easy to prove the assertion. We leave the detail of the proof to the reader. □

Bellman [68] showed the following variant of Theorem 1.1.2 to study the asymptotic behavior of the solutions of linear differential-difference equations.

Theorem 1.1.4 (The Bellman Inequality [68]) *Let u and f be continuous and non-negative functions on $J = [\alpha, \beta]$, and let $n(t)$ be a continuous, positive and non-*

decreasing function on J, and there holds that for all $t \in J$,

$$u(t) \leq n(t) + \int_{\alpha}^{t} f(s)u(s)ds, \tag{1.1.18}$$

then for all $t \in J$,

$$u(t) \leq n(t) \exp\left(\int_{\alpha}^{t} f(s)ds\right). \tag{1.1.19}$$

Proof Let $w(t) = u(t)/n(t)$. Then from (1.1.18), it follows that $w(t)$ solves

$$w(t) \leq 1 + \int_{\alpha}^{t} f(s)w(s)ds$$

which, by Theorem 1.1.2, implies

$$w(t) \leq \exp\left(\int_{\alpha}^{t} f(s)ds\right).$$

This gives us the required inequality (1.1.19). □

Remark 1.1.7 Clearly, Theorem 1.1.2 can be regarded as a special case $n(t) = const. = \eta$.

Theorem 1.1.5 (The Bellman Inequality [68]) *Let f be a non-negative continuous function defined on $\mathbb{R}_+$ such that $\int_0^{+\infty} f(s)ds < +\infty$ and $n(t) \geq 0$ be a continuous and decreasing function defined on $\mathbb{R}_+$. If $u(t) \geq 0$ is a bounded continuous function on $\mathbb{R}_+$ and satisfies that for all $t \in \mathbb{R}_+$,*

$$u(t) \leq n(t) + \int_{t}^{+\infty} f(s)u(s)ds, \tag{1.1.20}$$

then for all $t \in \mathbb{R}_+$,

$$u(t) \leq n(t) \exp\left(\int_{t}^{+\infty} f(s)ds\right). \tag{1.1.21}$$

Proof First we assume that $n(t) > 0$ for all $t \in \mathbb{R}_+$.Then from (1.2.20) it follows that

$$\frac{u(t)}{n(t)} \leq 1 + \int_{t}^{+\infty} f(s)\frac{u(s)}{n(s)}ds. \tag{1.1.22}$$

Define a function $z(t)$ by the right-hand side of (1.1.22), then $z(+\infty) = 1$, $\frac{u(t)}{n(t)} \leq z(t)$ and

$$z'(t) = -f(t)\frac{u(t)}{n(t)} \geq -f(t)z(t)$$

which implies

$$z(t) \leq \exp\Big(\int_t^{+\infty} f(s)ds\Big). \tag{1.1.23}$$

Using (1.1.23) in $\frac{u(t)}{n(t)} \leq z(t)$, we get the desired inequality (1.1.21).

If $n(t)$ is non-negative, we carry out the above procedure with $n(t) + \epsilon$ instead of $n(t)$, where $\epsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\epsilon \to 0$ to obtain (1.1.21). □

In 1980, Rodrigues [559] proved the following result which was once used to study the growth and decay of solutions of perturbed retarded linear equations.

Theorem 1.1.6 (Rodrigues [559]) *Let $f(t), g(t)$ be non-negative continuous functions defined for all $t \in \mathbb{R}_+$. Let $\gamma(t) > 0$ be a decreasing continuous function for all $t \geq \sigma$ and σ sufficiently large such that*

$$\beta = \int_\sigma^{+\infty} g(s)ds + \int_\sigma^{+\infty} f(s)ds < 1. \tag{1.1.24}$$

Suppose that u is a non-negative continuous function such that γu is bounded and for all $t \geq \sigma$, there holds that

$$u(t) \leq C + \int_\sigma^t f(s)u(s)ds + \frac{1}{\gamma(t)}\int_t^{+\infty} \gamma(s)g(s)u(s)ds, \tag{1.1.25}$$

where $C \geq 0$ is a constant. Then for all $t \in \mathbb{R}_+$,

$$u(t) \leq [C/(1-\beta)]\exp\left(1/(1-\beta)\int_t^{+\infty} g(s)ds\right). \tag{1.1.26}$$

Proof Let

$$v(t) = \max_{\sigma \leq s \leq t} u(s).$$

Then $v(t)$ is an increasing continuous function such that $u(t) \leq v(t)$ and $\gamma(t)v(t)$ is bounded for all $t \in \mathbb{R}_+$. For any given $t \geq \sigma$, there exists a $t_1 \in [\sigma, t]$ satisfying

$v(t) = u(t_1)$, which implies

$$v(t) \leq C + \int_{\sigma}^{t_1} f(s)v(s)ds + \frac{1}{\gamma(t_1)} \int_{t_1}^{+\infty} \gamma(s)g(s)v(s)ds.$$

Noting that

$$\begin{aligned}\int_{t_1}^{+\infty} \gamma(s)g(s)v(s)ds &= \int_{t_1}^{t} \gamma(s)g(s)v(s)ds + \int_{t}^{+\infty} \gamma(s)g(s)v(s)ds \\ &\leq \gamma(t_1)v(t) \int_{\sigma}^{+\infty} g(s)ds + \int_{t}^{+\infty} \gamma(s)g(s)v(s)ds,\end{aligned}$$

we may get

$$v(t) \leq C + v(t)\left(\int_{\sigma}^{+\infty} f(s)ds + \int_{\sigma}^{+\infty} g(s)ds\right) + \frac{1}{\gamma(t)} \int_{t}^{+\infty} \gamma(s)g(s)v(s)ds. \tag{1.1.27}$$

Hence

$$\gamma(t)v(t) \leq \Big(1/(1-\beta)\Big)\left(C\gamma(t) + \int_{t}^{+\infty} \gamma(s)g(s)v(s)ds\right). \tag{1.1.28}$$

Exploiting Theorem 1.1.5, we readily derive

$$\gamma(t)v(t) \leq \Big(1/(1-\beta)\Big)\gamma(t)\exp\left([1/(1-\beta)]\int_{t}^{+\infty} g(s)ds\right)$$

which completes the proof. □

1.2 Linear One-Dimensional Continuous Generalizations on the Gronwall-Bellman Inequalities

1.2.1 Linear One-Dimensional Continuous Integral Inequalities

The next inequality was established by Jones [305] in 1964.

Theorem 1.2.1 (The Jones Inequality [305]) *Let $y(t), f(t)$ and $g(t)$ be real-valued piecewise-continuous functions defined on a real interval $0 \leq t \leq \tau$ and let g be*

non-negative on this interval. If for all $t \in [0, \tau]$,

$$y(t) \leq f(t) + \int_0^t g(s)y(s)ds, \tag{1.2.1}$$

then for all $t \in [0, \tau]$,

$$y(t) \leq f(t) + \int_0^t g(s)f(s) \exp\left(\int_s^t g(\theta)d\theta\right) ds. \tag{1.2.2}$$

Proof Let $h = \int_0^t g(s)y(s)ds$. Then by (1.2.1), h satisfies

$$h'(t) = g(t)y(t) \leq g(t)f(t) + g(t)h(t)$$

which gives us

$$\frac{d}{dt}\left(h(t) \exp\left(-\int_0^t g(s)ds\right)\right) \leq g(t)f(t) \exp\left(-\int_0^t g(s)ds\right). \tag{1.2.3}$$

Thus integrating (1.2.3) with respect to t yields

$$h(t) \leq \int_0^t g(s)f(s) \exp\left(\int_s^t g(\theta)d\theta\right) ds$$

which, together with (1.2.1), implies (1.2.2). □

Note that the inequality (1.1.2) provides the best possible result in the sense that when we replace the inequality (1.2.1) by an equality, the same may be done in (1.2.2). Also, it is obvious that when $f(t) \equiv \eta$ (a constant), a straightforward integration in (1.2.3) yields

$$y(t) \leq \eta \exp\left(\int_0^t g(s)ds\right)$$

which is precisely (1.1.4).

Corollary 1.2.1 *Under assumptions of Theorem 1.2.1, let* $f(t)$ *be also non-decreasing on* $[0, \tau]$. *Then for all* $t \in [0, \tau]$,

$$y(t) \leq f(t) \exp\left(\int_0^t g(s)ds\right).$$

Proof In fact, (1.2.2) implies

$$\begin{aligned}
y(t) &\le f(t) + f(t)\int_0^t g(s)\exp\Big(\int_s^t g(\theta)d\theta\Big)ds \\
&= f(t)\Big(1 - \int_0^t \frac{d}{ds}[\exp(\int_0^t g(\theta)d\theta)]ds\Big) \\
&= f(t)\exp\left(\int_0^t g(s)ds\right).
\end{aligned}$$

The proof is thus complete. □

This corollary is just Theorem 1.1.4. Here we give its another proof.

An alternate form of (1.2.2) can be stated as follows when $y(t), f(t)$ possesses higher regularities.

Theorem 1.2.2 (The Generalized Jones Inequality [305]) *Assume that $g(t)$ is a non-negative integrable function on $[0, T]$ $(0 < T)$, $f(t)$ and $y(t)$ are non-negative absolutely continuous functions on $[0, T]$ verifying that for all $t \in [0, \tau]$,*

$$y(t) \le f(t) + \int_0^t g(s)y(s)ds. \tag{1.2.4}$$

Then we have

(1) for all $t \in [0, \tau]$,

$$y(t) \le f(0)\exp\left(\int_0^t g(s)ds\right) + \int_0^t \exp\left(\int_s^t g(\eta)d\eta\right)f'(s)ds. \tag{1.2.5}$$

(2) if $f(t) \equiv A = constant > 0$, then for all $t \in [0, \tau]$,

$$y(t) \le A\exp\left(\int_0^t g(s)ds\right). \tag{1.2.6}$$

Further, if $g(t) \equiv B = constant > 0$, then for all $t \in [0, \tau]$,

$$y(t) \le A\exp(Bt). \tag{1.2.7}$$

Proof Since $f(t)$ and $y(t)$ are non-negative absolutely continuous functions on $[0, T]$, we know that $y'(t), f'(t)$ exist almost all $t \in [0, T]$. Then if we set

$$h(t) = f(t) + \int_0^t g(s)y(s)ds, \tag{1.2.8}$$

then we derive from (1.2.4) and (1.2.8) that for almost all $t \in [0, T]$,

$$h'(t) = f'(t) + g(t)y(t) \le f'(t) + g(t)h(t)$$

which implies

$$\frac{d}{dt}\left[h(t)\exp\left(-\int_0^t g(s)ds\right)\right] \le f'(t)\exp\left(-\int_0^t g(s)ds\right). \tag{1.2.9}$$

Therefore integrating (1.2.9) with respect to t yields (1.2.5). Estimates (1.2.6) and (1.2.7) are direct results of (1.2.5). □

In what follows, we assume that all the integrals involved throughout the discussion exist on the respective domains of their definitions.

In 1968, Zadiraka [681] (see also [215]) showed the next linear generalization the Gronwall-Bellman inequality.

Theorem 1.2.3 (Zadiraka [681]) *Let a continuous function $u(t)$ satisfy*

$$|u(t)| \le |u(t_0)|\exp(-\alpha(t-t_0)) + \int_{t_0}^t (a|u(s)| + b)\, e^{-\alpha(t-s)}ds, \tag{1.2.10}$$

where a, b, and α are positive constants. Then

$$|u(t)| \le |u(t_0)|\exp\Big(-\alpha(t-t_0)\Big) + b(\alpha-a)^{-1}\Big(1-\exp\Big(-(\alpha-a)(t-t_0)\Big)\Big). \tag{1.2.11}$$

Proof It is easy to prove. □

The following theorem, due to Čandirov [124], was given by Filatov and Sarova [215] in 1976.

Theorem 1.2.4 (Čandirov [124]) *Let $u(t)$ be a non-negative continuous function on $\mathbb{R}_+$ such that*

$$u(t) \le ct^\alpha + mt^\beta \int_0^t \frac{u(s)}{s}ds, \tag{1.2.12}$$

where $c > 0, \alpha \ge 0, \beta \ge 0$. Then

$$u(t) \le ct^\alpha\left(1 + \sum_{n=1}^{+\infty}\frac{m^n t^{n\beta}}{\alpha(\alpha+\beta)+\cdots+(\alpha+(n-1)\beta)}\right). \tag{1.2.13}$$

Proof The proof is left to the reader as an exercise. □

In 1971, Filatov [214] proved the following result.

Theorem 1.2.5 (Filatov [214]) *Let $u(t)$ be a continuous non-negative function such that for all $t \geq t_0$,*

$$u(t) \leq a \int_{t_0}^{t} (bu(s) + c)ds, \tag{1.2.14}$$

where $a,\ b \neq 0,\ c$ are constants. Then for all $t \geq t_0$, there holds that

$$u(t) \leq \frac{c}{b}\Big(\exp\Big(b(t - t_0)\Big) - 1\Big) + a \exp\Big(b(t - t_0)\Big). \tag{1.2.15}$$

Proof The proof is left to the reader as an exercise. □

Remark 1.2.1 As the above proof of Filatov's inequality shows, the hypotheses on $u(t)$, a, and b are positive are irrelevant.

The next result is a generalization of Theorem 1.2.2.

Theorem 1.2.6 (Gollwitzer [231]) *Let u, f, g and h be non-negative continuous functions on $J = [\alpha, \beta]$, and for all $t \in J$,*

$$u(t) \leq f(t) + g(t) \int_{\alpha}^{t} h(s)u(s)ds. \tag{1.2.16}$$

Then for all $t \in J$,

$$u(t) \leq f(t) + g(t) \int_{\alpha}^{t} h(s)f(s) \exp\Big(\int_{s}^{t} h(\sigma)g(\sigma)d\sigma\Big)ds. \tag{1.2.17}$$

Proof Let

$$z(t) = \int_{\alpha}^{t} h(s)u(s)ds. \tag{1.2.18}$$

Then $z(\alpha) = 0$, $u(t) \leq f(t) + g(t)z(t)$ and

$$z'(t) = h(t)u(t) \leq h(t)f(t) + h(t)g(t)z(t). \tag{1.2.19}$$

Multiplying (1.2.19) by $\exp\big(- \int_{\alpha}^{t} h(\sigma)g(\sigma)d\sigma\big)$, we have

$$\frac{d}{dt}\left[z(t) \exp\left(- \int_{\alpha}^{t} h(\sigma)g(\sigma)d\sigma\right)\right] \leq h(t)f(t) \exp\left(- \int_{\alpha}^{t} h(\sigma)g(\sigma)d\sigma\right). \tag{1.2.20}$$

Setting $t = s$ in (1.2.20) and integrating the resulting equation over $[\alpha, t]$, we derive

$$z(t)\exp\left(-\int_\alpha^t h(\sigma)g(\sigma)d\sigma\right) \le \int_\alpha^t h(s)f(s)\exp\left(-\int_\alpha^t h(\sigma)g(\sigma)d\sigma\right). \tag{1.2.21}$$

Noting (1.2.16) and using (1.2.21), we finally derive (1.2.17). □

Remark 1.2.2 If $g(t) = 1$, then Theorem 1.2.6 reduces to Theorem 1.2.1 (Jones [305]). Moreover, some generalizations of Theorem 1.2.6 when $g(t) = 1$, the subsequent extensions to discrete and discontinuous functional equations are also contained in Jones [305].

A useful linear generalization of Theorem 1.1.2 may be stated as follows (see, e.g., Pachpatte [75]).

Theorem 1.2.7 (Willett [646]) *Let $x(t)$, $f(t)$, and $g(t)$ be real-valued non-negative continuous functions defined on $\mathbb{R}_+$, and $n(t)$ be a positive, monotonic, non-decreasing continuous function defined on $\mathbb{R}_+$, satisfying for all $t \in \mathbb{R}_+$,*

$$x(t) \le n(t) + g(t)\int_0^t f(s)x(s)ds. \tag{1.2.22}$$

Then for all $t \in \mathbb{R}_+$,

$$x(t) \le n(t)\left[1 + g(t)\left(\int_0^t f(s)\exp\left(\int_s^t g(\tau)f(\tau)d\tau\right)ds\right)\right]. \tag{1.2.23}$$

Proof Since $n(t)$ is positive, monotonic, non-decreasing, we observe from (1.2.22) that

$$\begin{aligned}\frac{x(t)}{n(t)} &\le 1 + g(t)\left(\int_0^t f(s)\frac{x(s)}{n(t)}ds\right)\\ &\le 1 + g(t)\left(\int_0^t f(s)\frac{x(s)}{n(s)}ds\right).\end{aligned} \tag{1.2.24}$$

Now we can complete the proof by setting $v(t)$ to be equal to the integral in the parentheses of (1.2.24) and following an argument similar to that in the proof of Theorem 1.2.1. □

Remark 1.2.3 This form (1.2.24) of Gronwall's inequality was given by Willett [646] who gave explicit bounds for $u(t)$ under more general assumptions, e.g.,

$$u(t) \le n(t) + \sum_{i=1}^n g_i(t)\int_\alpha^t h_i(s)u(s)ds. \tag{1.2.25}$$

We note that the integral inequality obtained in Theorem 1.2.6 is a generalization of Theorem 1.1.2 in [75].

Theorem 1.2.8 (Willett [647]) *Let u, p, q, f and g be non-negative continuous functions on $J = [\alpha, \beta]$, and for all $t \in J$,*

$$u(t) \le p(t) + q(t) \int_\alpha^t \Big(f(s)u(s) + g(s)\Big) ds. \tag{1.2.26}$$

Then for all $t \in J$,

$$u(t) \le p(t) + q(t) \int_\alpha^t \Big(f(s)p(s) + g(s)\Big) \exp\left(\int_s^t f(\sigma)q(\sigma)d\sigma\right) ds. \tag{1.2.27}$$

Proof Let

$$z(t) = \int_\alpha^t \Big(f(s)u(s) + g(s)\Big) ds.$$

Now we can follow the proof of Theorem 1.2.6 to get the desired inequality (1.2.27). □

Remark 1.2.4 In fact, Theorem 1.2.8 extends the result of Chandirov [127] where $q(t) = 1$. If we choose $g(t) = 0$ in Theorem 1.2.7, Theorem 1.2.8 reduces to Theorem 1.2.6.

The next result is due to Dhongade-Deo [182].

Theorem 1.2.9 (Dhongade-Deo [182]) *Suppose that*

(i) $\theta(x), h(x) : (0, +\infty) \to (0, +\infty)$,
(ii) $f(x) : (0, +\infty) \to (0, +\infty)$ *and monotonic non-decreasing in x,*
(iii) $g(x) : (0, +\infty) \to [1, +\infty)$,

and θ, h, f, and g are continuous functions on $\mathbb{R}_+$. Further, if for all $x \in \mathbb{R}_+$,

$$\theta(x) \le f(x) + g(x) \int_0^x h(s)\theta(s)ds, \tag{1.2.28}$$

then for all $x \in \mathbb{R}_+$,

$$\theta(x) \le f(x)g(x)\exp\Big(\int_0^x h(s)g(s)ds\Big). \tag{1.2.29}$$

Proof Since $f(x)$ is monotonic, non-decreasing, and $g(x) \geq 1$, it follows from (1.2.28) that for all $x \in \mathbb{R}_+$,

$$\frac{\theta(x)}{f(x)} \leq 1 + g(x) \int_0^x \frac{h(s)\theta(s)}{f(s)} ds$$
$$\leq g(x) \left[1 + \int_0^x \frac{h(s)\theta(s)}{f(s)} ds \right]. \tag{1.2.30}$$

Denoting the bracket on the right-hand side by $R(x)$, we obtain for all $x \in \mathbb{R}_+$,

$$\frac{R'(x)}{R(x)} \leq g(x)h(x),$$

which, on integration from 0 to x, reduces to (1.2.29). □

For $g(x) = 1$, (1.2.29) was obtained by Bellman [62] (see also Theorem 1.1.2).

Note that, (1.2.28) was also studied by Willett [647] under a more general hypothesis. In (1.2.28), we assume monotonicity on $f(x)$ and obtain a different estimate from that in [647].

Theorem 1.2.9 leads to the following more general inequality containing n-linear terms.

Theorem 1.2.10 (Dhongade-Deo [182]) *Suppose that*

(i) the functions $\theta(x)$,$f(x)$ are defined as in Theorem 1.2.9,
(ii) $g_i(x) : (0, +\infty) \to [1, +\infty)$ are continuous for $i = 1, 2, 3, .., n$,
(iii) $h_i(x) : (0, +\infty) \to (0, +\infty)$ are continuous for $i = 1, 2, \cdots, n$, and if for all $x \in \mathbb{R}_+$,

$$\theta(x) \leq f(x) + \sum_{i=1}^{n} g_i(x) \int_0^x h_i(s)\theta(s)ds, \tag{1.2.31}$$

then for all $x \in \mathbb{R}_+$,

$$\theta(x) \leq E^n f, \tag{1.2.32}$$

where E^k is defined inductively as follows:

$$\begin{cases} E^0 f = f, \\ E^k f = f(E^{k-1} g_k) \exp\Big(\int_0^x h_k E^{k-1} g_k ds \Big), \quad k = 1, 2, \cdots, n. \end{cases} \tag{1.2.33}$$

Proof The proof is by finite induction. Note that Theorem 1.2.10 reduces to Theorem 1.2.9 for $n = 1$ and hence is true. Let us assume that (1.2.32) is true

for given integer k, $1 < k \leq n-1$. Now

$$\theta(x) \leq E^k f \leq E^k f\Big[1 + g_{k+1}(x)\int_0^x \frac{h_{k+1}(s)\theta(s)}{f(s)}ds\Big]. \tag{1.2.34}$$

By (1.2.33), we observe that $E^k f/f \geq 1$. Since $g_i(x) \geq 1$, we can write (1.2.34) as

$$\frac{\theta(x)}{f(x)} \leq \frac{g_{k+1}(x)E^k f}{f(x)}\Big[1 + \int_0^x \frac{h_{k+1}(s)\theta(s)}{f(s)}ds\Big], \quad x \in I$$

which is of the form (1.2.30). Hence, as in Theorem 1.2.9, (1.2.29) takes the form

$$\theta(x) \leq g_{k+1}(x)E^k f\left(\exp\int_0^x \frac{h_{k+1}(s)g_{k+1}(s)E^k f}{f(s)}ds\right).$$

Thus, by (1.2.34), we get

$$\theta(x) \leq f(x)(E^k g_{k+1})\left(\exp\int_0^x h_{k+1}(s)E^k g_{k+1}(s)ds\right) = E^{k+1}f.$$

This proves that (1.2.32) holds for $k+1$. We conclude that (1.2.32) is true for $i = 1, 2\cdots, n$. $\square$

Corollary 1.2.2 *In Theorem 1.2.10, let $g_i(x) = 1, x \in I$ for $i = 1, 2, \cdots, n$, then*

$$\theta(x) \leq E^n f,$$

where E^k is defined inductively as follows:

$$\begin{cases} E^0 f = f, \\ E^k f = (E^{k-1}f)\exp\left(\int_0^x E^{k-1}h_k(s)ds\right), \quad x \in \mathbb{R}_+. \end{cases}$$

The proof can be written by following Theorem 1.2.10. This inequality is linear generalization of the lemma due to Bellman [62] for n terms. As an illustration of Theorem 1.2.10, we consider the inequality for all $x \in \mathbb{R}_+$,

$$\theta(s) \leq x^3 + \int_0^x (1+s)\theta(s)ds + e^x\int_0^x e^{-s^2/2}\theta(s)ds$$

where

$$\begin{cases} f(x) = x^3, \quad g_1(x) = 1, \quad h_1(s) = (1+s), \\ g_2(x) = e^x, \quad h_2(s) = e^{-s^2/2}. \end{cases}$$

In view of (1.2.33), we notice that

$$\theta(x) \le E^2 f = x^3 \exp\left(\frac{6x + x^2}{2}\right) \exp\left(\frac{e^{3x}}{3} - 1\right).$$

Theorem 1.2.11 (Gollwitzer [231]) *Let u, v, h and k be non-negative continuous functions on $J = [\alpha, \beta]$, and for all $\alpha \le x \le t \le \beta$,*

$$u(t) \ge v(x) - k(t) \int_x^t h(s)v(s)ds. \tag{1.2.35}$$

Then for all $\alpha \le x \le t \le \beta$,

$$u(t) \ge v(x) \exp\left(-k(t) \int_x^t h(s)ds\right). \tag{1.2.36}$$

Proof Set

$$z(x) = u(t) + k(t) \int_x^t h(s)v(s)ds, \quad \alpha \le x \le t \le \beta. \tag{1.2.37}$$

This, together with (1.2.35), gives us

$$z'(x) = -h(x)v(x)k(t) \ge -h(x)z(x)k(t), \quad \alpha \le x \le t \le \beta \tag{1.2.38}$$

due to $z(x) \ge v(x)$, here $z'(x)$ at the end points is taken to be the limit from the interior of $[\alpha, t]$. Then using the integral factor $r(x) = \exp(-k(t) \int_x^t h(s)ds)$, we have $(rz)'(x) \ge 0$ and hence $(rz)(t) \ge (rz)(x)$ on $[\alpha, t]$. This result is best possible in the sense that if equality holds in (1.2.35) on $[\alpha, t]$, the equality holds in (1.2.36) on $[\alpha, t]$. □

Remark 1.2.5 The conclusion in Theorem 1.2.11 also holds if in both (1.2.35) and (1.2.36) "$\ge$" is replaced by "$\le$".

Remark 1.2.6 Theorem 1.2.11 is similar to a special case of the Langenhop inequality (Langenhop [351]), and an estimate for u which is independent of x is obtained by taking $x = \alpha$.

Remark 1.2.7 Note that in Theorem 1.2.11 equality holds in (1.2.36) for a subinterval $J_1 = [\alpha, \beta_1]$ of J if equality holds in (1.2.35) for all $t \in J_1$. The results are still valid if "$\le$" is replaced by "$\ge$" in (1.2.35). Both (1.2.35) and (1.2.36) with "$\le$" therein replaced by "$\ge$" remain valid if $\int_\alpha^t$ is replaced by $\int_t^\beta$ and $\int_s^t$ by $\int_t^s$ throughout.

Remark 1.2.8 In 1975, Beesack [51] pointed out that if the integrals in Theorem 1.2.11 are Lebesgue integrals, then the hypotheses can be relaxed to: u, f, g and h are measurable functions such that $hu, hf, hg \in L(J)$. The equality and inequality

conditions are then to be understood to hold almost everywhere, and the stated condition for equality is necessary as well as sufficient. Similar remarks apply to all of our subsequent theorems, which we shall mostly state for the continuous case.

Corollary 1.2.3 (Sardarly [575]) *Let $u(t), a(t), b(t)$, and $q(t)$ be continuous functions in $J = [\alpha, \beta]$, let $c(t, s)$ be a continuous function for $\alpha \leq s \leq t \leq \beta$, let $b(t)$ and $q(t)$ be non-negative in J, and suppose that for all $t \in J$,*

$$u(t) \leq a(t) + \int_\alpha^t [q(t)b(s)u(s) + c(t, s)]ds. \tag{1.2.39}$$

Then for all $t \in J$,

$$\begin{aligned} u(t) \leq a(t) &+ \int_\alpha^t c(t, s)ds \\ &+ q(t) \int_\alpha^t \left(a(s) + \int_\alpha^s c(s, \tau)d\tau\right) \exp\left(\int_s^t b(\tau)q(\tau)d\tau\right) ds. \end{aligned} \tag{1.2.40}$$

Corollary 1.2.4 (Mitrinovic-Pečarić-Fink [409]) *Let $u(t)$, $k(t)$ be non-negative continuous functions and $a(t)$, $b(t)$ be Riemann integrable functions on $J = [\alpha, \beta]$ with $a(t)$, $b(t)$ and $k(t)$ being non-negative on J.*

(i) If for all $t \in J$,

$$u(t) \leq a(t) + b(t) \int_\alpha^t k(s)u(s)\, ds, \tag{1.2.41}$$

then we have for all $t \in J$,

$$u(t) \leq a(t) + b(t) \int_\alpha^t a(s)k(s) \exp\left(\int_s^t b(m)k(m)\, dm\right) ds. \tag{1.2.42}$$

(ii) If for all $t \in J$,

$$u(t) \leq a(t) + b(t) \int_t^\beta k(s)u(s)\, ds \tag{1.2.43}$$

then we have for all $t \in J$,

$$u(t) \leq a(t) + b(t) \int_t^\beta a(s)k(s) \exp\left(\int_t^\beta b(m)k(m)\, dm\right) ds. \tag{1.2.44}$$

In 1973, Pachpatte [445] proved the following general version of Theorem 1.1.2.

Theorem 1.2.12 (Pachpatte [445]) *Let $u(t)$, $f(t)$ and $g(t)$ be real-valued non-negative continuous functions defined on $\mathbb{R}_+$, satisfying that for all $t \in \mathbb{R}_+$,*

$$u(t) \le u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)(\int_0^s g(\tau)u(\tau)d\tau)ds, \tag{1.2.45}$$

where u_0 is a non-negative constant. Then for all $t \in \mathbb{R}_+$,

$$u(t) \le u_0\left(1 + \int_0^t f(s)\exp\left(\int_0^s (f(\tau) + g(\tau))d\tau\right)ds\right). \tag{1.2.46}$$

Proof Define a function $v(t)$ by the right-hand side of (1.2.45). Then

$$v'(t) = f(t)u(t) + f(t)\int_0^t g(\tau)u(\tau)d\tau, \quad v(0) = u_0,$$

which, in view of (1.2.45), implies

$$v'(t) \le f(t)\left(v(t) + \int_0^t g(\tau)v(\tau)d\tau\right). \tag{1.2.47}$$

If we put

$$m(t) = v(t) + \int_0^t g(\tau)v(\tau)d\tau, \quad m(0) = v(0),$$

then it follows from (1.2.47) and the fact that $v(t) \le m(t)$ that

$$m'(t) \le \Big(f(t) + g(t)\Big)m(t),$$

which, since $m(0) = u_0$, implies that

$$m(t) \le u_0 \exp\left(\int_0^t (f(s) + g(s))ds\right).$$

Then from (1.2.47), it follows

$$v'(t) \le u_0 f(t)\exp\Big(\int_0^t (f(s) + g(s))ds\Big). \tag{1.2.48}$$

Now, integrating both sides of (1.2.48) from 0 to t and substituting the value of $v(t)$ in (1.2.45), we can obtain the desired bound in (1.2.46). □

To prove the next theorem, we need the following lemma.

Lemma 1.2.1 (Kong and Zhang [317]) *The functions $A_k[x]$ $(k = 0, 1, \cdots, n)$ defined by*

$$\begin{cases} A_0[x] = x, & (1.2.49) \\ A_{k+1}[x] = A_k[x] + A_k[q_{k+1}] \int_\alpha^t b_{k+1} A_k[x] \exp\left(\int_s^t b_{k+1} A_k[q_{k+1}] d\tau\right) ds & (1.2.50) \end{cases}$$

satisfy the following conditions for $x(t) \geq 0, y(t) \geq 0$:

(1) $A_k[x+y] = A_k[x] + A_k[y]$;
(2) $A_k[xy](t) \leq (A_k[x])y(t)$ *if $y(t)$ is non-decreasing.*

Proof Clearly, conclusions (1) and (2) are true for $k = 0$. Suppose (1) and (2) are true for $k = i$ $(i = 0, \cdots, n-1)$, i.e.,

$$A_i[x+y] = A_i[x] + A_i[y], \quad A_i[xy] \leq A_i[x]y.$$

Then

$$\begin{aligned} A_{i+1}[x+y] &= A_i[x+y] + A_i[q_{i+1}] \int_\alpha^t b_{i+1} A_i[x+y] \exp\left(\int_s^t b_{i+1} A_i[q_{i+1}] d\tau\right) ds \\ &= A_i[x] + A_i[q_{i+1}] \int_\alpha^t b_{i+1} A_i[x] \exp\left(\int_s^t b_{i+1} A_i[q_{i+1}] d\tau\right) ds \\ &\quad + A_i[y] + A_i[q_{i+1}] \int_\alpha^t b_{i+1} A_i[y] \exp\left(\int_s^t b_{i+1} A_i[q_{i+1}] d\tau\right) ds \\ &= A_{i+1}[x] + A_{i+1}[y], \\ A_{i+1}[xy] &= A_i[xy] + A_i[q_{i+1}] \int_\alpha^t b_{i+1} A_i[xy] \exp\left(\int_s^t b_{i+1} A_i[q_{i+1}] d\tau\right) ds \\ &\leq \left\{A_i[x] + A_i[q_{i+1}] \int_\alpha^t b_{i+1} A_i[x] \exp\left(\int_s^t b_{i+1} A_i[q_{i+1}] d\tau\right) ds\right\} y \\ &= A_{i+1}[x]y. \end{aligned}$$

This proves that (1) and (2) are true for $k = i + 1$. □

Theorem 1.2.13 (Kong and Zhang [317]) *Let $a(t), b_i(t)$ and $q_i(t)$ $(i = 1, 2, \cdots, n)$ be non-negative continuous functions in $J = [\alpha, \beta]$, let $u(t)$ be a continuous function in J, and suppose that for all $t \in J$,*

$$u(t) \leq a(t) + \sum_{i=1}^n q_i(t) \int_\alpha^t b_i(s) u(s) ds. \tag{1.2.51}$$

Then for all $t \in J$,

$$u(t) \leq A_n[a](t), \tag{1.2.52}$$

where the function $A_n[x]$ *is defined in (1.2.49)–(1.2.50).*

Proof This proof is also by induction. For $n = 1$, (1.2.52) becomes

$$u(t) \leq a(t) + q_1(t) \int_\alpha^t b_1(s)a(s) \exp\left(\int_s^t b_1(\tau)q_1(\tau)d\tau\right) ds, \tag{1.2.53}$$

which indeed follows from Theorem 1.2.6 immediately.

Suppose (1.2.52) holds for $n = k \quad (1 \leq k \leq n-1)$. Then, in view of the induction assumption,

$$u \leq [a + q_{k+1} \int_\alpha^t b_{k+1}uds] + \sum_{i=1}^k q_i \int_\alpha^t b_i uds$$

implies

$$u \leq A_k[a + q_{k+1} \int_\alpha^t b_{k+1}uds]. \tag{1.2.54}$$

Since $\int_\alpha^t b_{k+1}uds$ is non-decreasing in t, (1.2.49) and (1.2.50) imply

$$u \leq A_k[a] + A_k[q_{k+1}] \int_\alpha^t b_{k+1}uds.$$

Now Theorem 1.2.6 implies

$$u \leq A_k[a] + A_k[q_{k+1}] \int_\alpha^t b_{k+1}A_k[a] \exp\left(\int_s b_{k+1}A_k[q_{k+1}]d\tau\right) ds = A_{k+1[a]}.$$

The proof is now complete. □

Remark 1.2.9 It is easy to show that the estimate here is better than that of Theorem 1 in [647]. Now we prove that under the condition of Theorem 1 in [181], the results here are better too.

By Theorem 1.2.13, $u(t) \leq A_n[a](t)$, which, according to Theorem 1 in [181], gives us

$$y(x) \leq E^n(f).$$

We shall show that $A_n(f) \leq E^n(f)$.

Clearly, $A_0(f) = E^0(f)$. Suppose $A_k(f) \leq E^k(f)$ for $0 \leq k \leq n-1$. Because $g_i(x) \geq 1$ $(i = 1, 2, \ldots, n)$, $f(x)$ and $A_k(u)$ are non-decreasing,

$$A_k(f) \leq fA_k(1) \leq fA_k(g_{k+1}) \leq fE^k(g_{k+1}).$$

Hence

$$\begin{aligned}
A_{k+1}(f) &= A_k(f) + A_k(g_{k+1}) \int_0^x h_{k+1}A_k(f) \exp\left(\int_s^x h_{k+1}A_k(g_{k+1})dt\right) ds \\
&= A_k(f) - A_k(g_{k+1}) \int_0^x \frac{A_k(f)}{A_k(g_{k+1})} ds \exp\left(\int_s^x h_{k+1}A_k(g_{k+1})dt\right) \\
&= A_k(f) - A_k(g_{k+1}) \Big[\frac{A_k(f)}{A_k(g_{k+1})} \exp\left(\int_s^x h_{k+1}A_k(g_{k+1})dt\right)\Big|_0^a \\
&\quad - \int_0^x \exp\left(\int_s^x h_{k+1}A_k(g_{k+1})dt\right) d\left(\frac{A_k(f)}{A_k(g_{k+1})}\right)\Big] \\
&\leq \frac{A_k(f)}{A_k(g_{k+1})}\Big|_{x=0} A_k(g_{k+1}) \exp\left(\int_0^x h_{k+1}A_k(g_{k+1})ds\right) \\
&\quad + A_k(f)(g_{k+1}) \exp\left(\int_0^x h_{k+1}A_k(g_{k+1})ds\right) \int_0^x d\left(\frac{A_k(f)}{A_k(g_{k+1})}\right) \\
&= A_k(f) \exp\left(\int_0^x h_{k+1}A_k(g_{k+1})ds\right) \\
&\leq fE^k(g_{k+1}) \exp\left(\int_0^x h_{k+1}E^k(g_{k+1})ds\right) = E^{k+1}(f).
\end{aligned}$$

Therefore, $A_n(f) \leq E^n(f)$.

As in Kong and Zhang [317], to see the difference among the three results, we next give an example.

Example 1.2.4 Let

$$y(x) \leq (1+x) + 2\int_0^x \frac{1}{1+s} y(s)ds + (1+x)\int_0^x \frac{3+4s}{(1+3s+2s^2)^2} y(s)ds.$$

We have

$$f(x) = g_2(x) = 1+x,\ g_1(x) = 2,\ h_1(x) = \frac{1}{1+x},\ h_2(x) = \frac{3+4x}{(1+3x+2x^2)^2}.$$

Then

$$A_1(f) = A_1(g_2) = (1+x) + 2\int_0^x \exp\left(2\int_s^x \frac{dt}{1+t}\right) ds = 1 + 3x + 2x^x.$$

By (1.2.50), we get

$$
\begin{aligned}
y(x) &\le (1+3x+2x^2) \\
&\quad +(1+3x+2x^2)\int_0^x \frac{3+4s}{(1+3s+2s^2)} \exp\left(\int_s^x \frac{3+4t}{1+3t+2t^2}dt\right) ds \\
&= (1+3x+2x^2)\left[1+(1+3x+2x^2)\left(1-\frac{1}{1+3x+2x^2}\right)\right] \\
&= (1+3x+2x^2)^2.
\end{aligned}
$$

By Theorem 1 in [647], we have

$$
\begin{aligned}
D_1W_0 &= (1+x)+2\left[\exp\left(2\int_0^x \frac{ds}{1+s}\right)\right]\int_0^x ds = 1+3x+4x^2+2x^3, \\
y(x) &\le E_2W_0 = D_2(D_1W_0) \\
&\le (1+3x+4x^2+2x^3)[1+\exp\left(\int_0^x \frac{3+4s}{(1+3s+2s^2)^2}(1+3s+4s^2+2s^3)ds\right) \\
&\quad \times \int_0^x \frac{3+4s}{(1+3s+2s^2)^2}(1+3s+4s^2+2s^3)ds].
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
E_2W_0 &> (1+3x+4x^2+2x^3)\left[1+\int_0^x \frac{3+4s}{1+3s+2s^2}\exp\left(\int_s^x \frac{3+4t}{1+3t+2t^2}dt\right) ds\right] \\
&= (1+3x+4x^2+2x^3)(1+3x+2x^2).
\end{aligned}
$$

By Theorem 1.2.13 (Theorem 1 in [182]), we have

$$
\begin{cases}
E^1(f) = E^1(g_2) = 2(1+x)\exp\left(2\int_0^x \dfrac{ds}{1+s}\right) = 2(1+x)^3, \\[2ex]
y(x) \le E^2(f) = 2(1+x)^4\exp\left(2\int_0^x \dfrac{(3+4s)(1+s)^3}{(1+3s+2s^2)^2}ds\right).
\end{cases}
$$

Obviously,

$$
E^2(f) \ge 2(1+x)^4e^{2x} \ge (1+3x+2x^2)^2 + 2(1+x)^4(e^x-x-1).
$$

The difference among the estimates are quite large.

The next result is a generation of Theorem 1.2.12.

Theorem 1.2.14 (El-Owaidy-Ragab-Abdeldaim [203]) *Let $x(t)$, $g(t)$, $f(t)$ and $h(t)$ be real-valued non-negative monotonic, non-decreasing continuous functions defined on $\mathbb{R}_+ = [0, +\infty)$, satisfying for all $t \in \mathbb{R}_+$,*

$$x(t) \leq x_0 + g(t) \int_0^t f(s)x(s)ds + \int_0^t f(s) \left(\int_0^s h(\tau)x(\tau)d\tau \right) ds \tag{1.2.55}$$

where x_0 is a non-negative constant. Then for all $t \in \mathbb{R}_+$,

$$x(t) \leq \Big(x_0 + \int_0^t f(s) \exp\Big(\int_0^s h(\tau)k(\tau)d\tau\Big)ds\Big)k(t) \tag{1.2.56}$$

where

$$k(t) = \left[1 + g(t) \int_0^t f(s) \exp\left(\int_0^s g(\tau)f(\tau)d\tau\right) ds\right].$$

Proof Define, for all $t \in \mathbb{R}_+$,

$$n(t) = x_0 + \int_0^t f(s)\Big(\int_0^s h(\tau)x(\tau)d\tau\Big)ds; \;\; n(0) = x_0.$$

Then (1.2.55) can be restated as, for all $t \in \mathbb{R}_+$,

$$x(t) \leq n(t) + g(t) \int_0^t f(s)x(s)ds.$$

Since $n(t)$ is a positive, monotonic, non-decreasing continuous function defined on $\mathbb{R}_+ = [0, +\infty)$, we derive from Theorem 1.2.7 that for all $t \in \mathbb{R}_+$,

$$x(t) \leq n(t)k(t). \tag{1.2.57}$$

Thus for all $t \in \mathbb{R}_+$,

$$n(t) \leq x_0 + \int_0^t f(s) \left(\int_0^s h(\tau)k(\tau)n(\tau)d\tau\right) ds.$$

Differentiating $n(t)$ with respect to t, we have for all $t \in \mathbb{R}_+$,

$$n'(t) \leq f(t) \int_0^t h(s)k(s)n(s)ds.$$

Let

$$m(t) = \int_0^t h(s)k(s)n(s)ds, \;\; m(0) = 0.$$

Then for all $t \in \mathbb{R}_+$,

$$n'(t) \leq f(t)m(t). \tag{1.2.58}$$

Differentiating $m(t)$ with respect to t, and using the fact that $n(t) \leq m(t)$, we have for all $t \in \mathbb{R}_+$,

$$m'(t) \leq h(t)k(t)n(t).$$

Integrating from 0 to t, we obtain, for all $t \in \mathbb{R}_+$,

$$m(t) \leq \exp\left(\int_0^t h(s)k(s)ds\right).$$

Thus it follows from (1.2.58) that for all $t \in \mathbb{R}_+$,

$$n'(t) \leq f(t)\exp(\int_0^t h(s)k(s)ds).$$

By integrating from 0 to t, we have for all $t \in \mathbb{R}_+$,

$$n(t) \leq x_0 + \int_0^t f(s)\exp(\int_0^s h(\tau)k(\tau)d\tau)ds. \tag{1.2.59}$$

Then the desired bound in (1.2.56) follows from (1.2.57) and (1.2.59). This completes the proof. □

Pachpatte [449, 457, 460, 462] showed the following theorem.

Theorem 1.2.15 (Pachpatte [449, 457, 460, 462]) *Let u, f, g and p be non-negative continuous functions defined on $\mathbb{R}_+$, and u_0 be a non-negative constant.*

(1) If for all $t \in \mathbb{R}_+$,

$$u(t) \leq u_0 + \int_0^t [f(s)u(s) + p(s)]ds + \int_0^t f(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds, \tag{1.2.60}$$

then for all $t \in \mathbb{R}_+$,

$$\begin{aligned} u(t) \leq u_0 + \int_0^t \Big[p(s) + f(s)\Big\{u_0\exp(\int_0^s [f(\sigma) + g(\sigma)]d\sigma) \\ + \int_0^s p(\sigma)\exp(\int_0^\sigma [f(\tau) + g(\tau)]d\tau)d\sigma\Big\}\Big]ds. \end{aligned} \tag{1.2.61}$$

(2) If for all $t \in \mathbb{R}_+$,

$$u(t) \le u_0 + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s [g(\sigma)u(\sigma) + p(\sigma)]d\sigma\right)ds, \tag{1.2.62}$$

then for all $t \in \mathbb{R}_+$,

$$u(t) \le u_0 + \int_0^t f(s)\Big[u_0 \exp\left(\int_0^s [f(\sigma) + g(\sigma)]d\sigma\right) + \int_0^t p(\sigma)\exp\left(\int_0^s [f(\tau) + g(\tau)]d\tau\right)\Big]ds. \tag{1.2.63}$$

(3) If for all $t \in \mathbb{R}_+$,

$$u(t) \le u_0 + \int_0^t f(s)u(s)ds + \int_0^t g(s)\left(u(s) + \int_0^s h(\sigma)u(\sigma)d\sigma\right)ds, \tag{1.2.64}$$

then for all $t \in \mathbb{R}_+$,

$$u(t) \le u_0\Big[\exp\left(\int_0^t f(\sigma)d\sigma\right) + \int_0^t g(s)\exp\left(\int_0^s [f(\sigma) + g(\sigma) + h(\sigma)]d\sigma\right) \times \exp\left(\int_s^t f(\sigma)d\sigma\right)ds\Big]. \tag{1.2.65}$$

(4) If for all $t \in \mathbb{R}_+$,

$$u(t) \le h(t) + p(t)\Big[\int_0^t f(s)u(s)ds + \int_0^t f(s)p(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)\Big], \tag{1.2.66}$$

then for all $t \in \mathbb{R}_+$,

$$u(t) \le h(t) + p(t)\Big[\int_0^t f(s)\Big\{h(s) + p(s)\int_0^s h(\sigma)[f(\sigma) + g(\sigma)] \times \exp\left(\int_\sigma^s p(\tau)[f(\tau) + g(\tau)]d\tau\right)d\sigma\Big\}ds\Big]. \tag{1.2.67}$$

Proof Since the proofs are similar one another, we only give the details of the proofs of (1) and (4).

(1) Define a function $z(t)$ by the right-hand side of (1.2.60). Then $z(0) = u_0, u(t) \leq z(t)$ and

$$\begin{aligned} z'(t) &= f(t)u(t) + p(t) + f(t)\int_0^t g(\sigma)u(\sigma)d\sigma \\ &\leq p(t) + f(t)\left[z(t) + \int_0^t g(\sigma)z(\sigma)d\sigma\right]. \end{aligned} \tag{1.2.68}$$

Define

$$v(t) = z(t) + \int_0^t g(\sigma)z(\sigma)d\sigma, \tag{1.2.69}$$

then we derive from (1.2.60),

$$v(0) = z(0) = u_0, z'(t) \leq p(t) + f(t)v(t), \tag{1.2.70}$$

From (1.2.69), it follows that $z(t) \leq v(t)$ and

$$\begin{aligned} v'(t) &= z'(t) + g(t)z(t) \\ &\leq p(t) + [f(t) + g(t)]v(t) \end{aligned} \tag{1.2.71}$$

which implies

$$\begin{aligned} v(t) \leq u_0 \exp&\left(\int_0^t \Big(f(\tau) + g(\tau)\Big)d\tau\right) \\ &+ \int_0^t p(\sigma)\exp\left(\int_\sigma^t \Big(f(\tau) + g(\tau)\Big)d\tau\right)d\sigma. \end{aligned} \tag{1.2.72}$$

Using (1.2.70) and (1.2.72), we get

$$\begin{aligned} z'(t) \leq p(t) + f(t)\Big[&u_0 \exp\left(\int_0^t [f(\tau) + g(\tau)]d\tau\right) \\ &+ \int_0^t p(\sigma)\exp\left(\int_\sigma^t [f(\tau) + g(\tau)]d\tau\right)d\sigma\Big]. \end{aligned} \tag{1.2.73}$$

Integrating the above inequality and using $u(t) \leq z(t)$, we can get the desired inequality (1.2.61).

(4) Set

$$v(t) = \int_0^t f(s)u(s)ds + \int_0^t f(s)p(s)\left(\int_0^s g(\sigma)u(\sigma)d\sigma\right)ds, \tag{1.2.74}$$

then $v(0) = 0, u(t) \leq h(t) + p(t)v(t)$ and

$$\begin{aligned} v'(t) &= f(t)u(t) + f(t)p(t)\int_0^t g(\sigma)u(\sigma)d\sigma \\ &\leq f(t)\left[h(t) + p(t)\left\{v(t) + \int_0^t g(\sigma)[h(\sigma) + p(\sigma)v(\sigma)]d\sigma\right\}\right]. \end{aligned} \tag{1.2.75}$$

If we set

$$m(t) = v(t) + \int_0^t g(\sigma)[h(\sigma) + p(\sigma)v(\sigma)]d\sigma, \tag{1.2.76}$$

then from (1.2.66) it follows that $m(0) = v(0) = 0, v'(t) \leq f(t)[h(t) + p(t)m(t)]$. From $v(t) \leq m(t)$, we have

$$\begin{aligned} m'(t) &= v'(t) + g(t)[h(t) + p(t)v(t)] \\ &\leq h(t)[f(t) + g(t)] + p(t)[f(t) + g(t)]m(t) \end{aligned}$$

which gives us

$$m(t) \leq \int_0^t h(\sigma)[f(\sigma) + g(\sigma)]\exp\left(\int_\sigma^t p(\tau)[f(\tau) + g(\tau)]d\tau\right)d\sigma. \tag{1.2.77}$$

Using (1.2.75) in (1.2.77), we have

$$\begin{aligned} v'(t) \leq f(t)\Big[h(t) + p(t)\int_0^t & h(\sigma)[f(\sigma) + g(\sigma)] \\ & \times \exp\left(p(\tau)[f(\tau) + g(\tau)]d\tau\right) d\sigma\Big]. \end{aligned} \tag{1.2.78}$$

Integrating (1.2.78) and using $u(t) \leq h(t) + p(t)v(t)$, we conclude the desired inequality (1.2.67). □

Corollary 1.2.5 (Pachpatte [443]) *Let $u(t), f(t)$ and $g(t)$ be real-valued non-negative continuous functions defined on $\mathbb{R}_+$, and $n(t)$ be a positive, monotonic, non-decreasing continuous function defined on $\mathbb{R}_+$, satisfying that for all $t \in \mathbb{R}_+$,*

$$u(t) \leq n(t) + \int_0^t f(s)u(s)ds + \int_0^t f(s)[\int_0^s g(\tau)u(\tau)d\tau]ds. \tag{1.2.79}$$

Then for all $t \in \mathbb{R}_+$,

$$u(t) \le n(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s (f(\tau) + g(\tau))d\tau\right)ds\right]. \qquad (1.2.80)$$

Proof Since $n(t)$ is positive, monotonic and non-decreasing continuous on $\mathbb{R}_+$, we deduce from (1.2.80) that

$$\begin{aligned}\frac{u(t)}{n(t)} &\le 1 + \int_0^t f(s)\frac{u(s)}{n(t)}ds + \int_0^t f(s)\left(\int_0^s g(\tau)\frac{u(\tau)}{n(t)}d\tau\right)ds \\ &\le 1 + \int_0^t f(s)\frac{u(s)}{n(s)}ds + \int_0^t f(s)\left(\int_0^s g(\tau)\frac{u(\tau)}{n(\tau)}d\tau\right)ds. \qquad (1.2.81)\end{aligned}$$

Thus by using Theorem 1.2.15(1), we may derive (1.2.81) from (1.2.80). □

Theorem 1.2.16 (Pachpatte [449, 457, 460, 462]) *Let u, k, p, f, g and h be non-negative continuous functions on $\mathbb{R}_+$ and u_0 is a non-negative constant.*

(1) If for all $t \in \mathbb{R}_+$,

$$\begin{aligned}u(t) \le u_0 &+ \int_0^t f(s)u(s)ds + \int_0^t f(s)g(\sigma)u(\sigma)d\sigma \\ &+ \int_0^t f(s)\left[\int_0^s g(\sigma)\left(\int_0^\sigma h(\tau)u(\tau)d\tau\right)d\sigma\right]ds, \qquad (1.2.82)\end{aligned}$$

then for all $t \in \mathbb{R}_+$,

$$\begin{aligned}u(t) \le u_0\Big[1 &+ \int_0^t f(s)\exp\left(\int_0^s f(\sigma)d\sigma\right) \\ &\times\left[1 + \int_0^s g(\sigma)\exp\left(\int_0^\sigma [g(\tau) + h(\tau)]d\tau\right)d\sigma\right]ds\Big]. \qquad (1.2.83)\end{aligned}$$

(2) If for all $t \in \mathbb{R}_+$,

$$\begin{aligned}u(t) \le k(t) + p(t)\Big[&\int_0^t f(s)u(s)ds + \int_0^t f(s)p(s)\left(\int_0^s g(\tau)u(\tau)d\tau\right)ds \\ &+ \int_0^t f(s)p(s)\left[\int_0^s g(\tau)p(\tau)\left(\int_0^\tau h(\sigma)u(\sigma)d\sigma\right)d\tau\right]ds\Big], \qquad (1.2.84)\end{aligned}$$

then for all $t \in \mathbb{R}_+$,

$$u(t) \le k(t) + p(t)\Big[\int_0^t f(s)\Big[k(s) + p(s)\Big\{\int_0^s \exp\left(\int_\tau^s f(\sigma)p(\sigma)d\sigma\right)$$
$$\times\Big(k(\tau)[f(\tau) + g(\tau)] + g(\tau)p(\tau)\int_0^\tau k(\sigma)[f(\sigma) + g(\sigma) + h(\sigma)]$$
$$\times\exp\left(\int_\sigma^\tau p(\xi)[f(\xi) + g(\xi) + h(\xi)]d\xi\right)d\sigma\Big)d\tau\Big\}\Big]ds\Big]. \tag{1.2.85}$$

Proof The proof is similar to those of (1) and (4) in Theorem 1.2.15. □

The next corollary is the classical Bellman inequality (e.g., Theorem 1.1.2).

Corollary 1.2.6 *Let* $u(t)$ *and* $b(t)$ *be continuous functions in* $J = [\alpha, \beta]$, *let* $b(t)$ *be non-negative in* J, *and suppose that for all* $t \in J$,

$$u(t) \le a + \int_\alpha^t b(s)u(s)ds, \tag{1.2.86}$$

where a *is a constant. Then for all* $t \in J$,

$$u(t) \le a\exp\left(\int_\alpha^t b(s)ds\right). \tag{1.2.87}$$

Corollary 1.2.7 *Let* $u(t)$ *be a continuous function in* $J = [\alpha, \beta]$, *and suppose that for all* $t \in J$,

$$u(t) \le a + \int_\alpha^t bu(s)ds, \tag{1.2.88}$$

where $b \ge 0$ *and* a *are constants. Then for all* $t \in J$,

$$u(t) \le a\exp\big(b(t-\alpha)\big). \tag{1.2.89}$$

Corollary 1.2.8 (Chandirov [127]) *Let* $a(t), b(t), c(t)$, *and* $u(t)$ *be continuous functions in* $J = [\alpha, \beta]$, *and let* $b(t)$ *and* $c(t)$ *be non-negative in* J, *and suppose that for all* $t \in J$,

$$u(t) \le a(t) + \int_\alpha^t [b(s)u(s) + c(s)]ds. \tag{1.2.90}$$

Then for all $t \in J$,

$$u(t) \le \left[\sup_{s\in[\alpha,t]} a(s) + \int_\alpha^t c(s)ds\right]\exp\left(\int_\alpha^t b(s)ds\right). \tag{1.2.91}$$

Proof It is easy to see from (1.2.90) that

$$u(t) \leq A(t) + \int_{\alpha}^{t} b(s)u(s)ds$$

where $A(t) = \sup_{s\in[\alpha,t]} a(s) + \int_{\alpha}^{t} c(s)ds$ is a non-decreasing function in J. Thus (1.2.91) follows from Theorem 1.1.4. □

The next theorem gives us the best possible estimate for a function $u(t)$ satisfying (1.2.92), which can be also regarded as a corollary of the Jones inequality (e.g., Theorem 1.2.12).

Corollary 1.2.9 (Chandirov [127]) *Let $a(t), b(t), c(t)$, and $u(t)$ be continuous functions in $J = [\alpha, \beta]$, and let $b(t)$ be non-negative in J, and suppose that for all $t \in J$,*

$$u(t) \leq a(t) + \int_{\alpha}^{t} \Big(b(s)u(s) + c(s)\Big)ds. \tag{1.2.92}$$

Then for all $t \in J$,

$$u(t) \leq a(t) + \int_{\alpha}^{t} [a(s)b(s) + c(s)] \exp\left(\int_{s}^{t} b(\tau)d\tau\right) ds, \tag{1.2.93}$$

$$u(t) \leq \int_{\alpha}^{t} c(s)ds + \sup_{\alpha\leq t\leq\beta} |a(t)| \exp\left(\int_{\alpha}^{t} b(s)ds\right). \tag{1.2.94}$$

Proof Let $v(t) = \int_{\alpha}^{t} \Big(b(s)u(s) + c(s)\Big)ds$. Then it follows from (1.2.92) that

$$\begin{aligned} v'(t) &= b(t)u(t) + c(t) \leq b(t)\Big(a(t) + v(t)\Big) + c(t) \\ &= a(t)b(t) + b(t)v(t) + c(t), \quad v(\alpha) = 0. \end{aligned}$$

Thus from Lemma 1.1.1 and noting that $u(t) \leq a(t) + v(t)$, we can easily derive (1.2.93) and (1.2.94) is a direct consequence of (1.2.93). □

Corollary 1.2.10 *If in Corollary 1.2.9, function $a(t)$ is non-decreasing in J, then for all $t \in J$,*

$$u(t) \leq a(t) \exp\left(\int_{\alpha}^{t} b(\tau)d\tau\right) + \int_{\alpha}^{t} c(s) \exp\left(\int_{s}^{t} b(\tau)d\tau\right) ds. \tag{1.2.95}$$

Corollary 1.2.11 *If in Corollary 1.2.10, $a(t) \equiv a =$ constant, then for all $t \in J$,*

$$u(t) \leq a \exp\left(\int_{\alpha}^{t} b(\tau)d\tau\right) + \int_{\alpha}^{t} c(s) \exp\left(\int_{s}^{t} b(\tau)d\tau\right) ds. \tag{1.2.96}$$

In 1942, Quade [540] showed the following result.

Corollary 1.2.12 (Quade [540]) *Let $u(t)$ be a continuous function for all $t \geq \alpha$, and suppose that for all $t \geq \alpha$,*

$$u(t) \leq ae^{-\gamma(t-\alpha)} + \int_\alpha^t e^{-\gamma(t-s)}[bu(s) + c]ds, \tag{1.2.97}$$

where $b \geq 0, a, c,$ and $\gamma \neq b$ are constants. Then for all $t \geq \alpha$,

$$u(t) \leq ae^{(b-\gamma)(t-\alpha)} + \frac{c}{\gamma - b}[1 - e^{(b-\gamma)(t-\alpha)}]. \tag{1.2.98}$$

Corollary 1.2.13 (Filatov [214]) *Let $u(t)$ be a continuous function in $J = [\alpha, \beta]$, and suppose that for all $t \in J$,*

$$u(t) \leq a + \int_\alpha^t [bu(s) + c]ds, \tag{1.2.99}$$

where $b > 0, a,$ and c are constants. Then for all $t \in J$,

$$u(t) \leq \frac{c}{b}\left(e^{b(t-\alpha)} - 1\right) + ae^{b(t-\alpha)}. \tag{1.2.100}$$

Corollary 1.2.14 (Rudakov [562]) *Let $a(t), b(t), c(t),$ and $u(t)$ be continuous functions in $J = [\alpha, \beta]$, and let $b(t)$ be non-negative in J, and suppose that for all $t, r \in J$,*

$$u(t) \leq u(r) + \int_r^t \Big(b(s)u(s) + c(s)\Big)|ds|. \tag{1.2.101}$$

Then for all $t \in J$,

$$\left[u(\alpha) - \int_\alpha^t c(s)\exp\left(\int_\alpha^s b(\tau)d\tau\right)ds\right]\exp\left(-\int_\alpha^t b(s)ds\right) \leq u(t), \tag{1.2.102}$$

$$u(t) \leq \left[u(\alpha) + \int_\alpha^t c(s)\exp\left(-\int_\alpha^s b(\tau)d\tau\right)ds\right]\exp\left(\int_\alpha^t b(s)ds\right). \tag{1.2.103}$$

Proof From (1.2.101) for $\alpha = r \leq t \leq \beta$, we can get (1.2.95) with $a(t) \equiv u(\alpha)$. From Corollary 1.2.10, we can obtain (1.2.102). From (1.2.101) for $\alpha \leq t \leq r = \beta$, we can derive

$$u(t) \leq u(r) + \int_t^r [b(s)u(s) + c(s)]ds \equiv v(t).$$

Then $v(r) = u(r)$ and for all $t \leq r$,

$$v'(t) \geq -b(t)v(t) - c(t),$$

which gives us

$$u(t) \leq v(t) \leq u(r) \exp\left(-\int_r^t b(s)ds\right) - \int_r^t c(s) \exp\left(-\int_s^t b(\tau)d\tau\right) ds.$$

Replacing t by α and subsequently r by t, we get for all $t \geq \alpha$,

$$u(\alpha) \leq u(t) \exp\left(-\int_t^\alpha b(s)ds\right) - \int_t^\alpha c(s) \exp\left(-\int_s^\alpha b(\tau)d\tau\right) ds,$$

which is equivalent to (1.2.103). □

The next result is a corollary which is the Bellman-Reid inequality (see also Theorem 1.1.3).

Corollary 1.2.15 (Bellman-Reid [211]) *Let $u(t)$ and $b(t) \geq 0$ be continuous functions in $J = [\alpha, \beta]$, and suppose that for all $t, r \in J$,*

$$u(t) \leq u(r) + \int_r^t b(s)u(s)|ds|. \tag{1.2.104}$$

Then for all $\alpha \leq t \leq \beta$,

$$u(\alpha) \exp\left(-\int_\alpha^t b(s)ds\right) \leq u(t) \leq u(\alpha) \exp\left(\int_\alpha^t b(s)ds\right). \tag{1.2.105}$$

In 1962, Bykov an Salpagarov [120] proved the next two results.

Theorem 1.2.17 (Bykov-Salpagarov [120]) *Let $u(t)$, $v(t)$, $h(t, r)$, and $H(t, r, x)$ be non-negative functions for all $t \geq r \geq x \geq a$ and c_1, c_2, and c_3 be non-negative constants not all zero. If*

$$\begin{aligned} u(t) \leq c_1 + c_2 \int_a^t \left[v(s)u(s) + \int_a^s h(s, r)u(r)dr\right] ds \\ + \quad c_3 \int_a^t \int_a^r \int_a^s H(s, r, x)u(x)dxdsdr, \end{aligned} \tag{1.2.106}$$

then for all $t \geq a$,

$$u(x) \leq c_2 \exp\left\{c_2 \int_a^t [v(s) + \int_a^s h(s, r)dr]ds + c_3 \int_a^t \int_a^s \int_a^r H(s, r, x)dxdrds\right\}. \tag{1.2.107}$$

Proof By $b(t)$ we denote the right-hand side of (1.2.106). Then $b(s) \leq b(t)$ for $s \leq t$ since all the terms are non-negative, we derive from (1.2.106)

$$\begin{aligned}\frac{b'(t)}{b(t)} &= c_2 v(t)\frac{u(t)}{b(t)} + c_2\int_a^t \frac{h(t,r)u(r)}{b(t)}dr + c_3\int_a^t\int_a^r \frac{H(t,r,x)u(x)}{b(t)}dxdr \\ &\leq c_2 v(t) + c_2\int_a^t h(t,r)dr + c_3\int_a^t\int_a^r H(t,r,x)dxdr. \qquad (1.2.108)\end{aligned}$$

Thus integration of (1.2.108) from a to t yields

$$\begin{aligned}\log b(t) - \log c_1 \leq c_2\int_0^b &\left[v(s) + \int_a^s h(s,r)dr\right]ds \\ &+ c_3\int_a^t\int_a^s\int_a^r H(s,r,x)dxdrds. \qquad (1.2.109)\end{aligned}$$

Writing this in terms of $b(t)$ and using $u(t) \leq b(t)$, we may complete the proof. □

Corollary 1.2.16 *Let $u(t)$, $f(t)$ be non-negative continuous functions in a real interval $I = [a, b]$. Suppose that $k(t, s)$ and its partial derivatives $k_t(t, s)$ exist and are non-negative continuous functions for almost every $t, s \in I$. If the following inequality holds for all $a \leq t \leq s \leq t \leq b$,*

$$u(t) \leq c + \int_a^t g(s)u(s)ds + \int_a^t f(s)\left(\int_a^s k(s,t)u(t)dt\right)ds, \qquad (1.2.110)$$

where c is a non-negative constant, then for all $t \geq a$,

$$u(t) \leq c\left[1 + \int_a^t f(s)\exp\left(\int_a^s (f(t) + k(t,t))dt\right)ds\right] \qquad (1.2.111)$$

Proof Define a function $v(t)$ by the right-hand side of (1.2.110). Then it follows that

$$u(t) \leq v(t). \qquad (1.2.112)$$

Therefore for $v(a) = c$, we have, by (1.2.112),

$$\begin{aligned}v'(t) &= f(t)u(t) + f(t)\int_a^t k(k,t)u(t)dt, \\ &\leq f(t)\left(v(t) + \int_a^t k(k,t)v(t)dt\right). \qquad (1.2.113)\end{aligned}$$

If we put

$$m(t) = v(t) + \int_a^t k(k,t)v(t)dt, \tag{1.2.114}$$

then

$$v(t) \leq m(t). \tag{1.2.115}$$

Therefore by $m(a) = v(a) = c$, we have, by (1.2.113) and (1.2.115),

$$\begin{aligned} m'(t) &= v'(t) + k(t,t)v(t) + \int_a^t k_t(k,t)v(t)dt, \\ &\leq v'(t) + k(t,t)v(t), \\ &\leq f(t)m(t) + k(t,t)v(t), \\ &\leq (f(t) + k(t,t))m(t). \end{aligned} \tag{1.2.116}$$

Integrating (1.2.116) from a to t, we obtain

$$m(t) \leq c \exp\left(\int_a^t (f(s) + k(s,s))ds\right). \tag{1.2.117}$$

Substituting (1.2.117) into (1.2.113), we have

$$v'(t) \leq cf(t) \exp\left(\int_a^t (f(s) + k(s,s))ds\right). \tag{1.2.118}$$

Integrating both sides of (1.2.118) from a to t, we obtain

$$u(t) \leq c\left[1 + \int_a^t f(s) \exp\left(\int_a^s (f(t) + k(t,t))dt\right) ds\right].$$

Therefore, by (1.2.112), we have the desired result. □

Remark 1.2.10 If in Corollary 1.2.16, we set $k(t,s) = g(s)$, then (1.2.111), corresponding to an estimate in Theorem 1, was obtained in [445].

Theorem 1.2.18 (Bykov-Salpagarov [120]) *Let the functions $u(t), \sigma(t), v(t)$, and $w(t,r)$ be non-negative and continuous for $a \leq r \leq t$, and let c_1, c_2, and c_3 be non-negative. If for all $t \in I = [a, +\infty)$,*

$$u(t) \leq c_1 + \sigma(t)\Big(c_2 + c_3 \int_a^t [v(s)u(s) + \int_a^s w(s,r)u(r)dr]ds\Big), \tag{1.2.119}$$

then for all $t \in I$,

$$u(t) \le c_1 + \sigma(t)\Big\{c_2 \exp\left[c_3 \int_a^t \left(v(s)\sigma(s) + \int_a^s w(s,r)\sigma(r)dr\right) ds\right] \\ +c_1 c_3 \int_a^t \left(v(s) + \int_a^s w(s,r)dr\right) \\ \times \exp\left[c_3 \int_s^t \left(v(\tau)\sigma(\tau) + \int_a^\tau w(\tau,r)\sigma(r)dr\right) d\tau\right] ds\Big\}. \quad (1.2.120)$$

Proof The proof is left to the reader as an exercise. □

Pachpatte [451] showed the next result.

Theorem 1.2.19 (Pachpatte [451]) *Let $x(t)$, $f_1(t)$, $f_2(t)$, $f_3(t)$ and $p(t)$ be real-valued non-negative continuous functions on $\mathbb{R}_+$ satisfying that for all $t \in \mathbb{R}_+$,*

$$x(t) \le p(t) + \int_0^t f_1(s)x(s)ds + \int_0^t f_2(s)\left[\int_0^s f_3(\tau)x(\tau)d\tau\right] ds. \quad (1.2.121)$$

Then for all $t \in \mathbb{R}_+$,

$$x(t) \le p(t) + \int_0^t \left[f_1(s)p(s) + f_2(s)\int_0^s f_3(\tau)p(\tau)d\tau\right] \\ \times \exp\left(\int_0^t \left[f_1(\tau) + f_2(\tau)\int_0^\tau f_3(\eta)d\eta\right] d\tau\right) ds. \quad (1.2.122)$$

Proof Define

$$R(t) = \int_0^t f_1(s)x(s)ds + \int_0^t f_2(s)\left[\int_0^s f_3(\tau)x(\tau)d\tau\right] ds, \; R(0) = 0,$$

then from (1.2.121) and the non-decreasing nature of $R(t)$, it follows that

$$R'(t) \le f_1(t)[p(t) + R(t)] + f_2(t)\left[\int_0^t f_3(s)p(s)ds + R(t)\int_0^t f_3(s)ds\right].$$

Integrating the above inequality, the desired result (1.2.122) follows readily. □

Corollary 1.2.17 (Agarwal [4]) *In Theorem 1.2.19, let $p(t)$ be also non-decreasing. Then we have*

$$x(t) \le p(t)\phi(t), \quad (1.2.123)$$

where

$$\phi(t) = \exp\left(\int_0^t \left[f_1(s) + f_2(s)\int_0^s f_3(\tau)d\tau\right]ds\right).$$

Theorem 1.2.20 (Agarwal [4]) *If h, f, g, u are non-negative continuous functions $(\mathbb{R} \to \mathbb{R}_+)$, and if, for all $x > 0$, there holds that*

$$u(x) \le h(x) + \int_0^x f(s)u(s)ds + \int_0^x f(s)\Big(\int_0^s g(t)u(t)dt\Big)ds, \qquad (1.2.124)$$

then there are continuous non-negative functions η, ϕ, ψ (not depending on u), satisfying the following inequality

$$u(x) \le \eta(x) + \phi(x)\int_0^x \psi(s)\eta(s)\exp\Big(\int_s^x \psi(t)\phi(t)dt\Big)ds \qquad (1.2.125)$$

with $\phi(x) = 1$, $\psi = f + g$, $\eta(x) = h(x)$.

Proof Let $z(x) = u(x) + \int_0^x g(s)u(s)ds$. Then

$$u(x) \le z(x) \le h(x) + \int_0^x f(s)\Big[u(s) + \int_0^s g(t)u(t)dt\Big]ds + \int_0^x g(s)u(s)ds$$
$$\le h(x) + \int_0^x [f(s) + g(s)]z(s)ds.$$

Then (1.2.125) follows from Theorem 1.2.1 on putting $\eta = h, \phi = 1$, and $\psi = f + g$ in (1.2.125). □

Using Riemann function methods, Yeh [669] also treated the case where (1.2.124) is replaced by

$$u(x) \le a(x) + \int_0^x b(s)u(s)ds + \int_0^x c(s)\Big\{\int_0^s g(t)u(t)dt\Big\}ds$$
$$+ \int_0^x c(s)\Big\{\int_0^s p(t)[\int_0^t q(r)u(r)dr]dt\Big\}ds \qquad (1.2.126)$$

where non-negativity and continuity of all the functions appearing in (1.2.126) hold.

The following result is a more general case of Theorem 1.2.18.

Theorem 1.2.21 (Agarwal [4]) *If there holds for all $x \ge 0$,*

$$u(x) \le a(x) + f_1(x)\int_0^x f_2(s)u(s)ds + g_1(x)\int_0^x g_2(x)\Big(\int_0^s g_3(t)u(t)dt\Big)ds$$
$$+ h_1(x)\int_0^x h_2(s)\Big(\int_0^s h_3(t)\Big(\int_0^t h_4(r)u(r)dr\Big)dt\Big)ds, \qquad (1.2.127)$$

and all the functions are continuous and non-negative for all $x \geq 0$*, then the conclusion of Theorem 1.2.19 holds.*

Proof Let $z = u + \int_0^x g_3 u dt + \int_0^x h_3(t)\Big(\int_0^t h_4(r)u(r)dr\Big)dt$. Then

$$\begin{aligned} u \leq z &\leq a + [f_1 + g_1 + h_1] \int_0^x (f_2 + g_2 + h_2) z ds \\ &\quad + \int_0^x g_3 u dt + \int_0^x h_3(t)\Big(\int_0^t h_4(r)u(r)dr\Big)dt \\ &\leq a + (f_1 + g_1 + h_1 + 1) \int_0^x (f_2 + g_2 + g_3 + h_2 + h_3) z dt. \end{aligned}$$

Applying Theorem 1.2.20 to the above inequality with $\eta = a, \phi = f_1 + g_1 + h_1 + 1$, and $\psi = f_2 + g_2 + h_2 + g_3 + h_3$ yields (1.2.127). □

The following result was proved by Qin [538].

Theorem 1.2.22 (Jones-Qin [538]) *Assume that* $f(t), g(t)$ *and* $y(t)$ *are non-negative continuous functions in* $[\tau, T]$ $(\tau < T)$ *verifying the following integral inequality for all* $t \in [\tau, T]$,

$$y(t) \leq f(t) + \int_\tau^t g(s)y(s)ds. \tag{1.2.128}$$

Then we have for all $t \in [\tau, T]$,

$$y(t) \leq f(t) + \int_\tau^t \exp\{\int_s^t g(\theta)d\theta\} g(s)f(s)ds. \tag{1.2.129}$$

In addition, if $f(t)$ *is a non-decreasing function in* $[\tau, T]$*, then for all* $t \in [\tau, T]$,

$$y(t) \leq f(t)\left[1 + \int_\tau^t \exp\left(\int_s^t g(\theta)d\theta\right) g(s)ds\right], \tag{1.2.130}$$

$$\leq f(t)\left[1 + \int_\tau^t g(s)ds \exp\left(\int_\tau^t g(\theta)d\theta\right)\right]. \tag{1.2.131}$$

If, further, $T = +\infty$ *and* $\int_\tau^{+\infty} g(s)ds < +\infty$*, then we have*

$$y(t) \leq Cf(t) \tag{1.2.132}$$

where $C = 1 + \int_\tau^{+\infty} g(s)ds \exp\left(\int_\tau^{+\infty} g(\theta)d\theta\right)$ *is a positive constant.*

Proof

(1) (1.2.129) is the Jones inequality in Theorem 1.2.1.
(2) If $f(t)$ is a non-decreasing function in $[\tau, T]$, then (1.2.130) and (1.2.131) easily follow from (1.2.129).
(3) If, further, $T = +\infty$ and $\int_\tau^{+\infty} g(s)ds < +\infty$, then (1.2.132) easily follows from (1.2.131). □

In 1984, Yang [657] showed the next result, which concerns the linear integral inequalities with iterated integrals.

Theorem 1.2.23 (Young [657]) *Let $u(t)$ be continuous and non-negative on $I = [0, h)$ and let $p(t)$ be continuous, positive, and non-decreasing on I. Suppose that $f_i(t, s)$, $(i = 1, 2, \cdots, n)$ are continuous non-negative functions on $I \times I$, and non-decreasing in t. If for all $t \in I$,*

$$u(t) \le p(t) + \int_0^t f_1(t, t_1) \int_0^{t_1} f_2(t_1, t_2) \cdots \int_0^{t_{n-1}} f_n(t_{n-1}, t_n)u(t_n)dt_n \cdots dt_1, \tag{1.2.133}$$

then for all $t \in I$,

$$u(t) \le p(t)U(t), \tag{1.2.134}$$

where $U(t) = V_n(t, t)$ and $V_n(T, t)$ is defined successively by

$$\begin{cases} V_1(T, t) = \exp\left\{\displaystyle\int_0^n \sum_{j=1}^{t} f_j(T, s)ds\right\}, \\ V_k(T, t) = F_{n-k+1}(T, t)\left\{1 + \displaystyle\int_0^t f_{n-k+1}(T, s)\frac{V_{k-1}(T_1, s)}{F_{n-k+1}(T, s)}ds\right\}, \end{cases} \tag{1.2.135}$$

where t, and T are in $k = 2, 3, \cdots, n$, and for $i = 1, \cdots, n-1$,

$$F_i(T, t) = \exp\left\{\int_0^t \left[\int_0^t \sum_{j=1}^{i-1} f_j(T, s) - f_i(T, s)\right] ds\right\}. \tag{1.2.136}$$

Proof Obviously, we have $U(0) = V_n(0, 0) = 1$ from (1.2.135) and (1.2.136), and hence the estimate for $u(t)$ in (1.2.134) trivially holds when $t = 0$. We define the following non-negative functionals on $C(I, \mathbb{R}_+)$ by

$$\begin{aligned} J_k(c, t) = f_k(c, t) \int_0^t f_{k+1}(c, t_{k+1}) \int_0^{t_{k+1}} f_{k+2}(c, t_{k+2}) \cdots \int_0^{t_{n-1}} f_n(c, t_n)y(t_n) \\ \times dt_n dt_{n-1} \cdots dt_{k+1}, \qquad k = 1, 2, \cdots, n-1, \end{aligned}$$

and

$$J_n(c,t)(y) = f_n(c,t)y(t),$$

where $y = y(t) \in C(I, \mathbb{R}_+)$, c is a constant and $t \in I$.

We note that here $J_i(c,t)(y)$, $i = 1, 2, \cdots, n$, are monotonic and non-decreasing in $y \in C(I, \mathbb{R}_+)$, that is, if $y_1, y_2 \in C(I, \mathbb{R}_+)$ and $y_1(t) \le y_2(t)$ on I, then for all $t \in I$,

$$J_i(c,t)(y_1) \le J_i(c,t)(y_2).$$

Now, fixing an arbitrary value T from $(0, h)$, then we have from the inequality (1.2.133) that for all $t \in [0, T]$,

$$u(t) \le p(T) + \int_0^t J_1(T, t_1)(x)dt_1.$$

If we set

$$\begin{cases} m_1(t) = p(T) + \displaystyle\int_0^t J_1(T, t_1)(x)dt_1, \\ m_k(t) = m_{k-1}(t) + \displaystyle\int_0^t J_k(T, t_k)(m_{k-1})dt, \quad k = 2, 3, \cdots, n, \end{cases} \tag{1.2.137}$$

then we have the relations

$$m_n(t) \ge m_{n-1}(t) \ge \cdots \ge m_1(t) \ge u(t), \quad t \in [0, T], \tag{1.2.138}$$

$$m_n(0) = m_{n-1}(0) = \cdots = m_1(0) = p(T) > 0. \tag{1.2.139}$$

We notice that the following differential inequalities for $m_i(t)$ are satisfied, for all $t \in [0, T]$,

$$\begin{aligned} m_i'(t) + f_i(T,t)m_i(t) \le &\sum_{j=1}^{i-1} f_j(T,t)m_i(t) \\ &+ f_i(T,t)m_{i+1}(t), \quad i = 1, 2, \cdots, n-1. \end{aligned} \tag{1.2.140}$$

We prove it by induction. First, using (1.2.138) and in view of the monotonicity of $J_k(T,t)(y)$, we obtain from the first equality in (1.2.137) that for all $t \in [0, T]$,

$$m_1'(t) \le J_1(T,t)(m_1),$$

and adding $f_1(T,t)m_1(t)$ to both sides of the above inequality, by (1.2.137), we have for all $t \in [0,T]$,

$$\begin{aligned} m_1'(t) + f_1(T,t)m_1(t) &\le f_1(T,t)m_1(t) + J_1(T,t)(m_1) \\ &= f_1(T,t)\left[m_1(t) + \int_0^t J_2(T,t_2)(m_1)dt_2\right] \\ &= f_1(T,t)m_2(t). \end{aligned}$$

The above inequality shows that (1.2.140) holds when $i = 1$. Now we suppose that (1.2.140) is established for $i = k$, where $1 \le k \le n-2$. Then by differentiating, we obtain from (1.2.137) that for all $t \in [0,T]$,

$$m_{k+1}'(t) = m_k' + J_{k+1}(T,t)(m_k).$$

Because (1.2.140) holds for $i = k$ and $f_k(T,t)m_k(t) \ge 0$. Using (1.2.138), we obtain for all $t \in [0,T]$,

$$\begin{aligned} m_{k+1}'(t) &\le \sum_{j=1}^{k-1} f_j(T,t)m_k(t) + f_k(T,t)m_{k+1}(t) + J_{k+1}(T,t)(m_k) \\ &\le \sum_{j=1}^{k} f_j(T,t)m_{k+1}(t) + J_{k+1}(T,t)(m_{k+1}), \end{aligned}$$

since $J_{k+1}(T,t)(y)$ is non-decreasing in $y \in C(I,\mathbb{R}_+)$. Adding $f_{k+1}(t,t)m_{k+1}(t)$ to both sides of the above inequality, we get for all $t \in [0,T]$,

$$\begin{aligned} m_{k+1}'(t) + f_{k+1}(t,t)m_{k+1}(t) &\le \sum_{j=1}^{k} f_j(T,t)m_{k+1}(t) + f_{k+1}(T,t)m_{k+1}(t) \\ &\quad + J_{k+1}(T,t)(m_{k+1}) \\ &= \sum_{j=1}^{k} f_j(T,t)m_{k+1}(t) + f_{k+1}(T,t)m_{k+2}(t). \end{aligned}$$

This proves (1.2.140).

We apply the relations (1.2.137) and (1.2.140) to derive the bounds on $m_i(t)$, here $i = 1, 2, \cdots, n$. We shall prove that the following estimates are true for all $t \in [0,T]$,

$$m_{n-k}(t) \le p(T)V_{k+1}(T,t), \;\; k = 0, 1, \cdots, n-1, \tag{1.2.141}$$

where $V_{k+1}(T,t)$ is given by (1.2.135). First, we consider the last equality in (1.2.137), for all $t \in [0,T]$,

$$m_n(t) = m_{n-1}(t) + \int_0^t f_n(T,t_n)m_{n-1}(t_n)dt_n.$$

Differentiating the above equality and using (1.2.138) and (1.2.139), in view of $f_{n-1}(T,t)$ and $m_{n-1}(t)$ being non-negative, we obtain for all $t \in [0,T]$,

$$\begin{aligned}
m_n'(t) &= m_{n-1}'(t) + f_n(T,t)m_{n-1}(t) \\
&\le \sum_{j=1}^{n-2} f_j(T,t)m_{n-1}(T,t)m_n(t) + f_n(T,t)m_{n-1}(t) \\
&\le \sum_{j=1}^{n} f_j(T,t)m_n(t).
\end{aligned}$$

Dividing both sides of the above inequality by $m_n(t) > 0$, and then integrating from 0 to t, using $m_n(0) = p(T)$, we obtain for all $t \in [0,T]$,

$$m_n(t) \le p(T)\exp\left(\int_0^t \sum_{j=1}^{n} f_j(T,s)ds\right) = p(T)V_1(T,t).$$

Here $V_1(T,t)$ is given by (1.2.135). Next, substituting this bound for $m_n(t)$ in (1.2.140) with $i = n-1$, we then get for all $t \in [0,T]$,

$$m_{n-1}'(t) + \left[f_{n-1}(T,t) - \sum_{j=1}^{n-2} f_j(T,t)\right] m_{n-1}(t) \le f_{n-1}(T,t)p(T)V_1(T,t).$$

Multiplying by $\exp\int_0^t [f_{n-1}(T,s) - \sum_{j=1}^{n-2} f_j(T,s)]ds$ both sides of the above inequality, and then integrating from 0 to t and using (1.2.141), we derive that for all $t \in [0,T]$,

$$\begin{aligned}
m_{n-1}(t) &\le P(T)F_{n-1}(T,t)\left\{1 + \int_0^t f_{n-1}(T,s)\frac{V_1(T,s)}{F_{n-1}(T,t)}ds\right\} \\
&= p(T)V_2(T,t),
\end{aligned}$$

where $F_{n-1}(T,t)$ and $V_2(T,t)$ are defined by (1.2.135) and (1.2.136).

Suppose that the inequality (1.2.141) is proved for $1 \le k \le n-2$, then by (1.2.140), we have

$$m'_{n-k-1}(t) + \left[f_{n-k-1}(T,t) - \sum_{j=1}^{n-k-2} f_j(T,t)(t) \right] m_{m-k-1}(t)$$
$$\le f_{n-k-1}(T,t)p(T)V_{k+1}(T,t).$$

Multiplying by $\exp \int_0^t [f_{n-k-1}(T,s) - \sum_{j=1}^{n-k-2} f_j(T,s)]ds$ both sides of the above inequality and integrating from 0 to t, and using (1.2.139), we obtain for all $t \in [0,T]$,

$$m_{n-k-1}(t) \le p(T)F_{n-k-1}(T,t) \left[1 + \int_0^t f_{n-k-1}(T,s) \frac{V_{k+1}(T,s)}{F_{n-k-1}(T,s)} ds \right]$$
$$= p(T)V_{k+2}(T,t),$$

where $F_{n-k-1}(T,t)$ and $V_{k+2}(T,t)$ are given by (1.2.135) and (1.2.136). Hence the inequality (1.2.141) is now completely proved. Finally, we observe from (1.2.138) that for all $t \in [0,T]$,

$$u(t) \le m_1(t) \le p(T)V_n(T,t).$$

Letting $t = T$ in this inequality and in view of $U(t) = V_n(t,t)$, we obtain

$$u(T) \le p(T)U(T).$$

Since T from $(0,h)$ is arbitrary, thus the proof of Theorem 1.2.23 is complete. □

Remark 1.2.11 In the above theorem, if $n = 2$, then we can get the following result

$$u(t) \le p(t) \exp\left(- \int_o^t f_1(t,s)ds \right)$$
$$\times \left\{ 1 + \int_0^t f_1(t,s) \left\{ \exp \int_0^s \Big(2f_1(s,r) + f_2(s,r) \Big) dr \right\} ds \right\}. \quad (1.2.142)$$

In order to investigate hyperbolicity, Hale [251] projected solutions of differential equations into invariant subspaces of exponential dichotomies, where the following inequality was applied, for all $t \ge 0$,

$$u(t) \le Ke^{-\alpha t} + L \int_0^t e^{-\alpha(t-s)} u(s)ds + M \int_0^{+\infty} e^{-\gamma s} u(t+s)ds. \quad (1.2.143)$$

In a certain sense, (1.2.143) can be regarded as a projected Gronwall-Bellman's inequality. In 1993, to study pseudo-hyperbolicity, the next generalization of (1.2.143) was discussed in [685, 686], for all $t \geq 0$,

$$u(t) \leq a + e^{-\alpha t}\sum_{i=0}^{m} a_i t^i + b\int_0^t e^{-\alpha(t-s)}u(s)ds + c\int_0^{+\infty} e^{-\gamma s}u(t+s)ds, \tag{1.2.144}$$

where $\alpha > 0, \gamma > 0, a, a_i, b, \in \mathbb{R}, i = 0, 1, \ldots, m$, and a, c, d non-negative. Later on, [687] considered a more general inequality, for all $t \geq 0$,

$$u(t) \leq a(t) + \int_0^t b(t-s)u(s)ds + \int_0^{+\infty} c(s)u(t+s)ds, \tag{1.2.145}$$

which was once mentioned by Hale [251]. The next result, due to Zhang [687], concerns the inequality (1.2.145).

Theorem 1.2.24 (Zhang [687]) *Suppose that $a(t), b(t)$, and $c(t)$ are continuous functions defined on $\mathbb{R}_+$ and valued in $\mathbb{R}_+$, and that $u : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded continuous solution of (1.2.145). If*

(i) both $a(t)$ and $b(t)$ are monotonically non-increasing and tend to 0 as $t \to +\infty$,
(ii) the derivative $b'(t) \leq \delta b(t)$, where δ is a real constant, and
(iii) $\int_0^{+\infty} b(s)ds < +\infty, \int_0^{+\infty} c(s)ds < +\infty, \beta := \int_0^{+\infty} b(s)ds + \int_0^{+\infty} c(s)ds < 1$, then for all $t \geq 0$,

$$u(t) \leq \frac{a(t)}{1-\beta} + \frac{b(0)}{(1-\beta)^2}\int_0^t a(s)\exp\Big((\delta + \frac{b(0)}{1-\beta})(t-s)\Big)ds. \tag{1.2.146}$$

Proof **Step 1:** We prove $\lim\limits_{t\to+\infty} u(t) = 0$. Suppose that $\lim\limits_{t\to+\infty} u(t)$ is positive, we denote it by γ. We can get $0 < \rho < +\infty$ from the boundedness of $u(t)$. Take any $\theta < 1$, $\lim\limits_{t\to+\infty} u(t) = \rho < \theta^{-1}\gamma$, that is, there is a non-negative t_0, when $t \geq t_0$, $u(t) \leq \theta^{-1}\rho$. By (1.2.143)for all $t \geq t_0$,

$$\begin{aligned} u(t) &\leq a(t) + \int_0^{t_0} b(t-s)u(s)ds + \int_{t_0}^{t} b(t-s)u(s)ds + \int_0^{+\infty} c(s)u(t+s)ds \\ &\leq a(t) + \int_0^{t_0} b(t-s)u(s)ds + \theta^{-1}\gamma\left(\int_0^{t-t_0} b(\tau)d\tau + \int_0^{+\infty} c(s)ds\right). \end{aligned} \tag{1.2.147}$$

Let $t \to +\infty$, and we can take any θ, we may take $\beta < \theta < 1$. This implies

$$\gamma \leq \theta^{-1}\gamma\left(\int_0^{+\infty} c(s)ds + \int_0^{+\infty} c(s)ds\right) = \theta^{-1}\gamma\beta < \gamma, \tag{1.2.148}$$

which is a contradiction, therefore, $\lim_{t\to+\infty} u(t) = 0$.

Step 2: To simplify (1.2.145), we denote $\nu(t) = \sup_{s\geq t} u(s)$. We can easily get $\nu(t) \geq u(t)$ and $\nu(t)$ is monotonically non-increasing result from the definition, $\lim_{t\to+\infty} \nu(t) = 0$, so for all $t \in [0, +\infty)$, there exists a $t_1 \geq t$ such that

$$\nu(s) \begin{cases} = \nu(t) = u(t_1), & \text{if } t \leq s \leq t_1, \\ < \nu(t_1), & \text{if } s > t_1. \end{cases} \tag{1.2.149}$$

Thus by the monotonicity of $a(t)$ and $b(t)$,

$$\begin{aligned} \nu(t) = u(t_1) &\leq a(t_1) + \int_0^t b(t_1 - s)\nu(s)ds + \int_t^{t_1} b(t_1 - s)\nu(s)ds \\ &\quad + \int_0^{+\infty} c(s)\nu(t_1 + s)ds \\ &\leq a(t_1) + \int_0^t b(t_1 - s)\nu(s)ds + \nu(t)\left[\int_t^{t_1} b(t_1 - s)ds + \int_0^{+\infty} c(s)ds\right] \\ &\leq a(t_1) + \int_0^t b(t_1 - s)\nu(s)ds + \nu(t)\left[\int_0^{+\infty} b(s)ds + \int_0^{+\infty} c(s)ds\right] \\ &\leq a(t) + \int_0^t b(t - s)\nu(s)ds + \nu(t)\beta, \end{aligned} \tag{1.2.150}$$

that is, for all $t \geq 0$,

$$\nu(t) \leq \frac{a(t)}{1-\beta} + \frac{1}{(1-\beta)}\int_0^t b(t-s)\nu(s)ds. \tag{1.2.151}$$

Step 3: We discuss the above inequality. Define

$$R(t) = \int_0^t \frac{b(t-s)}{(1-\beta)}\nu(s)ds. \tag{1.2.152}$$

Clearly, the derivative of $R(t)$ satisfies, for all $t \geq 0$,

$$\begin{aligned} R'(t) &= \int_0^t \frac{b'(t-s)}{(1-\beta)} v(s)ds + \frac{b(0)}{(1-\beta)} v(t) \\ &\leq \delta \int_0^t \frac{b(t-s)}{(1-\beta)} v(s)ds + \frac{b(0)}{(1-\beta)} v(t) \\ &\leq \delta R(t) + \frac{b(0)}{(1-\beta)} (\frac{a(t)}{(1-\beta)} + R(t)) \\ &= (\delta + \frac{b(0)}{(1-\beta)}) R(t) + \frac{b(0)}{(1-\beta)^2} a(t). \end{aligned} \tag{1.2.153}$$

By the comparison theorem, we get for all $t \geq 0$,

$$R(t) \leq \int_0^t \frac{b(0)}{(1-\beta)^2} a(s) \exp\Big((\delta + \frac{b(0)}{1-\beta})(t-s)\Big) ds. \tag{1.2.154}$$

Thus we derive from (1.2.151),

$$v(t) \leq \frac{a(t)}{1-\beta} + R(t) \leq \frac{a(t)}{1-\beta} + \frac{b(0)}{(1-\beta)^2} \int_0^t a(s) \exp\Big((\delta + \frac{b(0)}{1-\beta})(t-s)\Big) ds. \tag{1.2.155}$$

Thus the proof of is complete. □

In 1960s, another generalization was given by Hale [251] in the case of hyperbolic decomposition in order to discuss the stable or unstable manifolds for ordinary differential equations.

Theorem 1.2.25 (Hale [251]) *Let $\alpha > 0, \gamma > 0$ and K, L, M be non-negative constants. Suppose that $u(t)$ is a non-negative bounded continuous function satisfying that for all $t \geq 0$,*

$$u(t) \leq K e^{\alpha t} + L \int_t^0 e^{\alpha(t-s)} u(s) ds + M \int_{-\infty}^0 e^{\gamma s} u(t+s) ds, \tag{1.2.156}$$

or for all $t \geq 0$,

$$u(t) \leq K e^{-\alpha t} + L \int_0^t e^{-\alpha(t-s)} u(s) ds + M \int_0^{+\infty} e^{-\gamma s} u(t+s) ds. \tag{1.2.157}$$

If $\lambda := L/\alpha + M/\gamma < 1$, then for all $t \in \mathbb{R}$,

$$u(t) \leq \frac{K}{1-\lambda} \exp\left(-\left(\alpha - \frac{L}{1-\lambda}\right)|t|\right). \tag{1.2.158}$$

Proof We only prove (1.2.158) for the case when (1.2.156) holds. Since the discussion for (1.2.157) can be changed into the discussion for (1.2.157) by the change of variables $t \to -t$, $s \to -s$. First, we claim that as $t \to +\infty$, $u(t) \to 0$. In fact, if $\delta = \overline{\lim}_{t\to+\infty} u(t)$, then the boundedness of u implies that δ is finite. If θ satisfies $\lambda < \theta < 1$, then we derive from $\delta > 0$ that there exists a $t_1 \geq 0$ such that as $t \geq t_1$, $u(t) \leq \theta^{-1}\delta$. By (1.2.148), for all $t \leq t_1$, we get

$$u(t) \leq Ke^{-\alpha t} + Le^{-\alpha t}\int_0^{t_1} e^{\alpha s}u(s)ds + \Big(\frac{L}{\alpha} + \frac{M}{\gamma}\Big)\theta^{-1}\delta. \tag{1.2.159}$$

Note that as $t \to +\infty$, the upper limit of the right-hand side of (1.2.159) $< \delta$, this is a contradiction. Hence $\delta = 0$, and as $t \to +\infty$, $u(t) \to 0$.

If $v(t) = sup_{s\geq t}u(s)$, then as $t \to +\infty$, $u(t) \to 0$. Hence, for any $t \in [0, +\infty)$, there exists a $t_1 \geq t$, such that as $t \leq s \leq t_1$, $v(t) = v(s) = u(t_1)$, while as $s > t_1$, $v(s) < v(t_1)$. Thus it follows from (1.2.159) that

$$\begin{aligned} v(t) = u(t_1) &\leq Ke^{-\alpha t_1} + L\int_0^t e^{-\alpha(t_1-s)}v(s)ds \\ &\quad + L\int_t^{t_1} e^{-\alpha(t_1-s)}v(s)ds + M\int_0^{+\infty} e^{-\gamma s}v(t+s)ds \\ &\leq Ke^{-\alpha t_1} + L\int_0^t e^{-\alpha(t_1-s)}v(s)ds + \lambda v(t). \end{aligned}$$

If $z(t) = e^{\alpha t}v(t)$, then for all $t \leq t_1$,

$$z(t) \leq (1-\lambda)^{-1}K + (1-\lambda) + L\int_0^t z(s)ds. \tag{1.2.160}$$

By the Gronwall-Bellman inequality (e.g., Theorem 1.1.2), we obtain $z(t) \leq (1-\lambda)^{-1}K\exp((1-\lambda)^{-1})Lt$. This proves (1.2.158). $\square$

In the sequel, we shall consider the following more generalized integral inequalities

$$\begin{aligned} u(t) &\leq a + e^{\alpha t}\left(\sum_{i=0}^m b_i t^i\right) + c\int_t^0 e^{\alpha(t-s)}u(s)ds \\ &\quad + d\int_{-\infty}^t e^{-\gamma(t-s)}u(s)ds, \ \textit{for all } t \leq 0, \end{aligned} \tag{1.2.161}$$

$$\begin{aligned} u(t) &\leq a + e^{-\alpha t}\left(\sum_{i=0}^m b_i t^i\right) + c\int_0^t e^{-\alpha(t-s)}u(s)ds \\ &\quad + d\int_t^{+\infty} e^{\gamma(t-s)}u(s)ds, \ \textit{for all } t \geq 0. \end{aligned} \tag{1.2.162}$$

To generalize the above theorem, we need the following three lemmas.

Lemma 1.2.2 (Zhang [686]) *If the non-negative bounded continuous function $u(t)$ satisfies (1.2.162) for all $t \geq 0$, then $\bar{u}(t) := u(-t)$ satisfies an inequality like (1.2.161), i.e, for all $t \geq 0$,*

$$\bar{u}(t) = a + e^{\alpha t}\left(\sum_{i=0}^{m} \bar{b}_i t^i\right) + c\int_t^0 e^{\alpha(t-s)}u(s)ds + d\int_{-\infty}^t e^{-\gamma(t-s)}\bar{u}(s),$$

where $\bar{b}_i = (-1)^i b_i$.

Proof This lemma can be proved with the change of variables $t \to -t$ and $s \to -s$ in (1.2.162). □

Lemma 1.2.3 (Zhang [686]) $\int_{-\infty}^t e^{(\alpha+\gamma)s} \cdot s^i ds = e^{(\alpha+\gamma)i}\sum_{k=0}^i (-1)^k \frac{i!}{(i-k)!} \cdot \frac{t^{i-k}}{(\alpha+\gamma)^{k+1}}$.

Proof Let

$$I_i = \int_{-\infty}^t e^{(\alpha+\gamma)s} \cdot s^i ds.$$

Integrating by parts, we obtain

$$I_i = e^{(\alpha+\gamma)t} \cdot \frac{t^i}{\alpha+\gamma} - \frac{i}{\alpha+\gamma} I_{i-1}, \quad I_0 = \frac{1}{\alpha+\gamma}e^{(\alpha+\gamma)}.$$

Using this recursion formula and method of induction, we can easily prove this lemma. □

Lemma 1.2.4 (Zhang [686]) $\sum_{i=0}^{m-1} q_i \sum_{k=0}^i (-1)^k \cdot \frac{i!}{(i-k)!} \cdot \frac{t^{i-k}}{(\alpha+\gamma)^{k+1}} = -\sum_{i=0}^{m-1} Q_i t^i$, *where Q_i is defined by (1.2.164) below.*

Proof The proof consists of a simple but tedious induction and is thus omitted. □

Theorem 1.2.26 (Zhang [686]) *Let a, b_i, c, d, α, and γ be real numbers and $\alpha > 0, \gamma > 0, a \geq 0, c \geq 0$ and $d \geq 0$. Suppose $u(t)$ is a non-negative bounded continuous function, satisfying integral inequalities (1.2.161) or (1.2.162). If $\lambda := c/\alpha + d/\gamma < 1$, then*

$$u(t) \leq \frac{a}{1-\lambda} + e^{-\alpha|t|}\left(\sum_{i=0}^{m-1} q_i(-1)^i|t|^i\right) + \frac{|K|}{1-\lambda}\exp\left(-\left(\alpha - \frac{c}{1-\lambda}\right)|t|\right), \tag{1.2.163}$$

where constants $q_i, i = 0, 1, \ldots, m-1$*, and* K *are determined by relations*

$$\begin{cases} q_{m-1} = m\tilde{b}_m/c, \\ q_{i-1} = i(\tilde{b}_i - q_i - dQ_i)/c, \quad i = 1, 2, \ldots, m-1, \qquad 1 \leq i \leq N, \\ \tilde{b}_i = (-sgn(t))^i b_i, \quad \text{where } sgn(t) \text{ is the sign of } t, \\ Q_i = \sum_{k=0}^{m-1-i} q_{k+1} \left(\frac{-1}{\alpha+\gamma}\right)^{k+1} \frac{(k+i)!}{i!}, \quad i = 0, 1, \ldots, m-1, \\ K = b_0 - \frac{ac}{\alpha(1-\lambda)} - q_0 - dQ_0. \end{cases} \tag{1.2.164}$$

The sign function $sgn(t) = 1, 0$ *or* -1 *as* $t > 0, = 0$ *or* < 0*, respectively.*

Proof By Lemma 1.2.3, we see that it suffices to discuss (1.2.161) for $t \leq 0$, instead of (1.2.162) for $t \geq 0$. Let

$$v(t) = u(t) - p - e^{\alpha t}\left(\sum_{i=0}^{m-1} q_i t^i\right). \tag{1.2.165}$$

Then $v(t)$ satisfies that for all $t \leq 0$,

$$v(t) \leq Ke^{\alpha t} + L\int_t^0 e^{\alpha(t-s)}u(s)ds + \int_{-\infty}^t e^{-\gamma(t-s)}u(s)ds, \tag{1.2.166}$$

where p, q_i $(i = 0, 1, \ldots, m-1), K, L$ and M are assumed to be undetermined constants. Inserting $v(t)$ in (1.2.165) into (1.2.166), we have

$$\begin{aligned} u(t) - p - e^{\alpha t}\left(\sum_{i=0}^{m-1} q_i t^i\right) &\leq Ke^{\alpha t} - L\int_t^0 pe^{\alpha(t-s)}ds - \sum_{i=0}^{m-1} Lq_i \int_t^0 e^{\alpha(t-s)} \cdot e^{\alpha s} \cdot s^i ds \\ &\quad - M\int_{-\infty}^t pe^{-\gamma(t-s)}ds - \sum_{i=0}^{m-1} Mq_i \int_{-\infty}^t e^{-\gamma(t-s)} \cdot e^{\alpha s} \cdot s^i ds \\ &\quad + L\int_t^0 e^{\alpha(t-s)}ds + M\int_{-\infty}^t e^{-\gamma(t-s)}u(s)ds \\ &\leq Ke^{\alpha t} + \frac{Lp}{\alpha}(e^{\alpha t} - 1) + \frac{Mp}{-\gamma} + Le^{\alpha t}\sum_{i=0}^{m-1} \frac{q_i}{i+1} t^{i+1} \\ &\quad - Me^{-\gamma t}\sum_{i=0}^{m-1} q_i \int_{-\infty}^t e^{(\alpha+\gamma)s} \cdot s^i ds \\ &\quad + L\int_t^0 e^{\alpha(t-s)}u(s)ds + M\int_{-\infty}^t e^{-\gamma(t-s)}u(s)ds. \end{aligned} \tag{1.2.167}$$

It follows from Lemmas 1.2.3 and 1.2.4 that

$$\begin{aligned} u(t) - p - e^{\alpha t}\left(\sum_{i=0}^{m-1} q_i t^i\right) \leq &\ Ke^{\alpha t} + \frac{LP}{\alpha}(e^{\alpha t} - 1) + \frac{Mp}{-\gamma} + Le^{\alpha t}\sum_{i=0}^{m-1}\frac{q_i}{i+1}t^{i+1} \\ &+ Me^{\alpha t}\sum_{i=0}^{m-1} Q_i t^i + L\int_t^0 e^{\alpha(t-s)}u(s)ds \\ &+ M\int_{-\infty}^t e^{-\gamma(t-s)}u(s)ds. \end{aligned} \tag{1.2.168}$$

Comparing (1.2.168) with (1.2.158) in coefficients, we see that

$$\begin{cases} L = c, \quad M = d, \\ p\left(1 - \left(\dfrac{L}{\alpha} + \dfrac{M}{\gamma}\right)\right) = a, \\ K + \dfrac{Lp}{\alpha} + MQ_0 + q_0 = b_0, \\ \dfrac{Lq_{m-1}}{m} = b_m, \\ \dfrac{Lq_{i-1}}{i} + MQ_i + q_i = b_i, \quad i = 1, 2, \ldots, m-1. \end{cases} \tag{1.2.169}$$

Thus these undetermined constants p, q_i $(i = 0, 1, \ldots, m-1), K, L$ and M in (1.2.165) and (1.2.166) are now determined, that is,

$$L = c, \quad M = d, \quad p = \frac{a}{1-\lambda}, \tag{1.2.170}$$

where $\lambda := c/\alpha + d/\gamma < 1$, and

$$\begin{cases} q_{m-1} = \dfrac{mb_m}{c} = \dfrac{m\tilde{b}_m}{c}, \\ q_{i-1} = \dfrac{i(b_i - q_i - MQ_i)}{c} = \dfrac{i(\tilde{b}_i - q_i - dQ_i)}{c}, \quad i = 1, 2, \ldots, m-1, \\ K = b_0 - \dfrac{ac}{\alpha(1-\lambda)} - q_0 - dQ_0, \end{cases} \tag{1.2.171}$$

where

$$\begin{cases} Q_i = \sum_{k=0}^{m-i-1} q_{k+1} \left(\frac{-1}{\alpha+\gamma}\right)^{k+1} \frac{(k+i)!}{i!}, \quad i = 0, 1, \ldots, m-1, \\ \tilde{b}_i = (-sgn(t))^i b_i = b_i, \quad i = 0, 1, \ldots, m \textit{ for all } t < 0. \end{cases}$$

Obviously, the constructed function $v(t)$, defined by (1.2.165), satisfies the integral inequality (1.2.166) with appropriately chosen undetermined constants like (1.2.170) and (1.2.171). This implies, by Theorem 1.2.25, that

$$v(t) \leq |v(t)| \leq \frac{|K|}{1-\lambda} \exp\left(-\left(-\alpha - \frac{c}{1-\lambda}\right)|t|\right), \tag{1.2.172}$$

that is, for all $t \leq 0$,

$$\begin{aligned} u(t) &\leq p + e^{\alpha t}\left(\sum_{i=0}^{m-1} q_i t^i\right) + \frac{|K|}{1-\lambda} \exp\left(-\left(\alpha - \frac{c}{1-\lambda}\right)|t|\right) \\ &= \frac{a}{1-\lambda} + e^{-\alpha|t|}\left(\sum_{i=0}^{m-1} q_i (-1)^i |t|^i\right) \\ &\quad + \frac{|K|}{1-\lambda} \exp\left(-\left(\alpha - \frac{c}{1-\lambda}\right)|t|\right). \end{aligned} \tag{1.2.173}$$

Therefore, the inequality (1.2.163) has been proved. This completes the proof. □

Example 1.2.1 Theorem 1.2.20 can be deduced simply from this theorem, if we let $\alpha = 0$ and $b_i = 0$ ($i = 1, 2, \ldots, m$) in (1.2.161) or (1.2.162).

Example 1.2.2 Consider (1.2.161) or (1.2.162) for $m = 1$. From this theorem, we have

$$u(t) \leq \frac{a}{1-\lambda} - sgn(t)\frac{b_1}{c}e^{-\alpha|t|} + \frac{|K|}{1-\lambda} \exp\left(-\left(\alpha - \frac{c}{1-\lambda}\right)|t|\right), \tag{1.2.174}$$

where $K = b_0 - \frac{ac}{\alpha(1-\lambda)} + sgn(t)\frac{b_1}{c} - sgn(t)\frac{b_1 d}{c(\alpha+\gamma)}$. This result was used in [685] to discuss the almost-periodicity of weak hyperbolic manifolds for non-autonomous abstract differential equations under pseudo-hyperbolic conditions.

1.2.2 Linear One-Dimensional Gronwall-Bellman Integral Inequalities with Delays

In this section, we study the Gronwall-Bellman inequalities with delay or retardations.

We first introduce the following conditions:

(H1) $h : J = [\alpha, +\infty) \to \mathbb{R}$ is continuous and

$$h(t) \le t, \ t \in J, \ \lim_{t \to +\infty} h(t) = +\infty.$$

(H2) $h(t)$ is non-decreasing in J and has an inverse $h^{-1} : [h(\alpha), +\infty) \to J$.

Theorem 1.2.27 (Marušiak [389]) *Let $u(t), b(t), c(t)$ be non-negative continuous functions in J, and suppose that*

$$\begin{cases} u(t) \le a + \displaystyle\int_\alpha^t [b(s)u(s) + c(s)u(h(s))]ds, \ \textit{for all } t \in J, & (1.2.175) \\ u(t) = \phi(t), \ t \in E = \{t \in J : h(t) < \alpha\} \cup \{\alpha\}, & (1.2.176) \end{cases}$$

where $a \ge 0$ is a constant, $\phi(t)$ is a non-negative continuous function in E, and the function $h(t)$ satisfies the condition (H1). Then for all $t \in J$,

$$u(t) \le \left[a + \int_E c(s)\phi(h(s))ds\right] \exp\left(\int_\alpha^t [b(s) + c(s)]ds\right). \tag{1.2.177}$$

Proof Set

$$v(t) = \int_\alpha^t \Big(b(s)u(s) + c(s)u(h(s))\Big)ds + a.$$

Then

$$u(t) \le v(t), \tag{1.2.178}$$

and since $v(t)$ is non-decreasing in J, we have

$$\begin{aligned} v'(t) &= b(t)u(t) + c(t)u(h(t)) \\ &\le b(t)v(t) + c(t)\begin{cases} \phi(h(t)) & \textit{if } \ t \in E, \\ v(t) & \textit{otherwise}. \end{cases} \end{aligned} \tag{1.2.179}$$

Integration of the above inequality from α to t leads to

$$v(t) \leq a + \int_E c(s)\phi(h(s))ds + \int_\alpha^t [b(s) + c(s)]v(s)ds$$

which, along with Theorem 1.1.3 and (1.2.178), gives us (1.2.177). $\square$

The next two theorems take the forms suitable for immediate application to delay equations.

Let $C([a, b], \mathbb{R}^n)$ $(-\infty < a < b)$ the Banach space of continuous functions $y : [a, b] \to \mathbb{R}^n$ with the topology of uniform convergence, and $L^1([a, +\infty), \mathbb{R}^n)$ the Banach space of Lebesgue integral functions $z : [a, +\infty) \to \mathbb{R}^n$ with the norm $\|z\|_{L^1} = \int_a^{+\infty} |z(t)|dt$.

For any given $r_0 \geq r > 0, t \geq 0$, and $x \in C([t - r_0, t], \mathbb{R}^n)$, we put

$$\|x_t\|_r = \max_{t-r\leq s\leq t} |x(s)|.$$

Theorem 1.2.28 (Arino-Györi [31]) *Let $J = [\alpha, +\infty)$, and suppose that the following conditions hold:*

(1) $h : J \to \mathbb{R}$ is a function satisfying the above conditions (H1)–(H2);
(2) $r : J \to \mathbb{R}_+$ is a function such that $h(\alpha) \leq \alpha - r(\alpha) \leq t - r(t)$, for all $t \in J$;
(3) $\rho, p : J \to \mathbb{R}_+$ are locally Lebesgue integrable functions;
(4) $b : [h(\alpha), +\infty) \times [h(\alpha), +\infty) \to \mathbb{R}_+$ is a locally bounded, locally Lebesgue integrable functions, and there is a locally bounded, locally Lebesgue integrable function $a : [h(\alpha), +\infty) \to [1, +\infty)$ such that for all $s \geq h(\alpha)$,

$$\int_t^{h^{-1}(s)} b(t, s)a(t)dt \leq a(s) - 1, \tag{1.2.180}$$

where h^{-1} is the inverse function of h;
(5) $x : [h(\alpha), +\infty) \to \mathbb{R}^n$ is an absolutely continuous function satisfying for all $t \geq \alpha$,

$$|x'(t)| \leq \int_{h(t)}^t b(t, s)|x'(s)|ds + p(t)\|x_t\|_{r(t)} + \rho(t). \tag{1.2.181}$$

Then

(1) for all $t \geq \alpha$,

$$|x(t)| \leq \left[\|x_\alpha\|_{r(\alpha)} + \int_{h(\alpha)}^\alpha [a(s) - 1]|x'(s)|ds \right.$$
$$\left. + \int_\alpha^t a(\tau)\rho(\tau)d\tau \right] \exp\left(\int_\alpha^t a(\tau)p(\tau)d\tau \right). \tag{1.2.182}$$

(2) If, in addition to the above, we also suppose

$$\int_{\alpha}^{+\infty} a(t)\Big(p(t) + \rho(t)\Big)dt < +\infty, \tag{1.2.183}$$

then

$$\int_{\alpha}^{+\infty} |x'(t)|dt < +\infty. \tag{1.2.184}$$

Proof

(1) Let $c : [h(\alpha), +\infty) \to \mathbb{R}$ be an arbitrary, but fixed, locally Lebesgue integrable function. Then the function $a(t)\int_{h(t)}^{t} b(t,s)c(s)ds$ is defined and locally Lebesgue integrable in $[\alpha, +\infty)$. Changing the order of integration, we find

$$\Big|\int_{\alpha}^{t}\Big(a(\tau)\int_{h(\tau)}^{r} b(\tau,s)c(s)ds\Big)d\tau\Big| \le \int_{h(\alpha)}^{t}\left(\int_{s}^{h^{-1}(s)} b(\tau,s)a(\tau)d\tau\right)|c(s)|ds$$

which, along with (1.2.180), gives us for all $t \ge \alpha$,

$$\Big|\int_{\alpha}^{t}\Big(a(\tau)\int_{h(\tau)}^{\tau} b(\tau,s)c(s)ds\Big)\Big| \le \int_{h(t)}^{t}[a(s)-1]|c(s)|ds. \tag{1.2.185}$$

Applying (1.2.185) to $c(t) = |x'(t)|$, (1.2.181) gives us for all $t \ge \alpha$,

$$\int_{\alpha}^{t} a(\tau)|x'(\tau)|d\tau \le \int_{h(\alpha)}^{t}[a(\tau)-1]|x'(\tau)| + \int_{\alpha}^{t} a(\tau)\rho(\tau)d\tau$$

which implies that for all $t \ge \alpha$,

$$\int_{\alpha}^{t}|x'(\tau)|d\tau \le \int_{h(\alpha)}^{t}[a(\tau)-1]|x'(\tau)|d\tau$$
$$+\int_{\alpha}^{t} a(\tau)\rho(\tau)d\tau + \int_{\alpha}^{t} a(\tau)\rho(\tau)\|x_r\|_{r(\tau)}d\tau. \tag{1.2.186}$$

Noting that condition (2), we can deduce that for all $t \ge \alpha$,

$$\|x_t\|_{r(t)} = \max_{t-r(t)\le s\le t}|x(s)| \le \|x_\alpha\|_{r(\alpha)} + \int_{\alpha}^{t}|x'(s)|ds \equiv y(t), \tag{1.2.187}$$

and so (1.2.186) implies that for all $t \ge \alpha$,

$$y(t) \le A(t) + \int_{\alpha}^{t} a(\tau)p(\tau)u(\tau)d, \tag{1.2.188}$$

where

$$A(t) = \|x_\alpha\|_{r(\alpha)} + \int_{h(\alpha)}^{\alpha} [a(\tau) - 1]|x'(\tau)|d\tau + \int_{\alpha}^{t} a(\tau)\rho(\tau)d\tau.$$

By Theorem 1.1.4 and (1.2.187), we get that for all $t \geq \alpha$,

$$|x(t)| \leq y(t) \leq A(t) \exp\left(\int_{\alpha}^{t} a(\tau)p(\tau)d\tau\right), \tag{1.2.189}$$

which gives us (1.2.182). Under condition (1.2.183), inequality (1.2.184) follows from (1.2.186) and (1.2.187). □

Corollary 1.2.18 (Atkinson-Haddock [34]) *Let* $b, p : [\alpha - r_0, +\infty) \to \mathbb{R}_+$ $(r_0 > 0)$ *be continuous, and suppose that there is a continuous function* $a : [\alpha - r_0, +\infty) \to [1, +\infty)$ *such that for all* $s \geq \alpha - r_0$*,*

$$\int_{s}^{r_0+s} b(t)a(t)dt \leq a(s) - 1, \tag{1.2.190}$$

and

$$\int_{\alpha}^{+\infty} p(t)a(t)dt < +\infty. \tag{1.2.191}$$

If $x : [\alpha - r_0, +\infty) \to \mathbb{R}^n$ *is an absolutely continuous function satisfying that for all* $t \geq \alpha$*,*

$$|x'(t)| \leq b(t) \int_{t-r_0}^{t} |x'(s)|ds + p(t)\|x_t\|_{r_0}, \tag{1.2.192}$$

then for all $t \geq \alpha$*,*

$$|x(t)| \leq \left[\max_{\alpha - r_0 \leq s \leq \alpha} |x(s)| + \int_{\alpha - r_0}^{\alpha} [a(s) - 1]|x'(s)|ds\right] \exp\left(\int_{\alpha}^{t} a(\tau)p(\tau)d\tau\right), \tag{1.2.193}$$

and

$$\int_{\alpha}^{+\infty} |x'(t)|dt < +\infty. \tag{1.2.194}$$

Proof Indeed, (1.2.193) and (1.2.194) follow from Theorem 1.2.28 with $b(t, s) = b(s), h(t) = t - r_0, \rho(t) \equiv 0$. □

If, in the above corollary, $b(t) \equiv L, a(t) = 1/(1 - Lr_0)$, then we can get the following corollary.

Corollary 1.2.19 (Atkinson-Haddock [34]) *If $x : [\alpha - r_0, +\infty) \to \mathbb{R}^n$ is an absolutely continuous function satisfying that for all $t \geq \alpha$,*

$$|x'(t)| \leq L \int_{t-r_0}^{t} |x'(s)|ds + p(t)\|x_t\|_{r_0}, \tag{1.2.195}$$

where $r_0 > 0$ and $0 < L < 1/r_0$ are constants, and $p : [\alpha - r_0, +\infty) \to \mathbb{R}_+$ is a continuous function, then for all $t \geq \alpha$,

$$|x(t)| \leq \left[\max_{\alpha - r_0 \leq s \leq \alpha} |x(s)| + \frac{Lr_0}{1 - Lr_0} \int_{\alpha - r_0}^{\alpha} |x'(s)|ds\right] \exp\left(\frac{1}{1 - Lr_0} \int_{\alpha}^{t} p(\tau)d\tau\right). \tag{1.2.196}$$

Corollary 1.2.20 (Atkinson-Haddock [34]) *If $x : [\alpha - r_0, +\infty) \to \mathbb{R}^n$ is an absolutely continuous function satisfying that for all $t \geq \alpha$,*

$$|x'(t)| \leq L \int_{t-r_0}^{t} |x'(s)|ds + p(t)|x(t)|, \tag{1.2.197}$$

where $r_0 > 0$ and $0 < L < 1/r_0$ are constants, and $p : [\alpha - r_0, +\infty) \to \mathbb{R}_+$ is a continuous function, then for all $t \geq \alpha$,

$$|x(t)| \leq \left[|x(\alpha)| + \frac{Lr_0}{1 - Lr_0} \int_{\alpha - r_0}^{\alpha} |x'(s)|ds\right] \exp\left(\frac{1}{1 - Lr_0} \int_{\alpha}^{t} p(\tau)d\tau\right). \tag{1.2.198}$$

Proof Indeed, (1.2.198) follows from Theorem 1.2.28 with $b(s,t) \equiv L, h(t) = t - r_0, r(t) \equiv 0, \rho(t) \equiv 0, a(t) = 1/(1 - Lr_0)$. □

Remark 1.2.12 If assumptions (1)–(4) in Theorem 1.2.28 hold, and the absolutely continuous function $x : [\alpha - r_0, +\infty) \to \mathbb{R}^n$ satisfies (1.2.181), then the limit

$$\lim_{t \to +\infty} x(t) = x(+\infty) \in \mathbb{R}^n \tag{1.2.199}$$

exists and is finite, by (1.2.184).

However, Theorem 1.2.28 gives us no any information on the rate of convergence, which will be given in the next theorem.

Theorem 1.2.29 (Arino-Györi [31]) *Let conditions (1)–(3) in Theorem 1.2.28 hold, and let $b : [h(\alpha), +\infty) \times [h(\alpha), +\infty) \to \mathbb{R}_+$ be a locally bounded, locally Lebesgue integrable function such that for all $s \geq h(\alpha)$,*

$$\int_{s}^{h^{-1}(s)} b(t,s)dt \leq \mu, \tag{1.2.200}$$

where $\mu \in (0,1)$ is a constant.

If $x : [h(\alpha), +\infty) \to \mathbb{R}^n$ is an absolutely continuous function satisfying (1.2.181), then

(i) for all $t \geq \alpha$,

$$|x(t)| \leq \left[\|x_\alpha\|_{r(\alpha)} + \frac{\mu}{1-\mu} \int_{h(\alpha)}^{\alpha} |x'(s)|ds + \frac{\mu}{1-\mu} \int_{\alpha}^{t} \rho(\tau)d\tau \right] \exp\left(\frac{1}{1-\mu} \int_{\alpha}^{t} p(\tau)d\tau \right). \quad (1.2.201)$$

(ii) If, in addition to the above, we also suppose that

$$\int_{\alpha}^{+\infty} [p(t) + \rho(t)]dt < +\infty, \quad (1.2.202)$$

then for any $\beta \in L^1([h(\alpha), \alpha], \mathbb{R}_+)$, there is a function $\phi \in L^1([h(\alpha), +\infty], \mathbb{R}_+)$ such that for all $t \geq \alpha$,

$$\phi(t) \geq \begin{cases} \beta(t), & h(\alpha) \leq t \leq \alpha, \\ \displaystyle\int_{h(t)}^{t} b(t,s)\phi(s)ds + p(t) + \rho(t), & t \geq \alpha, \end{cases} \quad (1.2.203)$$

and such that for all $t \geq \alpha$,

$$|x'(t)| \leq \delta\phi(t), \quad (1.2.204)$$

provided that

$$|x'(t)| \leq \beta(t), \quad h(\alpha) \leq t \leq \alpha, \quad (1.2.205)$$

where

$$\delta = \max\left\{ 1, \left[\|x_\alpha\|_{r(\alpha)} + \frac{\mu}{1-\mu} \int_{h(\alpha)}^{\alpha} \beta(s)ds + \frac{1}{1-\mu} \int_{\alpha}^{+\infty} \rho(s)ds \right] e^{\frac{1}{1-\mu} \int_{\alpha}^{+\infty} p(s)ds} \right\}. \quad (1.2.206)$$

Proof

(i) (1.2.201) follows from Theorem 1.2.28 with $a(t) = 1/(1-\mu)$.

(ii) The existence of a solution of inequality (1.2.203) is shown if we can prove that equation

$$c(t) = K[c](t), \textit{ for all } t \geq h(\alpha), \quad (1.2.207)$$

has a solution $c_0 \in L^1([h(\alpha), +\infty), \mathbb{R}_+)$, where

$$K[c](t) = \begin{cases} \beta(t), & h(\alpha) \le t \le \alpha, \\ \int_{h(t)}^{t} b(t,s)c(s)ds + p(t) + \rho(t), & t \ge \alpha, \end{cases}$$

for each $c \in L^1([h(\alpha), +\infty), \mathbb{R}_+) \equiv L^1$. Conditions imposed on b, p, ρ imply that $K[c](t)$ is defined for all $(t, c) \in [h(\alpha), +\infty) \times L^1$, and is locally integrable in $[h(\alpha), +\infty)$. Moreover, (1.2.185) with $a(t) = 1/(1-\mu)$ implies that for all $t \ge \alpha$,

$$\left| \int_\alpha^t \left(\int_{h(\tau)}^{\tau} b(\tau,s)c(s)ds \right) d\tau \right| \le \int_\alpha^t \left(\int_{h(\tau)}^{\tau} b(\tau,s)|c(s)|ds \right) d\tau \le \mu \int_{h(\alpha)}^{t} |c(s)|ds. \tag{1.2.208}$$

Thus $K[c] \in L^1$, and for all $c_1, c_2 \in L^1$,

$$\begin{aligned} \int_{h(\alpha)}^{+\infty} |K[c_1](t) - K[c_2](t)|dt &\le \int_\alpha^{+\infty} \left[\int_{h(t)}^{t} b(t,s)|c_1(s) - c_2(s)|ds \right] dt \\ &\le \mu \int_\alpha^{+\infty} |c_1(t) - c_2(t)|dt \\ &\le \mu \int_{h(\alpha)}^{+\infty} |c_1(t) - c_2(t)|dt, \end{aligned} \tag{1.2.209}$$

i.e., $K : L^1 \to L^1$ is a contractive mapping.

Consequently, for any closed subset $B \subset L^1$, the inclusion $K(B) \subset B$ implies that (1.2.207) has only one solution $c_0 \in B$.

Let

$$B = \left\{ u \in L^1 : u(t) = \beta(t), h(\alpha) \le t \le \alpha; u(t) \ge 0, t \ge \alpha; \int_{h(\alpha)}^{+\infty} u(t)dt < d_0 \right\}$$

be a closed set in L^1, where

$$d_0 = \frac{1}{1-\mu} \left[\int_{h(\alpha)}^{\alpha} \beta(t)dt + \int_\alpha^{+\infty} [p(t) + \rho(t)]dt \right].$$

Since $K[c](t) \ge 0$ for all $c \in B$ and $t \ge h(\alpha)$, (1.2.208) implies

$$\begin{aligned} \int_{h(\alpha)}^{+\infty} K[c](t)dt &= \int_{h(\alpha)}^{\alpha} \beta(t)dt + \int_\alpha^{+\infty} K[c](t)dt \\ &\le \int_{h(\alpha)}^{\alpha} \beta(t)dt + \mu \int_{h(\alpha)}^{+\infty} |c(t)|dt + \int_\alpha^{+\infty} [p(t) + \rho(t)]dt \le d_0 \end{aligned}$$

i.e., $K(B) \subset B$. Hence (1.2.207) has only one solution $c_0 \in B \subset L^1$, which implies that (1.2.203) also has a solution in L^1.

We now compare $|x'(t)|$ and $\delta\phi(t)$, where $\phi(t) \in L^1$ is an arbitrary, fixed, solution of (1.2.203), when (1.2.205) holds.

Let $K_1 : L^1 \to L^1$ be the operator defined by

$$K_1[c](t) = \begin{cases} c\delta\beta(t), & h(\alpha) \leq t \leq \alpha, \\ \displaystyle\int_{h(t)}^{t} b(t,s)c(s)ds + \delta p(t) + \delta\rho(t), & t \geq \alpha. \end{cases} \tag{1.2.210}$$

Using (1.2.201) and (1.2.206), we find that $\|x_t\|_{r(t)} \leq \delta$, for all $t \geq \alpha$. Then (1.2.181) and (1.2.205) imply that for all $t \geq h(\alpha)$,

$$|x'(t)| \leq K_1[|x'|](t). \tag{1.2.211}$$

Furthermore, (1.2.203) and $\delta \geq 1$ imply that for all $t \geq h(\alpha)$,

$$K_1[\delta\phi](t) \leq \delta\phi(t). \tag{1.2.212}$$

Noting (1.2.208), we easily see that operator K_1 is a contraction, and hence equation

$$c(t) = K_1[c](t), \textit{ for all } t \geq h(\alpha), \tag{1.2.213}$$

has at most one solution in L^1. Consider the set

$$B_1 = \{u \in L^1 : 0 \leq u(t) \leq \delta\phi(t),\ t \geq h(\alpha)\},$$

it is closed in L^1. Since $b(t,s) \geq 0$, and (1.2.211) is valid, we have, for each $c \in B_1$ and $t \geq h(\alpha)$,

$$0 \leq K_1[c](t) \leq K_1[\delta\phi](t) \leq \phi(t).$$

The last inequality means that $K_1(B_1) \subset B_1$, and thus (1.2.213) has one solution $c_1 \in B_1$.

Moreover, (1.2.205), (1.2.202), and Theorem 1.2.28 imply that

$$\|x'\|_{L^1} = \int_{h(\alpha)}^{+\infty} |x'(t)|dt \leq \int_{h(\alpha)}^{\alpha} \beta(t)dt + \int_{\alpha}^{+\infty} |x'(t)|dt = M < +\infty. \tag{1.2.214}$$

We define a constant

$$d = \max\left\{M, \frac{\delta}{1-\mu}\left(\int_{h(\alpha)}^{\alpha} \beta(t)dt + \int_{\alpha}^{+\infty} [p(t) + \rho(t)]dt\right)\right\}$$

and a set

$$B_2 = \left\{u \in L^1 : u(t) = \delta\beta(t), h(\alpha) \le t \le \alpha; |x'(t)| \le u(t), t \ge \alpha; \|u\|_{L^1} \le d\right\}.$$

Obviously (1.2.214) implies that B_2 is nonempty, and B_2 is obviously closed in L^1. By (1.2.211), we may conclude that for all $c \in B_2,\ t \ge h(\alpha)$,

$$0 \le |x'(t)| \le K_1[|x'|](t) \le K_1[c](t).$$

By (1.2.208) and (1.2.209), we obtain, for all $c \in B_2$,

$$\int_{h(\alpha)}^{+\infty} K_1[c](t)dt \le \delta \int_{h(\alpha)}^{\alpha} \beta(t)dt + \mu \int_{h(\alpha)}^{+\infty} c(t)dt + \delta \int_{\alpha}^{+\infty} [p(t) + \rho(t)]dt \le d.$$

Hence $K(B_2) \subset B_2$, and (1.2.213) has only one solution $c_2 \in B_2$. But it has at most one solution in L^1, hence $c_1(t) = c_2(t)$, for all $t \ge h(\alpha)$, belongs to $B_1 \cap B_2$. The definitions of B_1 and B_2 then imply that

$$|x'(t)| \le c_2(t) = c_1(t) \le \delta\phi(t).$$

$\square$

Corollary 1.2.21 *Let $r_0 > 0, L \ge 0, m \ge 0, \mu \in (0,1)$, and $\gamma > 0$ be constants such that*

$$Lr_0 \le \mu, \ \ L(e^{\gamma r_0} - 1) < \gamma,$$

and support that $x : [\alpha - r_0, +\infty) \to \mathbb{R}^n$ is an absolutely continuous function satisfying that for all $t \ge \alpha$,

$$|x'(t)| \le L \int_{t-r_0}^{t} |x'(s)|ds + me^{-\gamma(t-\alpha)} \|x_t\|_{r_0}. \tag{1.2.215}$$

Then for all $t \ge \alpha$,

$$|x(t)| \le \left[\|x_\alpha\|_{r_0} + \frac{\mu}{1-\mu} \int_{\alpha-r_0}^{\alpha} |x'(s)|ds\right] e^{m/(\gamma(1-\mu))}, \tag{1.2.216}$$

and for all $t \ge \alpha$,

$$|x'(t)| \le \delta m_0 e^{-\gamma(t-\alpha)}, \tag{1.2.217}$$

if only for all $\alpha - r_0 \le t \le \alpha$,

$$|x'(t)| \le m_0 e^{-\gamma(t-\alpha)}, \tag{1.2.218}$$

where $m_0 \geq m\gamma/(\gamma - L(e^{\gamma_1\gamma_0} - 1))$ *and*

$$\delta = \max\left\{1, \left[\|x_\alpha\|_{r_0} + \frac{\mu m_0(e^{\gamma r_0} - 1)}{\gamma(1-\mu)}\right] e^{m/(\gamma(1-\mu))}\right\}.$$

1.2.3 Linear One-Dimensional Gronwall-Bellman Inequalities with Retardation

In this section, we shall introduce some linear one-dimensional Gronwall-Bellman inequalities with retardation.

In what follows, $\mathbb{R}_1 = [1, +\infty), I = [t_0, T)$. The following two theorems were obtained in [501].

Theorem 1.2.30 (Pachpatte [501]) *Let* $a, b \in C(I, \mathbb{R}_+), \alpha \in C^1(I, I)$ *be non-decreasing with* $\alpha(t) \leq t$ *on* I, *and* $k \geq 0, c \geq 1$, *and* $p > 1$ *are constants.*

(a_1) *If* $u \in C(I, \mathbb{R}_+)$ *and for all* $t \in I$,

$$u(t) \leq k + \int_{t_0}^{t} a(s)u(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u(s)ds, \tag{1.2.219}$$

then for all $t \in I$,

$$u(t) \leq k \exp(A(t) + B(t)), \tag{1.2.220}$$

where for all $t \in I$,

$$A(t) = \int_{t_0}^{t} a(s)ds, \quad B(t) = \int_{\alpha(t_0)}^{\alpha(t)} b(s)ds. \tag{1.2.221}$$

(a_2) *If* $u \in C(I, \mathbb{R}_1)$ *and for all* $t \in I$,

$$u(t) \leq c + \int_{t_0}^{t} a(s)u(s)\log u(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u(s)\log u(s)ds, \tag{1.2.222}$$

then for all $t \in I$,

$$u(t) \leq c^{\exp(A(t)+B(t))}, \tag{1.2.223}$$

where $A(t)$ *and* $B(t)$ *are defined by (1.2.221).*

(a_3) *If* $u \in C(I, \mathbb{R}_+)$ *and for all* $t \in I$,

$$u^p(t) \leq k + \int_{t_0}^{t} a(s)b(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u(s)ds, \tag{1.2.224}$$

then for all $t \in I$,

$$u(t) \leq \Big(k^{(p-1)/p} + (\frac{p-1}{p})[A(t) + B(t)]\Big)^{1/(p-1)}, \tag{1.2.225}$$

where $A(t)$ and $B(t)$ are defined by (1.2.221).

Proof From the hypotheses, we observe that $\alpha'(t) \geq 0$ for all $t \in I, \alpha'(x) \geq 0$ for all $x \in J_1$.

(a_1) Let $k > 0$ and defined a function $z(t)$ by the right-hand side of (1.2.221). Then, $z(t) > 0, z(t_0) = k, u(t) \leq z(t)$, and

$$\begin{aligned} z'(t) &= a(t)u(t) + b(\alpha(t))u(\alpha(t))\alpha'(t) \\ &\leq a(t)z(t) + b(\alpha(t))z(\alpha(t))\alpha'(t) \\ &\leq a(t)z(t) + b(\alpha(t))z(t)\alpha'(t), \end{aligned}$$

i.e.,

$$\frac{z'(t)}{z(t)} \leq a(t) + b(\alpha(t))\alpha'(t). \tag{1.2.226}$$

Integrating (1.2.226) from t_0 to $t, t \in I$, and the change of variable yield for all $t \in I$,

$$z(t) \leq k \exp(A(t) + B(t)). \tag{1.2.227}$$

Using (1.2.227) in $u(t) \leq z(t)$, we get the inequality in (1.2.220). If $k \geq 0$, we carry out the above procedure with $k + \epsilon$ instead of k, where $\epsilon > 0$ is an arbitrary small constant, and subsequently pass the limit as $\epsilon \to 0$ to obtain (1.2.220).
(a_2) Define a function $z(t)$ by the right-hand side of (1.2.222). Then $z(t) > 0, z(t_0) = c$, and $u(t) \leq z(t)$, and as in the proof of (a_1), we get

$$\frac{z'(t)}{z(t)} \leq a(t)\log z(t) + b(\alpha(t))\log z(\alpha(t))\alpha'(t). \tag{1.2.228}$$

Integrating (1.2.228) from t_0 to $t, t \in I$, and the change of variable yield

$$\log z(t) \leq \log c + \int_{t_0}^{t} a(s)\log z(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)\log z(s)ds. \tag{1.2.229}$$

Now applying the inequality given in (a_1) to (1.2.229), we get

$$\log z(t) \leq (\log c)\exp(A(t) + B(t)) = \log c^{\exp(A(t)+B(t))}. \tag{1.2.230}$$

Thus from (1.2.230), it follows that

$$z(t) \leq c^{\exp(A(t)+B(t))}. \tag{1.2.231}$$

Now using (1.2.231) in $u(t) \leq z(t)$, we get the required inequality in (1.2.223).
(a_3) Let $k > 0$ and define a function $z(t)$ by the right-hand side of (1.2.224). Then $z(t) > 0, z(t_0) = k, u(t) \leq \Big(z(t)\Big)^{1/p}$, and as in the proof of (a_1), we have

$$\{z(t)\}^{-1/p} z'(t) \leq a(t) + b(\alpha(t))\alpha'(t). \tag{1.2.232}$$

Integrating (1.2.232) from t_0 to $t, t \in I$, and the change of variable, we have

$$z(t) \leq \Big(k^{(p-1)/p} + \frac{p-1}{p}[A(t) + B(t)]\Big)^{1/(p-1)}. \tag{1.2.233}$$

Therefore, the desired inequality in (1.2.225) follows by using (1.2.233) in $u(t) \leq \{z(t)\}^{1/p}$. The case $k \geq 0$ can be completed as mentioned in the proof of (a_1). □

Theorem 1.2.31 (Pachpatte [501]) *Let a, b, α, k, c, p be as in Theorem 1.2.30. For $i = 1, 2$, let $g_i \in C(\mathbb{R}_+, \mathbb{R}_+)$ be non-decreasing functions with $g_i(u) > 0$ for all $u > 0$.*

(b_1) If $u \in C(I, \mathbb{R}_+)$ and for all $t \in I$,

$$u(t) \leq k + \int_{t_0}^{t} a(s)g_1(u(s))ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)g_2(u(s))ds. \tag{1.2.234}$$

(i) If $g_2(u) \leq g_1(u)$, then

$$u(t) \leq G_1^{-1}\Big(G_1(t) + A(t) + B(t)\Big). \tag{1.2.235}$$

(ii) If $g_1(u) \leq g_2(u)$, then

$$u(t) \leq G_2^{-1}\Big(G_2(t) + A(t) + B(t)\Big), \tag{1.2.236}$$

where $A(t)$ and $B(t)$ are defined by (1.2.221) and for $i = 1, 2,\ G_i^{-1}$ are inverse functions of

$$G_i(r) = \int_{r_0}^{r} \frac{ds}{g_i(s)}, \qquad r \geq r_0 > 0, \tag{1.2.237}$$

and $t_1 \in I$ *is chosen so that*

$$G_i(t) + A(t) + B(t) \in Dom\ (G_i^{-1}),$$

respectively.
(b_2) *If* $u \in C(I, \mathbb{R}_1)$ *and for all* $t \in I$,

$$u(t) \leq c + \int_{t_0}^{t} a(s)u(s)g_1(\log u(s))ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u(s)g_2(\log u(s))ds, \tag{1.2.238}$$

then for all $t_0 \leq t \leq t_2$,

(i) if $g_2(u) \leq g_1(u)$, *then*

$$u(t) \leq \exp\left(G_1^{-1}[G_1(\log c) + A(t) + B(t)]\right); \tag{1.2.239}$$

(ii) if $g_1(u) \leq g_2(u)$, *then*

$$u(t) \leq \exp\left(G_2^{-1}[G_2(\log c) + A(t) + B(t)]\right), \tag{1.2.240}$$

where $G_i, G_i^{-1}, A(t), B(t)$ *are as in* (b_1) *and* t *is chosen so that for* $i = 1, 2$,

$$G_i(\log c) + A(t) + B(t) \in Dom\ (G_i^{-1}),$$

respectively.

(b_3) *If* $u \in C(I, \mathbb{R}_+)$ *and for all* $t \in I$,

$$u^p(t) \leq k + \int_{t_0}^{t} a(s)g_1(u(s))ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)g_2(u(s))ds, \tag{1.2.241}$$

then for $t_0 \leq t \leq t_3$,

(i) if $g_2(u) \leq g_1(u)$, *then*

$$u(t) \leq \left(H_1^{-1}[H_1(k) + A(t) + B(t)]\right)^{1/p}; \tag{1.2.242}$$

(ii) if $g_1(u) \leq g_2(u)$, *then*

$$u(t) \leq \left(H_2^{-1}[H_2(k) + A(t) + B(t)]\right)^{1/p}; \tag{1.2.243}$$

where $A(t)$ and $B(t)$ are defined by (1.2.221) and for $i = 1, 2, H_i^{-1}$ are the inverse functions of

$$H_i(r) = \int_{r_0}^{r} \frac{ds}{g_i(s^{1/p})}, \qquad r \geq r_0 > 0, \tag{1.2.244}$$

and $t_3 \in I$ is chosen so that

$$H_i(k) + A(t) + B(t) \in Dom\ (H_i^{-1}),$$

respectively.

Proof (b_1) Let $k > 0$ and define a function $z(t)$ by the right-hand side of (1.2.234). Then $z(t) > 0, z(t_0) = k$, and $u(t) \leq z(t)$, and as in the proof of (a_1), we get

$$z'(t) \leq a(t)g_1(z(t)) + b(\alpha(t))g_2(z(\alpha(t)))\alpha'(t). \tag{1.2.245}$$

(i) when $g_2(u) \leq g_1(u)$, then from (1.2.245) it follows that

$$z'(t) \leq g_1(z(t))[a(t) + b(\alpha(t))\alpha'(t)]. \tag{1.2.246}$$

From (1.2.237) and (1.2.246) it follows that

$$\frac{d}{dt}G_1(z(t)) = \frac{z'(t)}{g_1(z(t))} \leq a(t) + b(\alpha(t))\alpha'(t). \tag{1.2.247}$$

Integrating (1.2.247) from t_0 to $t, t \in I$, and making the change of variable, we have

$$G_1(z(t)) \leq G_1(k) + A(t) + B(t). \tag{1.2.248}$$

Since $G_1^{-1}(z)$ is increasing, from (1.2.248) we derive

$$z(t) \leq G_1^{-1}[G_1(k) + A(t) + B(t)]. \tag{1.2.249}$$

Using (1.2.249) in $u(t) \leq z(t)$ gives us the required inequality in (1.2.235). The case $k \geq 0$ can be completed as mentioned in the proof of (a_1). The case when $g_1(u) \leq g_2(u)$ can be done similarly. The subinterval $t_0 \leq t \leq t_1$ is obvious. $\square$

Theorem 1.2.32 (Pachpatte [504]) *Let $u(t), a(t) \in C(I, \mathbb{R}_+), b(t, s) \in C(I^2, \mathbb{R}_+)$ for all $t_0 \leq s \leq t \leq T$ and $\alpha(t) \in C^1(I, I)$ be non-decreasing with $\alpha(t) \leq t$ on I and $k \geq 0$ be a constant. If for all $t \in I = [t_0, T)$,*

$$u(t) \leq k + \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s)u(s) + \int_{\alpha(t_0)}^{s} b(s, \sigma)u(\sigma)d\sigma \right] ds, \tag{1.2.250}$$

then for all $t \in I$,

$$u(t) \leq k \exp(A(t)), \tag{1.2.251}$$

where for all $t \in I$,

$$A(t) = \int_{\alpha(t_0)}^{\alpha(t)} \left[a(s) + \int_{\alpha(t_0)}^{s} b(s, \sigma) d\sigma \right]. \tag{1.2.252}$$

Proof Let $k > 0$ and define a function $z(t)$ by the right-hand side of (1.2.250). Then $z(t) > 0, z(t_0) = k, u(t) \leq z(t)$ and

$$\begin{aligned} z'(t) &= \left[a(\alpha(t))u(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} [b(\alpha(t), \sigma)u(\sigma)d\sigma] \right] \alpha'(t) \\ &\leq \left[a(\alpha(t))z(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} [b(\alpha(t), \sigma)z(\sigma)d\sigma] \right] \alpha'(t). \end{aligned} \tag{1.2.253}$$

Therefore, it follows from (1.2.253) that

$$\frac{z'(t)}{z(t)} \leq \left[a(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t), \sigma) d\sigma \right] \alpha'(t). \tag{1.2.254}$$

Integrating (1.2.254) from t_0 to $t, t \in I$ and making the change of variables, we get that for all $t \in I$,

$$z(t) \leq k \exp(A(t)). \tag{1.2.255}$$

Using (1.2.255) in $u(t) \leq z(t)$, we get the inequality in (1.2.251). If $k \geq 0$, we carry out the above procedure with $k + \varepsilon$ instead of k, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (1.2.251). □

Theorem 1.2.33 (Pachpatte [501]) *Let* $I = [t_0,\ T)$ $(t_0 < T)$, $a, b \in C(I, \mathbb{R}_+)$, $\alpha \in C^1(I, I)$ *be non-decreasing with* $\alpha(t) \leq t$ *on* I, *and* $k \geq 0$ *is a constant.*

If $u \in C(I, \mathbb{R}_+)$ *and for all* $t \in I$,

$$u(t) \leq k + \int_{t_0}^{t} a(s)u(s)ds + \int_{\alpha(t_0)}^{\alpha(t)} b(s)u(s)ds, \tag{1.2.256}$$

then for all $t \in I$,

$$u(t) \leq k \exp\left(A(t) + B(t)\right), \tag{1.2.257}$$

where for all $t \in I$,

$$A(t) = \int_{t_0}^{t} a(s)ds, \quad B(t) = \int_{\alpha(t_0)}^{\alpha(t)} b(s)ds. \tag{1.2.258}$$

Proof Let $k > 0$ and define a function $z(t)$ by the right-hand side of (1.2.256). Then, $z(t) > 0, z(t_0) = k, u(t) \leq z(t)$, and

$$\begin{aligned} z'(t) &= a(t)u(t) + b(\alpha(t))u(\alpha(t))\alpha'(t) \\ &\leq a(t)z(t) + b(\alpha(t))z(\alpha(t))\alpha'(t) \\ &\leq a(t)z(t) + b(\alpha(t))z(t)\alpha'(t), \end{aligned}$$

i.e.,

$$\frac{z'(t)}{z(t)} \leq a(t) + b(\alpha(t))\alpha'(t). \tag{1.2.259}$$

Integrating (1.2.259) from t_0 to $t, t \in I$, and the change of variable yield for all $t \in I$,

$$z(t) \leq k \exp(A(t) + B(t)). \tag{1.2.260}$$

Using (1.2.260) in $u(t) \leq z(t)$, we get the inequality in (1.2.257). If $k \geq 0$, we carry out the above procedure with $k + \epsilon$ instead of k, where $\epsilon > 0$ is an arbitrary small constant, and subsequently pass the limit as $\epsilon \to 0^+$ to obtain (1.2.257). □

Theorem 1.2.34 (Lipovan [366]) *Let $k \in C(\mathbb{R}_+, \mathbb{R}_+), \alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+), a \in C(\mathbb{R}_+^2, \mathbb{R}_+)$ with $(t, s) \mapsto \partial_t a(t, s) \in C(\mathbb{R}_+^2, \mathbb{R}_+)$. Assume, in addition, that α is non-decreasing and $\alpha(t) \leq t$ for all $t \geq 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies that for all $t \geq 0$,*

$$u(t) \leq k(t) + \int_0^{\alpha(t)} a(t, s)u(s)ds, \tag{1.2.261}$$

then for all $t \geq 0$,

$$u(t) \leq k(t) + e^{\int_0^{\alpha(t)} a(t,s)ds} \int_0^t e^{-\int_0^{\alpha(r)} a(r,s)ds} \partial_r \left(\int_0^{\alpha(r)} a(r, s)k(s)ds \right) dr. \tag{1.2.262}$$

Proof Denote $z(t) = \int_0^{\alpha(t)} a(t,s)u(s)ds$. Our assumptions on a and α imply that z is non-decreasing on $\mathbb{R}_+$. Hence, we have for all $t \geq 0$,

$$\begin{aligned}
z'(t) &= a(t,\alpha(t))u(\alpha(t))a'(t) + \int_0^{\alpha(t)} \partial_t a(t,s)u(s)ds \\
&\leq a(t,\alpha(t))\left[k(\alpha(t)) + z(\alpha(t))\right]\alpha'(t) + \int_0^{\alpha(t)} \partial_t a(t,s)\left(z(s) + k(s)\right)ds \\
&\leq a(t,\alpha(t))\left[k(\alpha(t)) + z(t)\right]\alpha'(t) + \int_0^{\alpha(t)} \partial_t a(t,s)k(s)ds + z(t)\int_0^{\alpha(t)} \partial_t a(t,s)ds,
\end{aligned}$$

or, equivalently,

$$z'(t) - z(t)\frac{d}{dt}\left(\int_0^{\alpha(t)} a(t,s)ds\right) \leq \frac{d}{dt}\left(\int_0^{\alpha(t)} a(t,s)k(s)ds\right).$$

Multiplying the above inequality by $e^{\int_0^{-\alpha(t)} a(t,s)ds}$, we can get

$$\frac{d}{dt}\left(z(t)e^{\int_0^{-\alpha(t)} a(t,s)ds}\right) \leq e^{\int_0^{-\alpha(t)} a(t,s)ds}\frac{d}{dt}\left(\int_0^{\alpha(t)} a(t,s)k(s)ds\right).$$

Thus integrating the above inequality with respect to t gives us for all $t \geq 0$,

$$z(t) \leq e^{\int_0^{\alpha(t)} a(t,s)ds}\int_0^t e^{-\int_0^{\alpha(r)} a(r,s)ds}\partial_r\left(\int_0^{\alpha(r)} a(r,s)k(s)ds\right)dr,$$

which, along with $u(t) \leq k(t) + z(t)$, implies (1.2.262) and, hence the proof is complete. □

Corollary 1.2.22 *Assume a, α are same as in Theorem 1.2.34 and $k(t) \equiv k > 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies (1.2.261), then for all $t \geq 0$,*

$$u(t) \leq ke^{\int_0^{\alpha(t)} a(t,s)ds}. \tag{1.2.263}$$

Proof Applying Theorem 1.2.34, we obtain for all $t \geq 0$,

$$\begin{aligned}
u(t) &\leq k + ke^{\int_0^{\alpha(t)} a(t,s)ds}\int_0^t e^{-\int_0^{\alpha(t)} a(t,s)ds}\partial_r\left(\int_0^{\alpha(r)} a(r,s)ds\right)dr \\
&= k + ke^{\int_0^{\alpha(t)} a(t,s)ds}\left(1 - e^{-\int_0^{\alpha(t)} a(t,s)ds}\right) = ke^{\int_0^{\alpha(t)} a(t,s)ds}.
\end{aligned}$$

□

Remark 1.2.13 We note that for $\partial_t a(t,s) \equiv 0$ in Corollary 1.2.22, we get an inequality in [362]. If, in addition, $\alpha(t) = t$, the inequality given in Corollary 1.2.22 reduces to Gronwall's inequality [239].

Corollary 1.2.23 *Assume a, α are as in Theorem 1.2.34 and $k(t) \equiv k > 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a solution to the Volterra integral equation*

$$u(t) = k + \int_0^{\alpha(t)} a(t,s)u(s)ds, \quad \textit{for all } t \geq 0. \tag{1.2.264}$$

If $\lim_{t\to+\infty} \int_0^{\alpha(t)} a(t,s)ds < +\infty$, then u is bounded on $\mathbb{R}_+$.

Proof The conclusion follows immediately from Corollary 1.2.22. Note that the limit

$$\lim_{t\to+\infty} \int_0^{\alpha(t)} a(t,s)ds < +\infty$$

always exists since the function $t \mapsto \int_0^{\alpha(t)} a(t,s)ds$ is non-decreasing on $\mathbb{R}_+$. $\square$

Example 1.2.3 The function $a(t,s) = t/[1 + 2t + (1+t)s^2]$, $t, s \geq 0$ satisfies the assumptions in Corollary 1.2.23 for any non-decreasing $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ with $\alpha(t) \leq t$, for all $t \geq 0$. In this case, all solutions $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ of integral equation (1.2.264) are bounded.

Theorem 1.2.35 (Lipovan [366]) *Let $a, b, k \in C(\mathbb{R}_+, \mathbb{R}_+), \alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ and assume that α is non-decreasing with $\alpha(t) \leq t$ for all $t \geq 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies that for all $t \geq 0$,*

$$u(t) \leq k(t) + a(t) \int_0^{\alpha(t)} b(s)u(s)ds. \tag{1.2.265}$$

Then for all $t \geq 0$,

$$u(t) \leq k(t) + a(t) \int_0^{\alpha(t)} e^{\int_r^{\alpha(t)} a(s)b(s)ds} b(r)k(r)dr. \tag{1.2.266}$$

Proof Denote $z(t) = \int_0^{\alpha(t)} b(s)u(s)ds$. Then for all $t \geq 0$,

$$\begin{aligned} z'(t) &= b(\alpha(t))u(\alpha(t))\alpha'(t) \leq b\alpha(t) \left[k(\alpha(t)) + a(\alpha(t))z(\alpha(t))\right] \alpha'(t) \\ &\leq b(\alpha(t)) \left[k(\alpha(t)) + a(\alpha(t))z(t)\right] \alpha'(t). \end{aligned}$$

Hence

$$z'(t) - z(t)b(\alpha(t))a(\alpha(t))\alpha'(t) \leq b(\alpha(t))k(\alpha(t))\alpha'(t).$$

Multiplying the above inequality by $e^{-\int_0^{\alpha(t)} a(s)b(s)ds}$, we get for all $t \geq 0$,

$$\frac{d}{dt}\left(z(t)e^{-\int_0^{\alpha(t)} a(s)b(s)ds}\right) \leq e^{-\int_0^{\alpha(t)} a(s)b(s)ds} b(\alpha(t))k(\alpha(t))\alpha'(t).$$

Integrating on the interval $[0, t]$, we may derive that for all $t \geq 0$,

$$\begin{aligned} z(t) &\leq e^{\int_0^{\alpha(t)} a(s)b(s)ds} \int_0^t e^{-\int_0^{\alpha(r)} a(s)b(s)ds} b(\alpha(r))k(\alpha(r))\alpha'(r)dr \\ &= \int_0^t e^{\int_{\alpha(r)}^{\alpha(t)} a(s)b(s)ds} b(\alpha(r))k(\alpha(r))\alpha'(r)dr \\ &= \int_0^{\alpha(t)} e^{\int_r^{\alpha(t)} a(s)b(s)ds} b(r)k(r)dr, \end{aligned}$$

which, after a change of variables performed in the last integral above, together with $u(t) \leq k(t) + a(t)z(t)$, implies (1.2.266). □

Considering $\alpha(t) = t$ in Theorem 1.2.35, we obtain Morro's inequality [417].

Corollary 1.2.24 (Morro [417]) *Assume a, b, k, α are as in Theorem 1.2.35. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a solution to the integral equation*

$$u(t) = k(t) + a(t) \int_0^{\alpha(t)} b(s)u(s)ds, \quad \textit{for all } t \geq 0.$$

If a, k are bounded on $\mathbb{R}_+$ and $\int_0^{\alpha(+\infty)} b(s)ds < +\infty$, then u is bounded on $\mathbb{R}_+$.

Corollary 1.2.25 *Assume a, b, k, α are as in Theorem 1.2.35 with $k(t) \to 0$ as $t \to +\infty$. Suppose $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a solution to the integral equation*

$$u(t) = k(t) + a(t) \int_0^{\alpha(t)} b(s)u(s)ds, \quad \textit{for all } t \geq 0. \tag{1.2.267}$$

If

$$\int_0^{\alpha(+\infty)} a(s)b(s)ds < +\infty, \qquad \lim_{t \to +\infty} a(t) \int_0^{\alpha(t)} b(r)k(r)dr = 0, \tag{1.2.268}$$

then $u(t) \to 0$ as $t \to +\infty$. In particular, if $a(t)$, $k(t) \to 0$ as $t \to +\infty$ and $\int_0^{\alpha(+\infty)} b(s)ds < +\infty$, then $u(t) \to 0$ as $t \to +\infty$.

Remark 1.2.14 To discuss the conditions in (1.2.268), we particularize $\alpha(t) = t$. The integral equation

$$u(t) = k(t) + a(t) \int_0^t b(s)u(s)ds, \quad \textit{for all } t \geq 0,$$

has the exact solution

$$u(t) = k(t) + a(t) \int_0^{\alpha(t)} e^{\int_r^t a(s)b(s)ds} b(r)k(r)ds, \quad for\ \ all\ \ t \geq 0.$$

So, in order to have $u(t) \to 0$ as $t \to +\infty$, both $\lim_{t\to+\infty} k(t) = 0$ and $\lim_{t\to+\infty} a(t) \int_0^t b(r)k(r)dr = 0$ must hold. Concerning the condition $\int_0^{+\infty} a(s)b(s)ds < +\infty$, the case $a(t) = k(t) = t^{-2}, b(t) = t^2$, shows that

$$\lim_{t\to+\infty} k(t) = 0, \quad \lim_{t\to+\infty} a(t) \int_0^t b(r)k(r)dr = 0, \quad \int_0^{+\infty} a(s)b(s)ds < +\infty,$$

can all hold simultaneously. Notice that in this setting, the solution equals

$$u(t) = (t+1)^{-2} + (e^t - 1)(t+1)^{-2} \to +\infty \text{ as } t \to +\infty.$$

This shows that both conditions in (1.2.268) are relevant.

In 2008, Ferreira and Torres [210] proved the following result.

Theorem 1.2.36 (Ferreira and Torres [210]) *Suppose that* $\alpha(\cdot) \in C^1([a,b],\mathbb{R})$ *is a non-decreasing function with* $a \leq \alpha(t) \leq t$, *for all* $t \in [a,b]$. *Assume that* $u(\cdot),\ a(\cdot),\ b(\cdot) \in C([a,b],\mathbb{R}_0)$ $(\mathbb{R}_0 \equiv (0,+\infty))$ *and let* $(t,s) \mapsto f(t,s) \in C([a,b] \times [a,\alpha(b)],\mathbb{R}_0)$ *be non-decreasing in* t *for every* s *fixed. If for all* $t \in [a,b]$,

$$u(t) \leq a(t) + b(t) \int_a^{\alpha(t)} f(t,s)u(s)ds, \tag{1.2.269}$$

then for all $t \in [a,b]$,

$$u(t) \leq a(t) + b(t) \int_a^{\alpha(t)} \exp\left(\int_s^{\alpha(t)} b(\tau)f(t,\tau)d\tau\right) f(t,s)a(s)ds. \tag{1.2.270}$$

Proof The result is obvious for $t = a$. Let t_0 be an arbitrary number in $(a,b]$ and define the function $z(t)$ as for all $t \in [a,t_0]$,

$$z(t) = \int_a^{\alpha(t)} f(t_0,s)u(s)ds.$$

Then, $u(t) \leq a(t) + b(t)z(t)$ for all $t \in [a,t_0]$, and $z(\cdot)$ is non-decreasing. Hence,

$$\begin{aligned} z'(t) &= f(t_0,\alpha(t))u(\alpha(t))\alpha'(t) \\ &\leq f(t_0,\alpha(t))\,[a(\alpha(t)) + b(\alpha(t))z(\alpha(t))]\,\alpha'(t) \\ &\leq f(t_0,\alpha(t))\,[a(\alpha(t)) + b(\alpha(t))z(t)]\,\alpha'(t) \end{aligned}$$

which implies

$$z'(t) - f(t_0, \alpha(t))b(\alpha(t))z(t)\alpha'(t) \leq f(t_0, \alpha(t))a(\alpha(t))\alpha'(t). \qquad (1.2.271)$$

Multiplying both sides of inequality (1.2.271) by $\exp\left(-\int_a^{\alpha(t)} b(s)f(t_0, s)ds\right)$, we get

$$\left[z(t)\exp\left(-\int_a^{\alpha(t)} b(s)f(t_0, s)ds\right)\right]'$$
$$\leq \exp\left(-\int_a^{\alpha(t)} b(s)f(t_0, s)ds\right)f(t_0, \alpha(t))a(\alpha(t))\alpha'(t).$$

Integrating from a to t and noting that $z(a) = 0$, we obtain successively that

$$\begin{aligned} z(t) &\leq \exp\left(\int_a^{\alpha(t)} b(s)f(t_0, s)ds\right)\int_a^t \exp\left(-\int_a^{\alpha(s)} b(\tau)f(t_0, \tau)d\tau\right) \\ &\quad \times f(t_0, \alpha(s))a(\alpha(s))\alpha'(s)ds \\ &= \int_a^t \exp\left(\int_{\alpha(s)}^{\alpha(t)} b(\tau)f(t_0, \tau)d\tau\right)f(t_0, \alpha(s))a(\alpha(s))\alpha'(s)ds \\ &= \int_a^{\alpha(t)} \exp\left(\int_s^{\alpha(t)} b(\tau)f(t_0, \tau)d\tau\right)f(t_0, s)a(s)ds. \end{aligned}$$

Since $u(t) \leq a(t) + b(t)z(t)$, we have, for $t = t_0$,

$$u(t_0) \leq a(t_0) + b(t_0)\int_a^{\alpha(t_0)} \exp\left(\int_s^{\alpha(t_0)} b(\tau)f(t_0, \tau)d\tau\right)f(t_0, s)a(s)ds.$$

Thus the required conclusion follows from the arbitrariness of t_0. □

In what follows, $I = [t_0, T), J_1 = [x_0, X), J_2 = [y_0, Y)$ are the given subsets of $\mathbb{R}$, $\Delta = J_1 \times J_2$ and $'$ denotes the derivative.

The next result generalizes Theorems 1.2.33–1.2.36.

Theorem 1.2.37 (Pachpatte [504]) *Let $u(t), a(t) \in C(I, \mathbb{R}_+), b(t, s) \in C(I^2, \mathbb{R}_+)$ for $t_0 \leq s \leq t \leq T$ and $\alpha(t) \in C^1(I, I)$ be non-decreasing with $\alpha(t) \leq t$ on I and $k \geq 0$ be a constant.*

(a_1) If for all $t \in I$,

$$u(t) \leq k + \int_{\alpha(t_0)}^{\alpha(t)} [a(s)u(s) + \int_{\alpha(t_0)}^{s} b(s, \sigma)u(\sigma)d\sigma]ds, \qquad (1.2.272)$$

then for all $t \in I$,

$$u(t) \leq k \exp(A(t)), \tag{1.2.273}$$

where for all $t \in I$,

$$A(t) = \int_{\alpha(t_0)}^{\alpha(t)} [a(s) + \int_{\alpha(t_0)}^{s} b(s, \sigma) d\sigma] ds. \tag{1.2.274}$$

(a_2) *Let $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a non-decreasing function with $g(u) > 0$ for all $u > 0$. If for all $t \in I$,*

$$u(t) \leq k + \int_{\alpha(t_0)}^{\alpha(t)} [a(s) g(u(s)) + \int_{\alpha(t_0)}^{s} b(s, \sigma) g(u(\sigma)) d\sigma] ds, \tag{1.2.275}$$

then for all $t_0 \leq t \leq t_1$,

$$u(t) \leq G^{-1}[G(k) + A(t)], \tag{1.2.276}$$

where $A(t)$ is defined by (1.2.274), G^{-1} is the inverse function of

$$G(r) = \int_{r_0}^{r} \frac{ds}{g(s)}, \quad r \geq r_0 > 0, \tag{1.2.277}$$

and $t \in I$ is chosen so that, for all $t \in [t_0, t_1]$,

$$G(k) + A(t) \in Dom\ (G^{-1}).$$

Proof From the hypotheses, we observe that $\alpha'(t) \geq 0$ for all $t \in I, \alpha'(x) \geq 0$ for all $x \in J_1, \beta'(y) \geq 0$ for all $y \in J_2$.

(a_1) Let $k > 0$ and define a function $z(t)$ by the right-hand side of (1.2.271). Then $z(t) > 0, z(t_0) = k, u(t) \leq z(t)$ and

$$\begin{aligned} z'(t) &= [a(\alpha(t)) u(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} [b(\alpha(t), \sigma) u(\sigma) d\sigma]] \alpha'(t) \\ &\leq [a(\alpha(t)) z(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} [b(\alpha(t), \sigma) z(\sigma) d\sigma]] \alpha'(t). \end{aligned} \tag{1.2.278}$$

Thus from (1.2.278) it follows

$$\frac{z'(t)}{z(t)} \leq [a(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t), \sigma) d\sigma] \alpha'(t). \tag{1.2.279}$$

Integrating (1.2.279) from t_0 to t, $t \in I$ and making the change of variables, we obtain that for all $t \in I$,

$$z(t) \leq k \exp(A(t)). \tag{1.2.280}$$

Using (1.2.280) in $u(t) \leq z(t)$, we get the inequality in (1.2.273). If $k \geq 0$, we carry out the above procedure with $k + \epsilon$ instead of k, where $\epsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\epsilon \to 0$ to obtain (1.2.273).

(a_2) Let $k > 0$ and define a function $z(t)$ by the right-hand side of (1.2.275). Then $z(t) > 0, z(t_0) = k, u(t) \leq z(t)$ and as in the proof of (a_1), we get

$$\frac{z'(t)}{g(z(t))} \leq [a(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t), \sigma) d\sigma] \alpha'(t). \tag{1.2.281}$$

From (1.2.277) and (1.2.281) it follows

$$\frac{d}{dt} G(z(t)) = \frac{z'(t)}{g(z(t))} \leq [a(\alpha(t)) + \int_{\alpha(t_0)}^{\alpha(t)} b(\alpha(t), \sigma) d\sigma] \alpha'(t). \tag{1.2.282}$$

Integrating (1.2.282) from t_0 to t, $t \in I$ and making the change of variables, we have

$$G(z(t)) \leq G(k) + A(t). \tag{1.2.283}$$

Since $G(z)$ is increasing, from (1.2.283) we derive

$$z(t) \leq G^{-1}\Big(G(k) + A(t)\Big). \tag{1.2.284}$$

Using (1.2.284) in $u(t) \leq z(t)$, we get (1.2.276). The case $k \geq 0$ can be completed in the same manner as in the proof of (a_1) . The subinterval $t_0 \leq t \leq t_1$ for t is obvious. □

1.2.4 Linear One-Dimensional Integral Inequalities of Volterra Type

In 1967, Chu and Metcalf [135] gave a linear generalization of the Volterra-type.

Theorem 1.2.38 (Chu-Metcalf [135]) *Let u and f be real continuous functions on* $[0, 1]$. *Let K be continuous and non-negative on the triangle $\Delta : 0 \leq y \leq x \leq 1$. If for all* $0 \leq x \leq 1$,

$$u(x) \leq f(x) + \int_0^x K(x, y) u(y) dy. \tag{1.2.285}$$

Then for all $0 \leq x \leq 1$*,*

$$u(x) \leq f(x) + \int_0^x H(x,y)f(y)dy, \tag{1.2.286}$$

where for $0 \leq y \leq x \leq 1$,

$$H(x,y) = \sum_{i=1}^{+\infty} K_i(x,y)$$

is the resolvent kernel and the K_i $(i = 1, 2, \cdots)$ *are the iterated kernels of* K.

Proof From (1.2.285), it follows that for all $0 \leq x \leq 1$,

$$\begin{aligned} u(x) &\leq f(x) + \int_0^x K(x,y)f(y)dy + \int_0^x K(x,y)\int_0^y K(y,z)u(z)dzdy \\ &= f(x) + \int_0^x K_1(x,y)f(y)dy + \int_0^x K_2(x,y)u(y)dy. \end{aligned}$$

The reminder of the proof is by induction and a standard estimation procedure showing the resulting series to be uniformly convergent. □

The previous results, in which an explicit upper bound for u was obtained, are merely those cases for which the resolvent kernel H can be summed in "closed form". For example, if $K(x,y) = g(x)h(y) \geq 0,\ 0 \leq y \leq x \leq 1$, then

$$\begin{aligned} H(x,y) &= \sum_{i=1}^{+\infty} \frac{g(x)h(y)}{(i-1)!}\left[\int_y^x g(z)h(z)dz\right]^{i-1} \\ &= g(x)h(y)\exp\left(\int_y^x g(z)h(z)dz\right), \end{aligned}$$

since we can show by induction that each K_i $(i = 1, 2, \cdots)$ is given by the appropriate term in the sum for sum H. □

Note that Beesack [49] extended Theorem 1.2.38 to the case that $u, f \in L^2(J)$ and $K \in L^2(\Delta)$ and the results are still valid if "$\leq$" is replaced by "$\geq$" in both (1.2.287) and (1.2.288). The inequality given in Theorem 1.2.38 includes as a special case the inequality given in Theorem 1.2.38. Concerning Theorem 1.2.38, there is another interesting linear generalization due to Willett [647] under the assumption that either $K(t,s)$ or $\partial K(t,s)/\partial t$ is degenerate or directly separable in the following sense

$$K(t,s) \leq \sum_{i=1}^{n} h_i(t)k_i(s)$$

or a similar relation holds for $\partial K(t,s)/\partial t$.

As pointed out by Chu and Metcalf [135], the cases in which we obtain an explicit bound on u are precisely those in which the resolvent kernel (or a majorant of it) can be summed in a closed form. This is, in fact, the case when $K(t,s) = h(s)g(s) \geq 0$. Of particular interest is the case $h \equiv 1$.

The following theorem provides a slight variant of the inequality given by Norbury and Stuart [432].

Theorem 1.2.39 (Norbury and Stuart [432]) *Let u and $K(t,s)$ be as in Theorem 1.2.38 and $K(t,s)$ be non-decreasing in t for each $s \in J$.*

(1) If for all $t \in J = [\alpha, T]$,

$$u(t) \leq C + \int_\alpha^t K(t,s)u(s)ds, \tag{1.2.287}$$

with $C \geq 0$ being a constant. Then for all $t \in J$,

$$u(t) \leq C\exp(\int_\alpha^t K(t,s)ds). \tag{1.2.288}$$

(2) Let $n(t)$ be a positive continuous and non-decreasing function for all $t \in J$. If for all $t \in J$,

$$u(t) \leq n(t) + \int_\alpha^t K(t,s)u(s)ds, \tag{1.2.289}$$

then for all $t \in J$,

$$u(t) \leq n(t)\exp(\int_\alpha^t K(t,s)ds). \tag{1.2.290}$$

Proof

(1) Fix any T, $\alpha \leq T \leq \beta$. Then for $\alpha \leq t \leq T$, we have

$$u(t) \leq C + \int_\alpha^t K(T,s)u(s)ds. \tag{1.2.291}$$

Define a function $z(t)$ by the right-hand side of (1.2.287), then $z(\alpha) = C$, $u(t) \leq z(t)$ for $\alpha \leq t \leq T$ and

$$z'(t) \leq K(T,t)u(t) \leq K(T,t)z(t), \quad \alpha \leq t \leq T. \tag{1.2.292}$$

Setting $t = s$ in (1.2.288) and integrating it with respect to s from α to t, we get

$$z(T) \leq C\exp\Big(\int_\alpha^T K(T,s)ds\Big). \tag{1.2.293}$$

Since T is arbitrary, from (1.2.293) with T replaced by t and $u(t) \leq z(t)$, we derive (1.2.288).

(2) Since $n(t)$ is a positive continuous and non-decreasing function for all $t \in J$, we derive from (1.2.289) that for all $t \in J$,

$$\frac{u(t)}{n(t)} \leq 1 + \int_{\alpha}^{t} K(t,s)\frac{u(s)}{n(s)}ds. \tag{1.2.294}$$

Now applying (1.2.293) to (1.2.294) yields (1.2.290). □

Remark 1.2.15 Note that the inequality given in (1.2.288) was obtained in Norbury and Stuart [432] under the assumptions of the existence and non-negativity of $(\partial/\partial t)K(t,s)$.

The next result is a generalization of Theorem 1.2.39.

Theorem 1.2.40 (Morro [419]) *Let $u(t)$ and $a(t)$ be non-negative continuous functions for all $t \geq \alpha$, let $k(t,s)$ and its partial derivative $k_t(t,s)$ be non-negative continuous functions for $\alpha \leq s \leq t$, and suppose that for all $t \geq \alpha$,*

$$u(t) \leq a(t) + \int_{\alpha}^{t} k(t,s)u(s)ds. \tag{1.2.295}$$

Then for all $t \geq \alpha$,

$$u(t) \leq a(t) + \int_{\alpha}^{t} A(s)\exp\left(\int_{s}^{t} B(\tau)d\tau\right)ds, \tag{1.2.296}$$

where

$$\begin{cases} A(t) = k(t,t)a(t) + \displaystyle\int_{\alpha}^{t} k_t(t,s)a(s)ds, \\ B(t) = k(t,t) + \displaystyle\int_{\alpha}^{t} k_t(t,s)ds. \end{cases}$$

Proof Putting $v(t) = \int_{\alpha}^{t} k(t,s)u(s)ds$, we find $v(\alpha) = 0$ and

$$v'(t) = k(t,t)u(t) + \int_{\alpha}^{t} k_t(t,s)u(s)ds. \tag{1.2.297}$$

Since $v(t)$ is non-decreasing and $u(t) \leq a(t) + v(t)$, from (1.2.297) we have for all $t \geq \alpha$,

$$\begin{aligned} v'(t) &\leq [k(t,t) + \int_{\alpha}^{t} k_t(t,s)ds]v(t) + [k(t,t)a(t) + \int_{\alpha}^{t} k_t(t,s)a(s)ds] \\ &= B(t)v(t) + A(t). \end{aligned}$$

Applying Lemma 1.1.1, we may obtain (1.2.296). □

Next theorem provides the derivation of a better estimate of u as long as the kernel $k(t,s)$ assumes the form $k(t,s) = \nu(t)h(s)$, which is also a special case of the Willett inequality (Theorem 1.2.7) with $a(t) \equiv 0$.

Theorem 1.2.41 (Morro [419]) *Let a, ν, h be real continuous functions on the interval $[t_0, T]$, t_0, $t \in \mathbb{R}$, $T > t_0$; $\nu h > 0$. If a continuous function u satisfies for all $t \in [t_0, T]$,*

$$u(t) \leq a(t) + \nu(t) \int_{t_0}^{t} h(s)u(s)ds, \tag{1.2.298}$$

then for all $t \in [t_0, T]$,

$$u(t) \leq a(t) + \nu(t) \int_{t_0}^{t} h(s)a(s) \exp\left(\int_{s}^{t} h(\tau)\nu(\tau)d\tau\right) ds. \tag{1.2.299}$$

Proof Letting ν, $h > 0$, put

$$x(t) = \int_{t_0}^{t} h(s)u(s)ds. \tag{1.2.300}$$

Hence it follows that x is differentiable and $x' = h(t)u(t)$. Then, in view of (1.2.298), we obtain

$$x'(t) - h(t)\nu(t)x(t) \leq h(t)a(t).$$

If we put

$$w(t) = x(t) \exp\left(-\int_{t_0}^{t} h(\tau)\nu(\tau)d\tau\right), \tag{1.2.301}$$

then we can write the last inequality as

$$w'(t) \leq h(t)a(t) \exp\left(-\int_{t_0}^{t} h(\tau)\nu(\tau)d\tau\right).$$

So, since $w(t_0) = 0$, a simple integration yields

$$w(t) \leq \int_{t_0}^{t} h(s)a(s) \exp\left(-\int_{t_0}^{s} h(\tau)\nu(\tau)d\tau\right) ds$$

which, along with (1.2.301), gives us

$$x(t) \leq \int_{t_0}^{t} h(s)a(s) \exp\left(\int_{s}^{t} h(\tau)\nu(\tau)d\tau\right) ds. \tag{1.2.302}$$

On the other hand, (1.2.298) and (1.2.300) lead to

$$u(t) \leq a(t) + \nu(t)x(t). \tag{1.2.303}$$

Thus inserting (1.2.302) into (1.2.303) yields (1.2.299). The proof of the case $\nu,\ h < 0$ can be done similarly. □

Now we give some comments for the inequality (1.2.299). First, as we should expect, on putting $\nu = 1$, the theorem reduces exactly to a well-known result (see, e.g., [137, p. 37]), sometimes referred to as Gronwall's lemma. Second, the inequality (1.2.299) has already been investigated in the literature; to my knowledge, the best estimate is the one obtained by Willett [646, 647] who arrived at

$$u(t) \leq a(t) + \nu(t)\left(\int_{t_0}^{t} h(s)a(s)ds\right)\left(\exp\int_{t_0}^{t} h(s)d(s)ds\right). \tag{1.2.304}$$

We note that, if a is non-negative, (1.2.304) is an immediate consequence of (1.2.303), but the converse is false. This is consistent with the fact that we cannot obtain Gronwall's inequality as a particular case of (1.2.304).

In 1969, Gamidov [222] showed the next two results.

Theorem 1.2.42 (Gamidov [222]) *Let u, f, g_i, h_i $(i = 1, 2, \cdots, n)$ be continuous functions defined on $J = [\alpha, \beta]$, let g_i and h_i be non-negative in J, and for all $t \in J$,*

$$u(t) \leq f(t) + \sum_{i=1}^{n} g_i(t)\int_{\alpha}^{t} h_i(s)u(s)ds. \tag{1.2.305}$$

Then for all $t \in J$,

$$u(t) \leq f(t) + g(t)\int_{\alpha}^{t} f(s)\sum_{i=1}^{n} h_i(s)\exp\left(\int_{s}^{t} g(\sigma)\sum_{i=1}^{n} h_i(\sigma)d\sigma\right)ds, \tag{1.2.306}$$

where $g(t) = \max_{1\leq i\leq n}\{g_i(t)\}$.

Proof It follows from (1.2.305) that

$$u(t) \leq f(t) + g(t)\int_{\alpha}^{t}\left(\sum_{i=1}^{n} h_i(s)\right)u(s)ds. \tag{1.2.307}$$

Applying Theorem 1.2.41 to (1.2.307) gives us the desired inequality (1.2.306). □

Theorem 1.2.43 (Gamidov [222]) *Let u, f, g_i, h_i $(i = 1, 2, \cdots, n)$ be non-negative continuous functions defined on $J = [\alpha, \beta]$, and*

$$u(t) \leq f(t) + g_1(t)\int_{t_1}^{t} h_1(s)u(s)ds + g_2(t)\sum_{i=2}^{n} c_i\int_{t_1}^{t} h_i(s)u(s)ds, \tag{1.2.308}$$

where $\alpha = t_1 \le t_2 \le \cdots \le t_n = \beta$, *and* c_i $(i = 1, 2, \ldots, n)$ *are constants, and*

$$\sum_{i=2}^{n} c_i \int_{t_1}^{t_i} h_i(s)\Big[g_2(s) + g_1(s)\int_{t_1}^{s} h_1(\tau)g_2(\tau) \times \exp\left(\int_{\tau}^{s} g_1(\sigma)h_1(\sigma)d\sigma\right) d\tau\Big]ds < 1, \tag{1.2.309}$$

then

$$u(t) \le p_1(t) + Mp_2(t) \tag{1.2.310}$$

where

$$\begin{cases} p_1(t) = f(t) + g_1(t)\displaystyle\int_{t_1}^{t} h_1(s)f(s)\exp\left(\int_{s}^{t} g_1(\tau)h_1(\tau)d\tau\right) ds, \\ p_2(t) = g_2(t) + g_1(t)\displaystyle\int_{t_1}^{t} h_1(s)g_2(s)\exp\left(\int_{s}^{t} g_1(\tau)h_1(\tau)d\tau\right) ds, \\ \quad M = \left(\displaystyle\sum_{i=2}^{n} c_i \int_{t_1}^{t_i} h_i(s)p_1(s)ds\right)\left(1 - \displaystyle\sum_{i=2}^{n} c_i \int_{t_1}^{t_i} h_i(s)p_2(s)ds\right)^{-1}. \end{cases} \tag{1.2.311}$$

Proof Put

$$m_i = c_i \int_{t_1}^{t_i} h_i(s)u(s)ds. \tag{1.2.312}$$

Then (1.2.308) can be rewritten as

$$u(t) \le \left(f(t) + g_2(t)\sum_{i=2}^{n} m_i\right) + g_1(t)\int_{t_1}^{t} h_1(t)u(s)ds$$

which, along with Theorem 1.2.42, yields

$$\begin{aligned} u(t) &\le \left(f(t) + g_2(t)\sum_{i=2}^{n} m_i\right) + g_1(t)\int_{t_1}^{t}\left(f(s) + g_2(s)\sum_{i=2}^{n} m_i\right) \\ &\quad \times\ h_1(s)\exp\left(\int_{s}^{t} h_1(\tau)g_1(\tau)d\tau\right) ds \\ &= p_1(t) + \sum_{i=2}^{n} m_i p_2(t). \end{aligned} \tag{1.2.313}$$

Noting that

$$\begin{aligned}\sum_{i=2}^{n} m_i &= \sum_{i=2}^{n} c_i \int_{t_1}^{t_i} h_i(s)u(s)ds \\ &\le \sum_{i=2}^{n} c_i \int_{t_1}^{t_i} h_i(s)\left[p_1(s) + \sum_{i=2}^{n} m_i p_2(s)\right] \\ &= \sum_{i=2}^{n} c_i \int_{t_1}^{t_i} p_1(s)h_i(s)ds + \sum_{i=2}^{n} c_i \int_{t_1}^{t_i} h_i(s)\left(\sum_{i=2}^{n} m_i\right) p_2(s)ds,\end{aligned}$$

we get

$$\sum_{i=2}^{n} m_i \left(1 - \sum_{i=2}^{n} c_i \int_{t_1}^{t_i} h_i(s)p_2(s)ds\right) \le \sum_{i=2}^{n} c_i \int_{t_1}^{t_i} h_i(s)p_1(s)ds$$

whence

$$\sum_{i=2}^{n} m_i \le M$$

which, together with (1.2.313), gives us (1.2.310). □

Next we shall prove a similar result to Theorem 1 in [95], using the notion of the resolvent kernel from the theory of the Volterra linear integral equations.

It is well-known that the solution of the equation,

$$u(x) = f(x) + \int_0^x k(x,s)u(s)ds, \tag{1.2.314}$$

where $u = u(x)$ is the unknown function, $f = f(x)$ and $k = k(x,s)$ are given continuous functions for all $x \ge 0$ and $x \ge s \ge 0$, respectively, is given by for all $x \ge 0$,

$$u(x) = f(x) + \int_0^x r(x,s)f(s)ds, \tag{1.2.315}$$

where the resolvent kernel $r = r(x,s)$ satisfies the relation, for all $x \ge s \ge 0$,

$$r(x,s) = \sum_{n=0}^{+\infty} k_n(x,s). \tag{1.2.316}$$

We recall that for all $n \geq 1$,

$$k_0(x,s) = k(x,s), \quad k_n(x,s) = \int_s^x k_{n-1}(x,\tau)k(\tau,s)d\tau, \tag{1.2.317}$$

and thus if $k(x,s) \geq 0$, then $r(x,s) \geq 0$.

Theorem 1.2.44 (Corduneanu [152]) *Assume that a continuous function $u = u(x)$ satisfies that for all $x \geq 0$,*

$$u(x) \leq f(x) + \int_0^x k(x,s)u(s)ds, \tag{1.2.318}$$

where f and k are continuous, $k \geq 0$. Then, writing

$$u(x) = f(x) - f_1(x) + \int_0^x k(x,s)u(s)ds, \quad f_1(x) \geq 0, \tag{1.2.319}$$

we get, for all $x \geq 0$,

$$u(x) \leq f(x) + \int_0^x r(x,s)f(s)ds. \tag{1.2.320}$$

If $k(x,s) = a(s) \geq 0$, then for all $x \geq 0$,

$$u(x) \leq f(x) + \int_0^x a(s)u(s)ds. \tag{1.2.321}$$

Proof Estimate (1.2.320) follows from (1.2.314)–(1.2.315). When $k(x,s) = a(s) \geq 0$, (1.2.321) follows from (1.2.320), and thus by Theorem 1.1.2, we have for all $x \geq 0$,

$$u(x) \leq f(x) + \int_0^x a(s)\exp\left(\int_s^x a(\tau)d\tau\right)f(s)ds, \tag{1.2.322}$$

because in this case, there holds that for all $x \geq s \geq 0$

$$r(x,s) = a(s)\exp\left(\int_s^x a(\tau)d\tau\right). \tag{1.2.323}$$

The proof is thus complete. □

Remark 1.2.16 The method of the resolvent kernel described in the above makes possible the proof of the following Gronwall inequality: if $u = u(x), f = f(x)$ and $a = a(x)$ are L^2-functions on every finite interval of the real half-axis $x \geq 0$ and if

$a(x) \geq 0$, and the following inequality holds for a.e. $x \geq 0$,

$$u(x) \leq f(x) + \int_0^x a(s)u(s)ds, \tag{1.2.324}$$

then for a.e. $x \geq 0$,

$$u(x) \leq f(x) + \int_0^x a(s)\exp\left(\int_s^x a(\tau)d\tau\right)f(s)ds. \tag{1.2.325}$$

Now we consider pointwise estimates of solutions to the following Volterra integral equations:

$$y(x) = f(x) + \int_0^x k(x,s)y(s)ds, \quad x \in \mathbb{R}_+, \tag{1.2.326}$$

where $f(x)$, $k(x,s)$, and $k^*(x,s)$ are non-negative known function.

Equation (1.2.326) is a linear Volterra integral equation and has been studied in many details [625].

In the sequel, we assume that Eq. (1.2.326) possess solutions on $\mathbb{R}_+$.

To study many properties such as existence, uniqueness of solutions, and asymptotic behavior periodic solutions, we need to establish some corresponding integral inequalities to the linear Volterra integral equation (1.2.326). The integral inequalities of such type have been found to be useful in several ways (see, e.g., Vidyasagar and Deo [631]), and this type of inequalities has been profitably employed in the study of bounded-input-bounded-output (BIBO) stability properties of some feedback systems.

Next, we obtain a pointwise estimate for the solution of Eq. (1.2.326) under the condition that the kernel $k(x,s)$ is differentiable and the first partial derivative is directly separable.

Theorem 1.2.45 (Dhongade-Deo [182]) *Let the function $k(x,s)$ $(x \geq s)$ be defined and continuous on $\mathbb{R}_+ \times \mathbb{R}_+$. Suppose that*

$$\begin{cases} \dfrac{\partial k(x,s)}{\partial x} \leq \displaystyle\sum_{i=1}^n g_i(x)h_i(s), & (1.2.327) \\ k(x,x) \leq m(x), & (1.2.328) \end{cases}$$

where $g_i(x)$ and $h_i(x)$ are as in Theorem 1.2.10 and $m(x) : (0,+\infty) \to (0,+\infty)$ is a continuous function.

Let $f(x)$ be defined as in Theorem 1.2.9. If $y(x)$ is the solution of (1.2.326) on $\mathbb{R}_+$, then for all $x \in \mathbb{R}_+$,

$$|\, y(x) \,| \leq p(x) + \sum_{i=1}^n Q_i(x) \int_0^x h_i(s)|y(s)|ds, \tag{1.2.329}$$

and further,

$$| y(x) | \leq E^n p, \tag{1.2.330}$$

where E^k is defined as in (1.2.33) replacing g_k by Q_k and

$$\begin{cases} p(x) = f(x) + \int_0^x m(s)f(s)\exp\Big(\int_s^x m(t)dt\Big)ds, & (1.2.331) \\ Q_i(x) = 1 + \int_0^x g_i(s)\exp\Big(\int_s^x m(t)dt\Big)ds. & (1.2.332) \end{cases}$$

Proof Let $y(x)$ be the solution of (1.2.326) existing on $\mathbb{R}_+$. Then

$$| y(x) | \leq f(x) + \int_0^x k(x,s)|y(s)|ds. \tag{1.2.333}$$

Define

$$R(x) = \int_0^x k(x,s)|y(s)|ds.$$

Now

$$R'(x) = k(x,x)|y(x)| + \int_0^x \frac{\partial k(x,s)}{\partial x}|y(s)|ds.$$

In view of (1.2.328) and (1.2.333), we get

$$R'(x) \leq m(x)f(x) + m(x)R(x) + \int_0^x \frac{\partial k(x,s)}{\partial x}|y(s)|ds.$$

Transposing $m(x)R(x)$ to the left-hand side and multiplying by the integrating factor $\exp(-\int_0^x m(s)ds)$, we obtain

$$\begin{aligned} &\Big[R(x)\exp\Big(-\int_0^x m(s)ds\Big)\Big]' \\ &\leq m(x)f(x)\exp\left(-\int_0^x m(s)ds\right) + \exp\left(-\int_0^x m(s)ds\right)\int_0^x \frac{\partial k(x,s)}{\partial x}|y(s)|ds. \end{aligned}$$

Now substituting for $\partial k(x,s)/\partial x$ from (1.2.327) and integrating from 0 to x, we obtain the bound for $R(x)$. Using the bound for $R(x)$ in (1.2.333) and then substituting the value of $p(x)$ and $Q_i(x)$ from (1.2.331) and (1.2.332), we obtain that for all $x \in \mathbb{R}_+$,

$$| y(x) | \leq p(x) + \sum_{i=1}^n Q_i(x)\int_0^x h_i(s)|y(s)|ds.$$

The conclusion (1.2.330) is a direct consequence of Theorem 1.2.10. As an illustration, consider the following Volterra integral equation of the form (1.2.326)

$$y(x) = \frac{x^3 + 3x}{e^3} + \int_0^x (1 + x^2)e^{-s^3/3} y(s)ds, \quad 0 < x < +\infty.$$

Now from Theorem 1.2.10, we assume that $m(x) = (1 + x^2)$. Furthermore, since

$$\frac{\partial k(x, s)}{\partial x} = 2x\exp\Big(-\frac{s^3}{3}\Big),$$

we may suppose that

$$g_1(x) = 2x, \quad h_1(s) = \exp\Big(-\frac{s^3}{3}\Big).$$

Thus from (1.2.331) and (1.2.332), we derive

$$p(x) = \frac{x^3 + 3x}{e^3}\left[1 + x + \frac{x^3}{3}\right], \quad Q_i(x) = 1 + 2\exp\left(\frac{x^3}{3}\right)[e^x - x - 1].$$

Now following the estimate (1.2.330),

$$\begin{aligned} |y(x)| \le E^1\Big(p(x)\Big) &= 2\exp\Big(\frac{x^3 + 3x}{3}\Big)\Big[1 + x + \frac{x^3}{3}\Big] \\ &\times\Big[1 + 2\exp\Big(\frac{x^3}{3}\Big)(e^x - x - 1)\Big][e^x - x - x^2]. \end{aligned}$$

□

In 1967, Bykov [120] showed the next theorem.

Theorem 1.2.46 (Bykov [120]) *Let $u(t), b(t), k(t, s)$, and $h(t, s, \sigma)$ be non-negative continuous functions for $\alpha \le \sigma \le s \le t \le \beta$, and assume that for all $t \in [\alpha, \beta]$,*

$$\begin{aligned} u(t) \le a &+ \int_\alpha^t b(s)u(s)ds + \int_\alpha^t \left(\int_\alpha^s k(s, \tau)u(\tau)d\tau\right) ds \\ &+ \int_\alpha^t \left(\int_\alpha^s \left(\int_\alpha^\tau h(s, \tau, \sigma)u(\sigma)d\sigma\right) d\tau\right) ds \end{aligned} \tag{1.2.334}$$

where $a \ge 0$ is a constant. Then for all $t \in [\alpha, \beta]$,

$$\begin{aligned} u(t) \le \exp\Big\{&\int_\alpha^t b(s)u(s)ds + \int_\alpha^t \left(\int_\alpha^s k(s, \tau)d\tau\right) ds \\ &+ \int_\alpha^t \left(\int_\alpha^s \left(\int_\alpha^\tau h(s, \tau, \sigma)d\sigma\right) d\tau\right) ds\Big\}. \end{aligned} \tag{1.2.335}$$

Proof We denote the right-hand side of (1.2.334) by $v(t)$. Then $v(\alpha) = a$, and

$$v'(t) = b(t)u(t) + \int_\alpha^t k(t,\tau)u(\tau)d\tau + \int_\alpha^t \left(\int_\alpha^\tau h(t,\tau,\sigma)u(\sigma)d\sigma\right)d\tau$$
$$\leq \left[b(t) + \int_\alpha^t k(t,\tau)d\tau + \int_\alpha^t (h(t,\tau,\sigma)d\sigma)\, d\tau\right] v(t) \qquad (1.2.336)$$

since $u(t) \leq v(t)$ and $v(t)$ is non-decreasing in $[\alpha,\beta]$. Applying Lemma 1.1.1, we can derive (1.2.335). □

In 1965, Ved [629] proved the following theorem.

Theorem 1.2.47 (Ved [629]) *Let $u(t), b(t), \sigma(t)$, and $k(t,s)$ be non-negative continuous functions for $\alpha \leq s \leq t \leq \beta$, and assume that for all $t \in [\alpha,\beta]$,*

$$u(t) \leq a_1 + \sigma(t)\left\{a_2 + \int_\alpha^t b(s)u(s)ds + \int_\alpha^t \left(\int_\alpha^s k(s,\tau)u(\tau)d\tau\right)ds\right\} \qquad (1.2.337)$$

where $a_1, a_2 \geq 0$ are constants. Then

$$u(t) \leq a_2 \exp\left\{\int_\alpha^t B(s)\sigma(s)ds\right\} + \int_\alpha^t B(s)\exp\left(\int_s^t B(\tau)\sigma(\tau)\right)ds \qquad (1.2.338)$$

where $B(s) = b(s) + \int_\alpha^s k(s,\tau)d\tau$.

Proof In the same manner as that in Theorem 1.2.45, we can easily prove this theorem. □

In 1962, Bykov and Salpagarov [42] showed the following theorem.

Theorem 1.2.48 (Bykov-Salpagarov [42]) *Let non-negative function $u(t)$ defined on $[t_0,+\infty)$ satisfy the inequality*

$$u(t) \leq c + \int_{t_0}^t k(t,s)u(s)ds + \int_{t_0}^t \int_{t_0}^s G(t,s,\sigma)u(\sigma)d\sigma ds \qquad (1.2.339)$$

where $k(t,s)$ and $G(t,s,\sigma)$ are continuously differentiable non-negative functions for $t \geq s \geq \sigma \geq t_0$, and $c > 0$. Then

$$u(t) \leq c\exp\left\{\int_{t_0}^t \left[k(s,s) + \int_{t_0}^s \Big(k_s(s,\sigma) + G(s,s,\sigma)\Big)d\sigma + \int_{t_0}^s \int_{t_0}^\sigma G_s(s,\sigma,r)drd\sigma\right]ds\right\}. \qquad (1.2.340)$$

Proof The proof is similar to that of Theorem 1.2.45. □

Remark 1.2.17 We note that the estimate (1.2.340) can be replaced by

$$u(t) \le c \exp\left\{ \int_{t_0}^{t} k(t,s)ds + \int_{t_0}^{t} \left(\int_{t_0}^{s} G(t,s,\tau)d\tau \right) ds \right\}, \quad t_0 \le t < +\infty. \tag{1.2.341}$$

Moreover, (1.2.339) still holds if, in Theorem 1.2.48 the condition that the partial derivatives are non-negative is replaced by the condition that the functions $k(t,s)$ and $G(t,s,\sigma)$ are non-decreasing in t for fixed s, σ, while in (1.2.334) and (1.2.336) the constant a is replaced by a non-negative, non-decreasing function $a(t)$. In this case, the proof of the theorem can be given along the lines of that of Theorem 1.9 (Movlyankulov and Filatov [420]).

Let $\alpha < \beta$, and set $J_i = \{(t_1, \cdots, t_i) \in \mathbb{R}^i : \alpha \le t_i \le \cdots \le t_1 \le \beta\}, i = 1, \cdots, n$. In 1979, Ráb [541] showed the next result.

Theorem 1.2.49 (Ráb [541]) *Let $u(t), a(t)$, and $b(t)$ be non-negative continuous functions in $J = [\alpha, \beta]$, and suppose that for all $t \in J$,*

$$\begin{aligned} u(t) \le a(t) + b(t) \Big\{ &\int_{\alpha}^{t} k_1(t,t_1)u(t_1)dt_1 + \cdots + \\ &+ \int_{\alpha}^{t} \left(\int_{\alpha}^{t_1} \cdots \left(\int_{\alpha}^{t_{n-1}} k_n(t, t-1, \cdots, t_n)u(t_n)dt_n \right) \cdots \right) dt_1 \Big\} \end{aligned} \tag{1.2.342}$$

where $k_i(t, t_1, \cdots, t_i)$ are non-negative continuous functions in J_{i+1}, $i = 1, \cdots, n$. Suppose that the partial derivatives $\partial k_i(t, t_1, \cdots, t_i)/\partial t$ exist and are non-negative, continuous in J_{i+1}, $i = 1, \cdots, n$. Then for all $t \in J$,

$$u(t) \le a(t) + b(t) \int_{\alpha}^{t} (R[a] + Q[a])(s) \exp\left(\int_{s}^{t} (R[b] + Q[b])(\tau)d\tau \right) ds, \tag{1.2.343}$$

where, for all $t \in J$ and for each continuous function $w(t)$ in J,

$$\begin{cases} R[w](t) = k_1(t,t)w(t) + \displaystyle\int_{\alpha}^{t} k_2(t,t,t_2)w(t_2)dt_2 \\ \qquad + \displaystyle\sum_{i=3}^{n} \int_{\alpha}^{t} \left(\int_{\alpha}^{t_2} \cdots \left(\int_{\alpha}^{t_{i-1}} k_i(t,t,t_2,\cdots,t_i)w(t_i)dt_i \right) \cdots \right) dt_2, \\ Q[w](t) = \displaystyle\int_{\alpha}^{t} \frac{\partial k_1}{\partial t}(t,t_1)w(t_1)dt_1 \\ \qquad + \displaystyle\sum_{i=2}^{n} \int_{\alpha}^{t} \left(\int_{\alpha}^{t_1} \cdots \left(\int_{\alpha}^{t_{i-1}} \frac{\partial k_i}{\partial t}(t,t_1,\cdots,t_i)w(t_i)dt_i \right) \cdots \right) dt_1. \end{cases}$$

Proof First we note that $R[w]$ and $Q[w]$ are linear functions, and if $w_1(t) \le w_2(t)$, for all $t \in J$,

$$R[w_1] \le R[w_2], \quad Q[w_1] \le Q[w_2] \tag{1.2.344}$$

and if $w_1(t)$ is non-negative in J, and $w_2(t)$ is non-decreasing and continuous in J,

$$R[w_1 w_2] \le R[w_1]w_2, \ \ Q[w_1 w_2] \le Q[w_1]w_2, \tag{1.2.345}$$

we get for all $t \in J$,

$$\begin{aligned} v'(t) &= R[u](t) + Q[u](t) \le R[a + bv](t) + Q[a + bv](t) \\ &\le (R[a] + Q[a])(t) + (R[b] + Q[b])(t)v(t), \end{aligned}$$

which, together with Lemma 1.1.1, gives us (1.2.343). □

Theorem 1.2.50 (Ráb [541]) *Let $u(t), a(t)$, and $b(t)$ be non-negative continuous functions in $J = [\alpha, \beta]$, and suppose for all $t \in J$,*

$$\begin{aligned} u(t) \le a(t) + b(t)\Big[&\int_\alpha^t k_1(t, t_1)u(t_1)dt_1 + \cdots \\ &+ \int_\alpha^t \left(\int_\alpha^{t_1} \cdots \left(\int_\alpha^{t_{n-1}} k_n(t, t_1, \cdots, t_n)u(t_n)dt_n \right) \cdots \right) dt_1 \Big] \end{aligned} \tag{1.2.346}$$

where $k_i(t, t_1, \cdots, t_i)$ are non-negative continuous functions in J_{i+1}, $\ i = 1, \cdots, n$, which are non-decreasing in $t \in J$ for all fixed $(t_1, \cdots, t_i) \in J_i$, $\ i = 1, \cdots, n$. Then for all $t \in J$,

$$u(t) \le a(t) + b(t) \int_\alpha^t \hat{R}[a](t, s) \exp\left(\int_s^t \hat{R}[b](t, \tau)d\tau \right) ds, \tag{1.2.347}$$

where, for all $(t, s) \in J_2$ and for each continuous function $w(t)$ in J,

$$\begin{aligned} \hat{R}[w](t, s) = \ &k_1(t, s)w(s) + \int_\alpha^s k_2(t, s, t_2)w(t_2)dt_2 \\ &+ \sum_{i=3}^n \int_\alpha^s \left(\int_\alpha^{t_2} \cdots \left(\int_\alpha^{t_{i-1}} k_i(t, s, t_2, \cdots, t_i)w(t_i)dt_i \cdots \right) dt_2 \right). \end{aligned} \tag{1.2.348}$$

Proof For a fixed $T \in (\alpha, \beta]$ and $\alpha \le t \le T$, we have

$$\begin{aligned} u(t) &\le a(t) + b(t)w(t) \\ &\equiv a(t) + b(t)\Big[\int_\alpha^t k_1(T,t_1)u(t_1)dt_1 + \cdots \\ &\quad + \int_\alpha^t \left(\int_\alpha^{t_1} \cdots \left(\int_\alpha^{t_{n-1}} k_n(T,t_1,\cdots,t_n)u(t_n)dt_n\right)\cdots\right)dt_1\Big]. \end{aligned} \tag{1.2.349}$$

Since $\frac{\partial k_i}{\partial t}(T, t_1, \cdots, t_i) \equiv 0$ for $i = 1, \cdots, n$ and for all $t \in [\alpha, T]$, we have

$$w'(t) \le \hat{R}[a](T,t) + \hat{R}[b](T,t)w(t) \tag{1.2.350}$$

from which, by Lemma 1.1.1, it follows

$$u(t) \le a(t) + b(t)\int_\alpha^t \hat{R}[a](T,s)\exp\left(\int_s^t \hat{R}[b](T,\tau)d\tau\right)ds, \quad \alpha \le t \le T. \tag{1.2.351}$$

In particular, for $T = t$, we obtain (1.2.347). □

Corollary 1.2.26 *If, under the conditions of Theorem 1.2.50, the functions $a(t)$ and $b(t)$ are also non-decreasing in J, then for all $t \in J$,*

$$\begin{aligned} u(t) \le a(t)\exp\Big\{b(t)\Big[&\int_\alpha^t k_1(t,t_1)dt_1 + \cdots \\ &+ \int_\alpha^t \left(\int_\alpha^{t_1} \cdots \left(\int_\alpha^{t_{n-1}} k_n(t,t_1,\cdots,t_n)dt_n\right)\cdots\right)dt_1\Big]\Big\}. \end{aligned} \tag{1.2.352}$$

Proof Indeed, inequality (1.2.346) implies that

$$\begin{aligned} u(t) &\le a(t) + b(t)\int_\alpha^t \hat{R}[a](t,s)\exp\left\{\int_s^t \hat{R}[b](t,\tau)d\tau\right\}ds \\ &\le a(t) + b(t)\int_\alpha^t a(s)\hat{R}[1](t,s)\exp\left\{\int_s^t b(\tau)\hat{R}[1](t,\tau)d\tau\right\} \\ &\le a(t)\left[1 + \int_\alpha^t b(t)\hat{R}[1](t,s)\exp\left\{b(t)\hat{R}[1](t,\tau)d\tau\right\}ds\right] \\ &= a(t)\exp\left\{b(t)\int_\alpha^t \hat{R}[1](t,\tau)d\tau\right\}, \end{aligned}$$

which is just (1.2.352). □

Theorem 1.2.51 (Ráb [541]) *Let $u, f_1, \cdots, f_n$ be non-negative continuous functions in $J = [\alpha, \beta]$, and suppose that for all $t \in J$,*

$$u(t) \leq a + \int_\alpha^t f_1(t_1)u(t_1)dt_1 + \cdots + \int_\alpha^t f_1(t_1)\left(\int_\alpha^{t_1} f_2(t_2)\cdots\left(\int_\alpha^{t_{n-1}} f_n(t_n)u(t_n)dt_n\right)\cdots\right)dt_1 \quad (1.2.353)$$

where $a \geq 0$ is a constant. Then for all $t \in J$,

$$u(t) \leq aR_1(t), \quad (1.2.354)$$

where for all $t \in J$,

$$\begin{cases} R_n(t) = \exp\left(\int_\alpha^t f_n(s)ds\right), \\ R_i(t) = 1 + \int_\alpha^t f_i(t)R_{i+1}(s)\exp\left(\int_\alpha^s f_i(\tau)d\tau\right)ds, \quad s \in J, \quad i = n-1. \end{cases}$$

Proof Set, for all $t \in J, j = 1, \cdots, n-1$,

$$u_1(t) = a + L_1[u](t), \quad u_{j+1}(t) = u_j(t) + L_{j+1}[u](t),$$

where for all $t \in J, k = 1, \cdots, n$,

$$L_k[u](t) = \int_\alpha^t f_k(t_k)u(t_k)dt_k + \cdots + \int_\alpha^t f_k(t_k)\left(\int_\alpha^{t_k} f_{k+1}(t_{k+1})\cdots\left(\int_\alpha^{t_{n-1}} f_n(t_n)u(t_n)dt_n\right)\cdots\right)dt_k. \quad (1.2.355)$$

Now (1.2.353) implies

$$u(t) \leq u_1(t). \quad (1.2.356)$$

Noting that

$$u_k(t) \leq u_{k+1}(t), \quad (L_k[u])' = f_k(u + L_{k+1}[u]), \quad k = 1, \cdots, n-1, (L_n[u])' = f_n u, \quad (1.2.357)$$

we successively find

$$\begin{cases} u_1' \le f_1 u_2, & (1.2.358)\\ u_k' \le (f_1 + \cdots + f_{k-1})u_k + f_k u_{k+1}, \quad k = 2, \cdots, n-1, & (1.2.359)\\ u_n' \le (f_1 + \cdots + f_n)u_n. & (1.2.360) \end{cases}$$

Since $u_k(\alpha) = a, \quad k = 1, \cdots, n$, (1.2.358)–(1.2.360) imply, by successive application of Lemma 1.1.1,

$$u_k(t) \le aR_k(t) \exp\left(\int_\alpha^t \sum_{j=1}^{k-1} f_j(s)ds\right), \quad k = n, n-1, \cdots, 1. \tag{1.2.361}$$

For $k = 1$, this and (1.2.356) imply (1.2.354). □

Theorem 1.2.52 (Young [680]) *Let $u, a, f_i, \quad i = 1, \cdots, n$, be non-negative continuous functions in $J = [\alpha, \beta]$, and suppose that for all $t \in J$,*

$$u(t) \le a(t) + \int_\alpha^t f_1(t_1)u(t_1)dt_1 + \cdots + \int_\alpha^t f_1(t_1)\left(\int_\alpha^{t_1} f_2(t_2)\cdots\left(\int_\alpha^{t_{n-1}} f_n(t_n)u(t_n)dt_n\right)\cdots\right)dt_1. \tag{1.2.362}$$

Then

$$u(t) \le a(t) + \int_\alpha^t f_1(s)[a(s) + v_2(s)]ds, \tag{1.2.363}$$

where

$$\begin{cases} v_n(t) = \int_\alpha^t \Big(f_1(s) + \cdots + f_n(s)\Big)a(s)e^{\int_s^t (f_1(\tau)+\cdots+f_n(\tau))d\tau}ds,\\ v_k(t) = \int_\alpha^t \Big((f_1(s) + \cdots + f_k(s))a(s) + f_k(s)v_{k+1}(s)\Big)e^{\int_s^t (f_1(\tau)+\cdots+f_{k-1}(\tau))}ds. \end{cases}$$

Proof Let $L_k[u](t)$ be defined as in (1.2.355) of Theorem 1.2.51, and put

$$\begin{cases} v_1(t) = L_1[u](t), & (1.2.364)\\ v_{k+1}(t) = v_k(t) + L_k[u](t), \quad k = 1, \cdots, n-1. & (1.2.365) \end{cases}$$

Then $v_k(\alpha) = 0, k = 1, \cdots, n$, and

$$\begin{cases} v_1' \le f_1(a + v_2), & (1.2.366) \\ v_k' \le (f_1 + \cdots + f_{k-1})v_k + (f_1 + \cdots + f_k)a + f_k v_{k+1}, \quad k = 2, \cdots, n-1, & (1.2.367) \\ v_n' \le (f_1 + \cdots + f_n)v_n + (f_1 + \cdots + f_n)a. & (1.2.368) \end{cases}$$

Solving the system (1.2.366)–(1.2.368) "backward", and applying Lemma1.1.1, we arrive at (1.2.363). □

Remark 1.2.18 For $a(t) \equiv a = const.$, the estimates (1.2.356), and (1.2.367)–(1.2.368) coincide. This follows from the fact that if in (1.2.367)–(1.2.368) we set $v_k + a = u_k$, we obtain (1.2.358)–(1.2.360).

Corollary 1.2.27 (Pachpatte [452]) *If u, p, q, h are non-negative continuous functions in J, $u_0 \ge 0$ is a constant, and for all $t \in J$,*

$$u(t) \le u_0 + \int_\alpha^t \big[p(s)u(s) + h(s)\big]ds + \int_\alpha^t p(s)\bigg(\int_\alpha^s q(\tau)u(\tau)d\tau\bigg)ds, \qquad (1.2.369)$$

then for all $t \in J$, we have

$$u(t) \le u_0 + \int_\alpha^t h(s)ds + \int_\alpha^t p(s)\big[u_0 e^{\int_\alpha^s p(\tau)+q(\tau)d\tau} + \int_\alpha^s h(\tau)e^{\int_\alpha^s \big(p(r)+q(r)\big)dr}d\tau\big]ds. \qquad (1.2.370)$$

Proof Indeed, (1.2.370) follows readily from Theorem 1.2.52 with $f_1 = p, f_2 = q$, and $a(t) = u_0 + \int_\alpha^t h(s)ds$. □

The next theorem is a more general result.

Theorem 1.2.53 (Bykov-Salpagarov [42]) *Let $u(t)$ and $a(t)$ be continuous functions in $J = [\alpha, \beta]$, let $b_k(t, s_1, \cdots, s_k)$ be non-negative continuous functions for $\alpha \le s_k \le \cdots \le s_1 \le t \le \beta$, and suppose that for all $t \in J$,*

$$u(t) \le a(t) + \sum_{k=1}^n \int_\alpha^t \bigg(\int_\alpha^{s-1} \cdots \bigg(\int_\alpha^{s_{k-1}} b_k(t, s_1, \cdots, s_k)u(s_k)ds_k\bigg)\cdots\bigg)ds_1. \qquad (1.2.371)$$

Then for all $t \in J$,

1)

$$u(t) \le \mu(t), \qquad (1.2.372)$$

where $\mu(t)$ is a solution of the equation

$$u(t) = a(t) + \sum_{k=1}^{n} \int_{\alpha}^{t} \left(\int_{\alpha}^{s-1} \cdots \left(\int_{\alpha}^{s_{k-1}} b_k(t, s_1, \cdots, s_k) u(s_k) ds_k \right) \cdots \right) ds_1, \tag{1.2.373}$$

2) *the solution $u(t)$ of (1.2.371) is unique and can be expressed as the sum of the series*

$$u_0(t) + u_1(t) + \cdots + u_m(t) + \cdots$$

where $u_0(t) \equiv a(t)$,

$$u_m(t) = \sum_{k=1}^{n} \int_{\alpha}^{t} \left(\int_{\alpha}^{s-1} \cdots \left(\int_{\alpha}^{s_{k-1}} b_k(t, s_1, \cdots, s_k) u_{m-1}(s_k) ds_k \right) \cdots \right) ds_1. \tag{1.2.374}$$

Usually finding an exact solution of a linear system of differential equations (1.2.374) or of an integral equation (1.2.372) often proves very difficult. Therefore such solutions are estimated as, e.g., in the following two theorems.

Theorem 1.2.54 (Agarwal-Thandapani [18]) *Under the conditions of Theorem 1.2.53, (1.2.371) implies*

$$u(t) \le a(t) + b(t) Q_k(t), \quad k = 1, \cdots, n, \quad t \in J, \tag{1.2.375}$$

where

$$\begin{cases} Q_k(t) = \displaystyle\int_{\alpha}^{t} \left[a(s) \sum_{i=1}^{k} f_i(s) + g_k(s) Q_{k+1}(s) \right] \exp\left(\int_{s}^{t} [M_k(\tau) - g_k(\tau)] \right) ds, \quad k = 1, \cdots, n, \\ M_k(t) = \max \left[b(t) \displaystyle\sum_{i=1}^{k} f_i(s), g_1(t), \cdots, g_{k-1}(t) \right], \quad k = 2. \cdots, n, \\ M_1(t) = b(t) f_1(t), \ Q_{n+1}(t) \equiv 0, \ g_n(t) \equiv 0. \end{cases} \tag{1.2.376}$$

Proof We start from the relations below,

$$\begin{cases} u \le a + b z_1, \\ z_k(\alpha) = 0, \quad k = 1, \cdots, n, \\ z_k' = f_k u + g_k z_{k+1} \le f_k b z_1 + g_k z_{k+1} + f_k a, \quad k = 1, \cdots, n-1, \\ z_n' = f_n u \le f_n b z_n + f_n a. \end{cases}$$

which can be obtained as in Corollary 3.1.3. Set

$$s_k = \sum_{i=1}^{k} z_i, \quad k = 1, \cdots, n, \quad z_{n+1} \equiv 0.$$

Adding up the first k inequalities in (1.2.376) and using inequalities $b \sum_{i=1}^{k} f_i \leq M_k$ and $g_j \leq M_k$ for $1 \leq j \leq k-1$, we arrive at

$$s_k' \leq M_k s_k + a \sum_{i=1}^{k} f_i + g_k z_{k+1}, \quad k = 1, \cdots, n, \tag{1.2.377}$$

$$s_k(\alpha) = 0, \quad k = 1, \cdots, n. \tag{1.2.378}$$

The estimates (1.2.375) follows immediately by solving "backwards" (1.2.375), and using the inequality $z_{k+1} \leq Q_{k+1} - s_k$, for each $k = n-1, \cdots, 2, 1$. □

Corollary 1.2.28 (Pachpatte [443]) *Let u, p, q, h be non-negative continuous functions for t, and suppose that for all $t \geq \alpha$,*

$$u(t) \leq u_0 + \int_{\alpha}^{t} p(s)u(s)ds + \int_{\alpha}^{t} p(s)\Big[u(s) + \int_{\alpha}^{s} r(\tau)u(\tau)d\tau\Big]ds. \tag{1.2.379}$$

Then for all $t \geq \alpha$,

$$u(t) \leq u_0 e^{\int_{\alpha}^{s} p(s)ds}\Big[1 + \int_{\alpha}^{t} q(s) e^{\int_{\alpha}^{s}\left(p(\tau)+r(tau)\right)d\tau} ds\Big]. \tag{1.2.380}$$

Proof In fact, (1.2.380) follows immediately from Theorem 1.2.54 for $n = 2$, $a(t) \equiv u_0$, $b(t) \equiv 1$, $f_1(t) = p(t) + q(t)$, $f_2 = r(t)$, $q(t) = q(t)$. □

Theorem 1.2.55 (Agarwal-Thandapani [18]) *Let $u(t)$ and $a(t)$ be non-negative continuous functions in $J = [\alpha, \beta]$, with $a(t)$ non-decreasing in J, and let $f_i(t,s)$, $i = 1, \cdots, n$, be non-negative continuous functions for $\alpha \leq s \leq t \leq \beta$ which are non-decreasing in t for fixed $s \in J$. If for all $t \in J$,*

$$u(t) \leq a(t) + \int_{\alpha}^{t} f_1(t_1,t_2)\left(\int_{\alpha}^{t_1} f_2(t_1,t_2)\cdots\left(\int_{\alpha}^{t_{n-1}} f_n(t_{n-1},t_n)u(t_n)dt_n\right)\cdots\right)dt_1, \tag{1.2.381}$$

then for all $t \in J$,

$$u(t) \leq R_1(t,t), \tag{1.2.382}$$

where $R_1(T,t)$ can be successively determined from the formulas

$$\begin{cases} R_n(T,t) = a(T)\exp\{\int_\alpha^t \sum_{i=1^n} f_i(T,s)ds\}, \\ R_k(T,t) = E_k(T,t)\left[a(T) + \int_\alpha^t f_k(T,s)\frac{R_{k+1}(T,s)}{R_k(T,s)}ds\right], \\ E_k(T,t) = \exp\left(\int_\alpha^t [\sum_{i=1}^{k-1} f_i(T,t) - f_k(T,\tau)]d\tau\right), \end{cases}$$

for $k = n-1,\cdots,1, \alpha \le t \le T \le \beta$.

Proof Fix $T \in (\alpha,\beta]$. For $\alpha \le t \le T$, we obtain from (1.2.381),

$$u(t) \le a(T) + \int_\alpha^t f_1(T,t_1)\left(\int_\alpha^{t_1} f_2(T,t_2)\cdots\left(\int_\alpha^{t_{n-1}} f_n(T,t_n)u(t_n)dt_n\right)\cdots\right)dt_1. \tag{1.2.383}$$

Now we introduce the functions for all $t \in [\alpha,T]$ and $k = 2,\cdots,n$,

$$\begin{cases} m_1(t) = a(T) + \int_\alpha^t f_1(T,t_1)\left(\int_\alpha^{t_1} f_2(T,t_2)\cdots\left(\int_\alpha^{t_{n-1}} f_n(T,t_n)u(t_n)dt_n\right)\cdots\right)dt_1, \\ m_k(t) = m_{k-1}(t) + \int_\alpha^t f_k(T,t_k)\left(\int_\alpha^{t_k}\cdots\left(\int_\alpha^{t_{n-1}} f_n(T,t_n)m_{k-1}(t_n)dt_n\right)\cdots\right)dt_k. \end{cases}$$

Then (1.2.383) implies that $m_k(\alpha) = a(T)$, $\ k = 1,2,\cdots,n$, and for all $t \in [\alpha,T]$,

$$u(t) \le m_1(t) \le \cdots \le m_n(t).$$

Thus induction with respect to k implies that for all $t \in [\alpha,T]$, $\ k = 1,\cdots,n-1$,

$$\begin{cases} m_k'(t) \le \left[\sum_{i=1}^{k-1} f_i(T,t) - f_k(T,t)\right]m_k(t) + f_k(T,t)m_{k+1}(t), & (1.2.384) \\ m_n'(t) \le \sum_{i=1}^{n} f_i(T,t)m_n(t). & (1.2.385) \end{cases}$$

Lemma 1.1.1 and (1.2.385) imply that for all $\alpha \le t \le T$,

$$m_n(t) \le a(T)\exp\left(\int_\alpha^t \sum_{i=1}^n f_i(T,s)ds\right) = R_n(T,t). \tag{1.2.386}$$

Applying Lemma 1.1.1 again to (1.2.386) for $k = n-1, \cdots, 2, 1$, we can obtain, for all $\alpha \le t \le T \le \beta$,

$$u(t) \le m_1(t) \le R_1(T,t),$$

which readily implies (1.2.382) for $T = t$. □

Corollary 1.2.29 (Yang [657]) *Under the conditions of Theorem 1.2.55, (1.2.386) implies that for all $t \in [\alpha, \beta]$,*

$$u(t) \le a(t)Q_1(t,t), \tag{1.2.387}$$

where $Q_1(T,t)$ can be successively determined from the formulas

$$\begin{cases} Q_n(T,t) = \exp\left(\int_\alpha^t \sum_{i=1}^n f_i(T,s)ds\right), \\ Q_k(T,t) = E_k(T,t)\left[1 + \int_\alpha^t f_k(T,s)\dfrac{Q_{k+1}(T,s)}{E_k(T,s)}ds\right], \end{cases}$$

for $k = n-1, \cdots, 1$ and $\alpha \le t \le \beta$.

Pachpatte [443] extended the above result of Norbury and Stuart [432].

Theorem 1.2.56 (Pachpatte [443]) *Let u, q, r and f be non-negative continuous function defined on $J = [\alpha, \beta]$. Let $K(t,s)$ and its partial derivative $(\partial/\partial t)K(t,s)$ be non-negative continuous functions for all $\alpha \le t \le \beta$, and for all $t \in J$,*

$$u(t) \le p(t) + q(t)\int_\alpha^t K(t,s)\left(r(s)u(s) + f(s)\right)ds. \tag{1.2.388}$$

Then for all $t \in J$,

$$u(t) \le p(t) + q(t)\int_\alpha^t B(\sigma)\exp\left(\int_\sigma^t A(\tau)d\tau\right)d\sigma, \tag{1.2.389}$$

where

$$\begin{cases} A(t) = K(t,t)r(t)q(t) + \int_\alpha^t \dfrac{\partial}{\partial t}K(t,s)r(s)q(s)ds, & (1.2.390) \\ B(t) = K(t,t)\left(r(t)p(t) + f(t)\right) + \int_\alpha^t \dfrac{\partial}{\partial t}K(t,s)\left(r(s)p(s) + f(s)\right)ds. & (1.2.391) \end{cases}$$

Proof Define

$$z(t) = \int_\alpha^t K(t,s)\,(r(s)u(s) + f(s))\,ds. \tag{1.2.392}$$

Differentiating (1.2.392) and using the inequality $u(t) \le p(t) + q(t)z(t)$ and the fact that $z(t)$ is monotonic non-decreasing in t, (1.2.388) and (1.2.392), we arrive at

$$\begin{aligned}
z'(t) &= K(t,t)\,(r(t)u(t) + f(t)) + \int_\alpha^t \frac{\partial}{\partial t}K(t,s)\,(r(s)u(s) + f(s))\,ds \\
&\le K(t,t)\,(r(t)(p(t) + q(t)z(t)) + f(t)) \\
&\quad + \int_\alpha^t \frac{\partial}{\partial t}K(t,s)\,(r(s)(p(s) + q(s)z(s)) + f(s))\,ds \\
&\le z(t)\left(K(t,t)r(t)q(t) + \int_\alpha^t \frac{\partial}{\partial t}K(t,s)r(s)q(s)ds\right) \\
&\quad + K(t,t)\,(r(t)p(t) + f(t)) + \int_\alpha^t \frac{\partial}{\partial t}K(t,s)\,(r(s)p(s) + f(s))\,ds \\
&= A(t)z(t) + B(t),
\end{aligned}$$

which implies

$$z(t) \le \int_\alpha^t B(\sigma)\exp\left(\int_\sigma^t A(\tau)d\tau\right)d\sigma. \tag{1.2.393}$$

Using (1.2.393) in $u(t) \le p(t) + q(t)z(t)$, we get the desired estimate (1.2.389). □

Note that the special version of the above inequality with $r(t) = 1$ and $f(t) = 0$ in Theorem 1.2.56 was obtained by Movlyankulov and Filatov [420], which is also a generalization of Theorem 1.2.40.

Theorem 1.2.57 (Movlyankulov-Filatov [420]) *Let $u(t)$ be a continuous function in $J = [\alpha, \beta]$, let $b(t)$ be a non-negative continuous function in J, let $k(t,s)$ be a non-negative continuous function for $\alpha \le s \le t \le \beta$, and suppose that for all $t \in J$,*

$$u(t) \le a(t) + b(t)\int_\alpha^t k(t,s)u(s)ds. \tag{1.2.394}$$

Then for all $t \in J$,

$$u(t) \le A(t)\exp\left(B(t)\int_\alpha^t K(t,s)ds\right), \tag{1.2.395}$$

where

$$A(t) = \sup_{s\in[\alpha,t]} a(s), \quad B(t) = \sup_{s\in[\alpha,t]} b(s), \quad K(t,s) = \sup_{\tau\in[s,t]} k(\tau,s).$$

Proof The functions $A(t), B(t)$, and $K(t,s)$ are non-decreasing in $t \in J$. Thus (1.2.394) implies that for all $\alpha \le t \le \tau \le \beta$,

$$u(t) \le A(\tau) + B(\tau)\int_\alpha^t K(\tau,s)u(s)ds. \tag{1.2.396}$$

Therefore, Theorem 1.1.2 implies

$$u(t) \le A(\tau)\exp\left\{B(\tau)\int_\alpha^t K(\tau,s)ds\right\} \tag{1.2.397}$$

and taking $\tau = t$ in (1.2.397), we obtain (1.2.395). □

Remark 1.2.19 Inequality (1.2.394) becomes (1.2.295) when $b(t) \equiv 1$ in Theorem 1.2.40.

In 1970, Daletskii and Krein [159] showed the following result.

Theorem 1.2.58 (Daletskii-Krein [159]) *Let J be a finite or an infinite interval in $\mathbb{R}$. Let $k(t,s)$ be a non-negative function in J^2 such that the integral operator*

$$K[u](t) = \int_J k(t,s)u(s)ds$$

leaves invariant the space $C(J)$ of bounded continuous functions in J and has, in this space, spectral radius less than one. Suppose the function $u \in C(J)$ satisfies the inequality

$$u(t) \le a(t) + \int_J k(t,s)u(s)ds, \tag{1.2.398}$$

where $a \in C(J)$. Then for all $t \in J$,

$$u(t) \le v(t), \tag{1.2.399}$$

where $v \in C(J)$ is the unique solution of the integral equation, for all $t \in J$,

$$v(t) = a(t) + \int_J k(t,s)v(s)ds.$$

Proof This theorem follows from the theorem on integral inequalities in a space with a cone. □

Corollary 1.2.30 *Suppose that for all $t \geq \alpha$,*

$$u(t) \leq c + \int_{\alpha}^{t} h(\tau)u(\tau)d\tau, \tag{1.2.400}$$

where $h(t)$ is a continuous non-negative function, and $c \geq 0$ is a constant. Then for all $t \geq \alpha$,

$$u(t) \leq ce^{\int_{\alpha}^{t} h(\tau)d\tau}. \tag{1.2.401}$$

Corollary 1.2.31 *Suppose that for all $t \geq \alpha$,*

$$u(t) \leq \alpha e^{-\nu(t-\alpha)} + \beta \int_{\alpha}^{t} e^{-\nu(t-\tau)} p(\tau)u(\tau)d\tau, \tag{1.2.402}$$

where $p(t)$ is a continuous non-negative function and $\alpha \geq 0$, $\beta \geq 0$, $\nu \geq 0$ are constants. The for all $t \geq \alpha$,

$$u(t) \leq \alpha e^{-\nu(t-\alpha)+\beta \int_{\alpha}^{t} p(\tau)d\tau}. \tag{1.2.403}$$

If we reverse the inequality signs in (1.2.400) and (1.2.402), we obtain estimates (1.2.401) and (1.2.403) with their inequality signs revered.

Remark 1.2.20 Suppose that $k(t, \tau)$ is a continuous non-negative kernel on J and

$$\sup_{t \geq \alpha} \int_{J} k(t, \tau) = q < 1. \tag{1.2.404}$$

Then in $C(\mathbb{R}, J)$ the spectral radius $r(K) \leq \|K\| = q < 1$, and hence Theorem 1.2.58 is applicable.

Corollary 1.2.32 *Let $u(t)$ be a bounded continuous function in $J = [\alpha, +\infty)$, and suppose that for all $t \in J$,*

$$u(t) \leq ae^{-\gamma(t-\alpha)} + \int_{\alpha}^{+\infty} be^{-\gamma|t-s|}u(s)ds, \tag{1.2.405}$$

where $a \geq 0, b \geq 0$, and $\gamma > 0$ are constants and $b < \gamma/2$. Then for all $t \in J$,

$$u(t) \leq \frac{a}{b}(\gamma - \delta)e^{-\delta(t-\alpha)}, \tag{1.2.406}$$

where $\delta = \sqrt{\gamma^2 - 2b\gamma}$.

Proof Let $C(J)$ be the Banach space of functions v which are bounded and continuous in $J = [\alpha, +\infty)$ with norm $\|v\| = \sup_{t \in J} |v(t)|$.

Consider the linear operator $K: \; C(J) \to C(J)$ defined by

$$K[v](t) = \int_{\alpha}^{+\infty} be^{-\gamma|t-s|}v(s)ds, \quad t \in J.$$

If $v \in C(J)$ and $\|v\| = L$, it is easy to see that

$$|K[v](t)| \leq \int_{\alpha}^{t} bLe^{-\gamma(t-s)}ds + \int_{t}^{+\infty} bLe^{\gamma(t-s)}ds \leq \frac{2b}{\gamma}L = \frac{2b}{\gamma}\|v\|$$

whence we can conclude that $K[v] \in C(J)$ and that K is a contraction with $q = 2b/\gamma < 1$. Thus $u(t) \leq v(t)$ where $v(t) \in C(J)$ is the unique solution of the integral equation

$$v(t) = ae^{-\gamma(t-\alpha)} + K[v](t), \quad t \in J.$$

By a straightforward calculation, we can verify that $v(t)$ is equal to the right-hand side of (1.2.406). $\square$

Corollary 1.2.33 *Suppose that for* $0 < \beta < \nu/2$*, for all* $t \geq \alpha$*,*

$$u(t) \leq \alpha e^{-\nu(t-\alpha)} + \beta \int_{\alpha}^{+\infty} e^{-\nu|t-\tau|}u(\tau)d\tau. \tag{1.2.407}$$

Then for all $t \geq \alpha$*,*

$$u(t) \leq \frac{2\alpha\nu}{\nu + \sqrt{\nu^2 - 2\beta\nu}} e^{-\sqrt{\nu^2-2\beta\nu}(t-\alpha)}, \tag{1.2.408}$$

where α*,* β *and* ν *are positive constants.*

Proof Here $k(t,\tau) = \beta e^{-\nu|t-\tau|}$ and

$$\int_{\alpha}^{+\infty} k(t,\tau)d\tau = \beta\{\int_{\alpha}^{t} e^{-\nu(t-\tau)}d\tau + \int_{t}^{+\infty} e^{-\nu(\tau-t)}d\tau\} \leq \frac{2\beta}{\nu}.$$

Therefore when $\beta < \nu/2$, condition (1.2.404) is satisfied and Theorem 1.2.58 is applicable.

We consider the equation

$$\psi(t) = \alpha e^{-\nu(t-\alpha)} + \beta \int_{\alpha}^{+\infty} e^{-\nu|t-\tau|}\psi(\tau)d\tau. \tag{1.2.409}$$

Differentiating it twice with respect to t, we obtain

$$\begin{cases} \psi'(t) = -\alpha\nu e^{-\nu(t-\alpha)} - \beta\nu \int_\alpha^t e^{-\nu(t-\tau)}\psi(\tau)d\tau + \beta\nu \int_t^{+\infty} e^{-\nu(\tau-t)}\psi(\tau)d\tau \\ \psi''(t) = \alpha\nu^2 e^{-\nu(t-\alpha)} + \beta\nu^2 \int_\alpha^{+\infty} e^{-\nu|t-\tau|}\psi(\tau)d\tau - 2\beta\nu\psi(t) \end{cases}$$

which, together with (1.2.409), give us the differential equation

$$\psi''(t) - (\nu^2 - 2\beta\nu)\psi(t) = 0.$$

Since we are interested in the solution of (1.2.409) that is bounded on $[\alpha, +\infty)$, we have

$$\psi(t) = ce^{-\sqrt{\nu^2-2\beta\nu}(t-\alpha)} \quad \left(\nu^2 - 2\beta\nu \geq 0 \ \textit{for} \ \beta \leq \frac{\nu}{2}\right).$$

The constant c can be found by substituting this expression in (1.2.409). Carrying out the calculations, we obtain the equality $c = 2\alpha\nu/(\nu + \sqrt{\nu^2 - 2\beta\nu})$ from which (1.2.408) follows. □

Now consider the more complicated inequality, for all $t \geq \alpha$,

$$u(t) \leq \alpha e^{-\nu(t-\alpha)} + \beta \int_\alpha^{+\infty} e^{-\nu|t-\tau|}p(\tau)u(\tau)d\tau, \tag{1.2.410}$$

with a constants $\nu > 0, \alpha > 0, \beta > 0$.

Theorem 1.2.59 (Daletskii-Krein [159]) *Suppose that for all $t \geq \alpha$, (1.2.410) holds, where $p(t)$ is a continuous non-negative function, and $\nu > 0$, $\beta > 0$ are constants.*

For any $\mu < \nu$, τ_0 and $q < 1$, there exists a constant $\delta > 0$ such that if the condition

$$M_{\tau_0} = \sup_{t\geq\alpha} \frac{1}{\tau_0} \int_t^{t+\tau_0} p(\tau)d\tau < \delta, \tag{1.2.411}$$

is satisfied, then for all $t \geq \alpha$,

$$u(t) \leq \Big(\alpha/(1-q)\Big)e^{-\mu(t-\alpha)}. \tag{1.2.412}$$

Proof It immediately reduces to previous inequality if $p(t)$ is a bounded function on $[\alpha, +\infty)$. We shall need to estimate $u(t)$, however, under the more general

assumption that

$$\sup_{t\geq\alpha}\frac{1}{\tau_0}\int_t^{t+\tau_0}p(\tau)d\tau = M_{\tau_0} < +\infty, \tag{1.2.413}$$

for some fixed τ_0.

In this case, Eq. (1.2.409) has the form

$$\psi(t) = \alpha e^{-\nu(t-\alpha)} + \beta\int_\alpha^{+\infty} e^{-\nu|t-\tau|}p(\tau)\psi(\tau)d\tau. \tag{1.2.414}$$

It can be shown in the same way as above that the solution of this equation reduces to finding a bounded solution of the differential equation $\psi''(t) - (\nu^2 - 2\nu\beta p(t))\psi(t) = 0$.

In order to avoid the task of estimating the solution of the equation, we proceed differently. We put $\psi(t) = u(t)e^{-\mu(t-\alpha)}$, where the number $\mu > 0$ will be closen later. The function $\phi(t)$ must satisfy the equation

$$\phi(t) = \alpha e^{-(\nu-\mu)(t-\alpha)} + \beta\int_\alpha^{+\infty} e^{-\nu|t-\tau|+\mu(t-\tau)}p(\tau)\phi(\tau)d\tau. \tag{1.2.415}$$

which we consider in the space $C(\mathbb{R}^1, [\alpha, +\infty))$ of continuous bounded functions on $[\alpha, +\infty)$.

We consider in this space the operator

$$(A\phi)(t) = \beta\int_\alpha^{+\infty} e^{-\nu|t-\tau|+\mu(t-\tau)}p(\tau)\phi(\tau)d\tau.$$

It is easy to check that

$$\|A\| = \beta\sup_{t\geq\alpha}\int_\alpha^{+\infty} e^{-\nu|t-\tau|+\mu(t-\tau)}p(\tau)d\tau. \tag{1.2.416}$$

We estimate the integral in this inequality. Let $n = [t/\tau_0]$. Then

$$\begin{aligned}\int_\alpha^{+\infty} e^{-\nu|t-\tau|+\mu(t-\tau)}p(\tau)d\tau &= \sum_{k=1}^{+\infty}\int_{\alpha+(k-1)\tau_0}^{\alpha+k\tau_0} e^{-\nu\|t-\tau\|+\mu(t-\tau)}p(\tau)d\tau \\ &\leq \sum_{k=1}^{+\infty}\max_{[\alpha+(k-1)\tau_0,\alpha+k\tau_0]} e^{-\nu\|t-\tau\|+\mu(t-\tau)}\int_{\alpha+(k-1)\tau_0}^{\alpha+k\tau_0}p(\tau)d\tau \\ &\leq \tau_0 M_{\tau_0}\left\{\sum_{k=1}^{n} e^{-(\nu-\mu)(n-k)\tau_0} + 1 + \sum_{k=n+2}^{+\infty} e^{-(\nu+\mu)(k-n-2)\tau_0}\right\} \\ &\leq M_{\tau_0}C(\mu,\nu,\tau_0),\end{aligned}$$

where, for $\mu < \nu$,

$$C(\mu, \nu, \tau_0) = \tau_0 + \frac{\tau_0}{1 - e^{-(\nu-\mu)\tau_0}} + \frac{\tau_0}{1 - e^{-(\nu+\mu)\tau_0}} < +\infty.$$

Thus

$$\|A\| \leq \beta M_{\tau_0} C(\mu, \nu, \tau_0). \tag{1.2.417}$$

We require that

$$\beta M_{\tau_0} C(\mu, \nu, \tau_0) \leq q < 1. \tag{1.2.418}$$

Under this requirement, the equation

$$\phi - A\phi = \alpha e^{-(\nu-\mu)(t-\alpha)} \tag{1.2.419}$$

will be solvable in $C(\mathbb{R}^1, [\alpha, +\infty))$, i.e., the function $\phi(t)$ is bounded:

$$\sup_{t \geq \alpha} \phi(t) = c < +\infty, \tag{1.2.420}$$

which gives us the following estimate for $\psi(t)$:

$$\psi(t) \leq c e^{\mu-(t-\alpha)}. \tag{1.2.421}$$

We shall assume that τ_0 is sufficiently small. Then

$$C(\mu, \nu, \tau_0) = \frac{1}{\nu - \mu} + \frac{1}{\nu + \mu} + O(\tau_0) = \frac{2\nu}{\nu_2 - \mu_2} + O(\tau_0),$$

and condition (1.2.418) reduces to the relation

$$\beta M_{\tau_0} \left[\frac{2\nu}{\nu_2 - \mu_2} + O(\tau_0) \right] < 1$$

or

$$\mu < \sqrt{\nu_2 - \frac{2\nu\beta M_{\tau_0}}{1 - O(\tau_0 M_{\tau_0})}}. \tag{1.2.422}$$

Thus an estimate of form (1.2.421) holds under condition (1.2.422), and it only remains to find the constant c. This can be done by considering (1.2.415),

which implies

$$c \le \alpha + \beta c \sup_{t\ge\alpha} \int_{\alpha}^{+\infty} e^{-\nu|t-\tau|+\mu(t-\tau)} p(\tau)d\tau \le \alpha + \beta c M_{\tau_0} C(\mu,\nu,\tau_0)$$

and, finally,

$$c \le \frac{\alpha}{1-\beta M_{\tau_0} C(\mu,\nu,\tau_0)} \le \frac{\alpha}{1-q}.$$

□

Remark 1.2.21 If τ_0 is sufficiently small, relation (1.2.422) indicates the values of μ for which the desired estimate is valid for a given M_{τ_0}.

If the condition $\sup p(t) \le M$ is satisfied, inequality (1.2.410) with β replaced by βM will hold and Corollary 1.2.33 can be applied.

In the same manner, we can also show the next result (see, Coppel [150]).

Corollary 1.2.34 (Coppel [150]) *Let $u(t)$ be a continuous function for all $\alpha \le t \le \beta$, and suppose that for all $\alpha \le t \le \beta$,*

$$u(t) \le ae^{-\gamma(\beta-t)} + \int_{\alpha}^{\beta} be^{-\gamma|t-s|}u(s)ds, \tag{1.2.423}$$

where $a \ge 0, b \ge 0$, and $\gamma > 0$ are constants and $b < \gamma/2$. Then for all $\alpha \le t \le \beta$,

$$u(t) \le \frac{a}{b}(\gamma-\delta)e^{-\delta(\beta-t)}, \tag{1.2.424}$$

where $\delta = \sqrt{\gamma^2 - 2b\gamma}$.

To show the next theorem, we need the following lemma.

Lemma 1.2.5 (Coppel [150]) *Let $g(t)$ be a non-negative locally integrable function for $\tau \ge \alpha$ such that for some fixed $t > 0$ and for all $t \ge \alpha$,*

$$\frac{1}{\tau}\int_{t}^{t+\tau} g(s)ds \le m < +\infty. \tag{1.2.425}$$

Then, for any $\gamma > 0$ and for all $t \ge \alpha$,

$$\int_{\alpha}^{t} e^{-\gamma(t-s)}g(s)ds \le m\tau(1-e^{-\gamma\tau})^{-1}, \tag{1.2.426}$$

$$\int_{t}^{+\infty} e^{-\gamma(s-t)}g(s)ds \le m\tau(1-e^{-\gamma\tau})^{-1}. \tag{1.2.427}$$

Proof The proof is left to the reader as an exercise. □

Theorem 1.2.60 (Coppel [150]) *Let $u(t)$ be a bounded continuous function in $J = [\alpha, +\infty)$, let $g(t)$ be a non-negative continuous function in J, and suppose that for all $t \in J$,*

$$u(t) \le ae^{-\gamma(t-\alpha)} + \int_\alpha^{+\infty} be^{-\gamma|t-s|}g(s)u(s)ds, \tag{1.2.428}$$

where $a \ge 0, b \ge 0$, and $\gamma > 0$ are constants. If there are numbers $\tau > 0$ and $m = \sup_{t\in J}\int_t^{t+\tau} g(s)ds < +\infty$ such that

$$q_0 = \frac{2bm}{1-e^{-\gamma\tau}} < 1,$$

then there is a constant $\mu \in (0, \gamma)$ such that

$$q(\mu) \equiv bm\mu\left[\frac{1}{1-e^{-(\gamma-\mu)\tau}} + \frac{1}{1-e^{-(\gamma+\mu)\tau}}\right] < 1 \tag{1.2.429}$$

and for all $t \in J$,

$$u(t) \le \frac{a}{1-q(\mu)}e^{-\mu(t-\alpha)}, \quad . \tag{1.2.430}$$

Proof Since $q(\mu)$ is a continuous function for $\mu \in (0, \gamma)$ and $\lim_{\mu\downarrow 0} q(\mu) = q_0 < 1$, there is a $\mu \in (0, \gamma)$ satisfying (1.2.429).

Consider the operator K defined on the space $C(J)$ of bounded continuous functions in J, for all $t \in J$,

$$K[u](t) = \int_\alpha^{+\infty} be^{-\gamma|t-s|}g(s)u(s)ds.$$

For $u \in C(J)$, $\|u\| = \sup_{t\in J}|u(t)| = L$, we find, by Lemma 1.2.5,

$$|K[u](t)| \le \int_\alpha^t bLe^{-\gamma(t-s)}g(s)ds + \int_t^{+\infty} bLe^{-\gamma(s-t)}g(s)ds \le \frac{2bm\tau}{1-e^{-\gamma\tau}}\|u\|,$$

i.e., $\|K[u] \le q_0\|u\|$. Consequently, K is a contraction and the comparison integral equation

$$v(t) = ae^{-\gamma(t-\alpha)} + K[v](t)$$

has a unique solution. Moreover,

$$u(t) \le v(t) = r(t) + K[r](t) + \cdots + K^n[r](t) + \cdots,$$

where $r(t) = ae^{-\gamma(t-\alpha)}$. But since $r(t) \leq ae^{-\mu(t-\alpha)} = R(t)$, we have

$$u(t) \leq R(t) + K[R](t) + \cdots + K^n[R](t) + \cdots . \tag{1.2.431}$$

Using Lemma 1.2.5, we conclude

$$\begin{aligned} K[R](t) &= \int_\alpha^t abe^{-\gamma(t-s)}e^{-\mu(s-\alpha)}g(s)ds + \int_t^{+\infty} abe^{-\gamma(s-t)}e^{-\mu(s-\alpha)}g(s)ds \\ &= abe^{-\mu(t-\alpha)}\left[\int_\alpha^t e^{-(\gamma-\mu)(t-s)}g(s)ds + \int_t^{+\infty} e^{-(\gamma+\mu)(s-t)}g(s)ds\right] \\ &\leq bm\tau\left[\frac{1}{1-e^{-(\gamma-\mu)\tau}} + \frac{1}{1-e^{-(\gamma+\mu)\tau}}\right]ae^{-\mu(t-\alpha)} = q(\mu)R(t), \end{aligned} \tag{1.2.432}$$

whence

$$u(t) \leq (1 + q(\mu) + \cdots + q^n(\mu) + \cdots)\, R(t) = \frac{a}{1-q(\mu)}e^{-\mu(t-\alpha)}.$$

□

Next, we consider integral equations of the form

$$y(t) = \phi(t) + \int_0^t \int_0^{t_m} \cdots \int_0^{t_1} \frac{k(t,s,y(s))}{(t_1-s)^\alpha}dsdt_1\cdots dt_m, \tag{1.2.433}$$

where $\alpha < 1$ and m is a natural number. The functions $\phi,\ k$ are assumed to satisfy suitable continuity conditions and $k(t,s,y)$ is assumed Lipschitz continuous with respect to y.

Integral equations of this form or, the corresponding integral inequalities, may be used to demonstrate uniqueness and boundedness of the solution of integro-differential equations with an Abel's type singularity,

$$y^{(m)}(t) = \psi(t) + \int_0^t \frac{k(t,s,y(s))}{(t-s)^\alpha}ds, \quad 0 \leq t \leq T, \tag{1.2.434}$$

where $\alpha < 1$ and $m \geq 1$. Here $y^{(m)}(t)$ denotes the m-th derivative of y with respect to t.

Lemma 1.2.6 *Let the function x be continuous and non-negative on the interval* $[0,T]$. *If*

$$x(t) \leq \phi(t) + M\int_0^t \int_0^{t_m} \cdots \int_0^{t_1} \frac{k(t,s,y(s))}{(t_1-s)^\alpha}dsdt_1\cdots dt_m, \quad 0 \leq t \leq T, \tag{1.2.435}$$

where $\alpha < 1$ $m \geq 1$, $M > 0$ *is a constant, and* $\phi(t)$ *is a non-negative, non-decreasing continuous function in* t, $0 \leq t \leq T$, *then*

$$x(t) \leq \phi(t)E_{1-(\alpha-m)}(M\Gamma(1-\alpha)t^{1-(\alpha-m)}),\ 0 \leq t \leq T, \tag{1.2.436}$$

where $E_{1-\beta}(z)$ *is the Mittag-Leffler function defined for any* β *by*

$$E_{1-\beta}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(n(1-\beta)+1)}$$

and $\Gamma(a)$ *is the Gamma function defined for* $Re a > 0$ *by*

$$\Gamma(a) = \int_0^{+\infty} w^{a-1}e^{-w}dw.$$

The exponential function, which is obtained when $\beta = 0$, *is a special case of the Mittag-Leffler function.*

The Mittag-Leffler function has been studied in some detail in the literature; for references, see Erdelyi [205] (1955, Chap. 18).

Proof For $m \geq 1$, by interchanging the order of integration, we get

$$\int_0^t \int_0^{t_m} \cdots \int_0^{t_1} \frac{x(s)}{(t_1-s)^\alpha} ds dt_1 \cdots dt_m = \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} \int_0^t (t-s)^{m-\alpha} x(s) ds.$$

Consequently, the inequality (1.2.435) is equivalent to

$$x(t) \leq \phi(t) + \int_0^t k(t,s)x(s)ds$$

where the kernel $k(t,s)$ given by

$$k(t,s) = \frac{\Gamma(1-\alpha)(t-s)^{m-\alpha}}{\Gamma(1-\alpha+m)}, \quad 0 \leq s \leq t \leq T, \tag{1.2.437}$$

with $\alpha < 1$ $m \geq 1$, is continuous and non-negative.

Using Theorem 1.2.37,

$$x(t) \leq y(t)$$

where $y(t)$ is the solution of (1.2.433) is given by

$$y(t) = \phi(t) + \int_0^t \Gamma(t,s)y(s)ds, \ \ 0 \leq t \leq T,$$

where

$$\Gamma(t,s) = \sum_{n=1}^{+\infty} k^{(n)}(t,s), \quad 0 \le t \le T,$$

is the resolvent kernel of $k(t,s)$ and $k^{(n)}(t,s)$ are the iterated kernels of $k(t,s)$ defined by

$$\begin{cases} k^{(1)}(t,s) = k(t,s), \\ k^{(n)}(t,s) = \int_s^t k(t,u)k^{(n-1)}(t,u)du, \quad n \ge 2. \end{cases}$$

Using mathematical induction, it can be shown that the iterated kernels satisfy

$$k^{(n)}(t,s) = \frac{M^n \Gamma(1-\alpha)^n (t-s)^{n(m+1-\alpha)-1}}{\Gamma(n(1-\alpha+m))}, \quad n = 1, 2, \cdots \tag{1.2.438}$$

Hence, we obtain for all $t \in [0, T]$,

$$x(t) \le \phi(t) + \sum_{n=1}^{+\infty} \frac{(M\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha+m))} \int_0^t (t-s)^{n(m+1-\alpha)-1}\phi(s)ds \tag{1.2.439}$$

$$\le \phi(t) E_{1-(\alpha-m)}(M\Gamma(1-\alpha)t^{1-(\alpha-m)}). \tag{1.2.440}$$

The proof is thus complete. □

Note that if $\alpha = 0$, then (1.2.436) reduces to

$$x(t) \le \phi(t) \cos h(M^{\frac{1}{2}} t).$$

Note that in the case $\phi(t) = \phi,\ 0 \le t \le T$, (1.2.436) is the best possible result since equality in (1.2.435) implies equality in (1.2.436). For a more general $\phi(t)$, the best possible result is given by

$$x(t) \le \frac{d}{dt} \int_0^t E_{1-(\alpha-m)}(M\Gamma(1-\alpha)(t-s)^{1-(\alpha-m)})\phi(s)ds, \tag{1.2.441}$$

where the right-hand side of (1.2.436) is the solution of the integral equation (1.2.433) with kernel (1.2.437).

We also note that if $\alpha \le 0$, Lemma 1.2.6 remains valid if $m = 0$, that is, if (1.2.435) involves a single, rather that repeated, integral and in this case Lemma 1.2.6 is an example of Theorem 1.2.38. If $0 < \alpha < 1$ and $m = 0$, the kernel $k(t,s) = M/(t-s)^\alpha$ is weakly singular; Gronwall inequalities where the

kernel of the associated integral equation is weakly singular can be found in Dixon and McKee [187].

In what follows, $\mathbb{R}_+ = [0, +\infty)$, $I = [\alpha, \beta]$ are the given subsets of $\mathbb{R}$ and $\mathbb{Z}$ is the set of integers. For $\alpha,\ \beta \in \mathbb{Z},\ \alpha \leq \beta$. We denote by

$$\Delta = \{(t, s) \in I^2 : \alpha \leq s \leq t \leq \beta\},$$

and $C(A, B)$ denotes the class of continuous functions from A to B. We use the usual conventions that the empty sums and products are taken to be 0 and 1 respectively. We shall also assume that all the integrals, sums and products involved throughout the discussion exist in the respective domains of their definitions.

Theorem 1.2.61 (Pachpatte [500]) *Let* $u(t) \in C(I, \mathbb{R}_+)$, $a(t, s)$, $b(t, s) \in C(\Delta, \mathbb{R}_+)$ *and* $a(t, s)$, $b(t, s)$ *be non-decreasing in* t, *for each* $s \in I$ *and suppose that for all* $t \in I$,

$$u(t) \leq c + \int_\alpha^t a(t, s)u(s)ds + \int_\alpha^\beta b(t, s)u(s)ds, \tag{1.2.442}$$

where $c \geq 0$ *is a constant. If*

$$p(t) = \int_\alpha^\beta b(t, s) \exp\left(\int_\alpha^s a(s, \sigma)d\sigma\right) ds < 1, \tag{1.2.443}$$

then for all $t \in I$,

$$u(t) \leq \frac{c}{1-\alpha} \exp\left(\int_\alpha^t a(t, s)ds\right). \tag{1.2.444}$$

Proof Fix any $T,\ \alpha \leq T \leq \beta$, then for $\alpha \leq t \leq T$, we have

$$u(t) \leq c + \int_\alpha^t a(T, s)u(s) + \int_\alpha^\beta b(T, s)u(s)ds. \tag{1.2.445}$$

Define a function $z(t),\ \alpha \leq t \leq T$ by the right-hand side of (1.2.442). Then $u(t) \leq z(t),\ \alpha \leq t \leq T$,

$$z(\alpha) = c + \int_\alpha^\beta b(T, s)u(s)ds, \tag{1.2.446}$$

and

$$z'(t) = a(T, t)u(t) \leq a(T, t)z(t) \tag{1.2.447}$$

for all $\alpha \le t \le T$. By setting $t = \sigma$ in (1.2.447) and integrating it with respect to σ from α to T, we get

$$z(T) \le z(\alpha) \exp\left(\int_\alpha^T a(T,\sigma)d\sigma\right). \tag{1.2.448}$$

Since T is arbitrary, from (1.2.447) and (1.2.448) with T replaced by t and $u(t) \le z(t)$, it follows

$$u(t) \le z(\alpha) \exp\left(\int_\alpha^T a(T,\sigma)d\sigma\right), \tag{1.2.449}$$

where

$$z(\alpha) = c + \int_\alpha^\beta b(t,s)u(s)ds. \tag{1.2.450}$$

Using (1.2.449) on the right-hand side of (1.2.450) and (1.2.443), we have

$$z(\alpha) \le \frac{c}{1-\alpha}. \tag{1.2.451}$$

Using (1.2.451) in (1.2.449), we get the desired inequality in (1.2.444). The proof is complete. □

Note that in the special case when $a(t,s) = b(s)$, $b(t,s) = c(s)$, the inequality given in Theorem 1.2.61 reduces to the inequality in Corollary 1.2.2, in case $u(t)$ and a therein are non-negative.

Corollary 1.2.35 (Bainov-Simeonov [42]) *Let $u(t), b(t)$, and $c(t)$ be continuous functions in $J = [\alpha, \beta]$, let $b(t)$ and $c(t)$ be non-negative in J, and suppose that for all $t \in J$,*

$$u(t) \le a + \int_\alpha^t b(s)u(s)ds + \int_\alpha^\beta c(s)u(s)ds, \tag{1.2.452}$$

where $a \ge 0$ is a constant. If

$$q = \int_\alpha^\beta c(s) \exp\left(\int_\alpha^s b(\tau)d\tau\right) ds < 1,$$

then for all $t \in J$,

$$u(t) \le \frac{a}{1-q} \exp\left(\int_\alpha^t b(s)ds\right). \tag{1.2.453}$$

1.3 Linear One-Dimensional Systems of Integral Inequalities of the Gronwall-Bellman Type

Greene [235] showed the following interesting inequality, called simultaneous inequalities, which can be used in analysis of various problems in the theory of some systems of simultaneous differential and integrals equations.

Theorem 1.3.1 (Greene [235]) *Let K_1, K_2 and μ be non-negative constants, and let f, g, h_j $(1 \leq j \leq 4)$ be continuous non-negative functions for all $t \geq 0$ with h_i such that*

$$\begin{cases} f(t) \leq K_1 + \int_0^t h_1(s)f(s)ds + \int_0^t h_2(s)g(s)\exp(\mu s)ds, & (1.3.1) \\ g(t) \leq K_2 + \int_0^t h_3(s)f(s)\exp(-\mu s)ds + \int_0^t h_4(s)g(s)ds. & (1.3.2) \end{cases}$$

Then there exist positive constants c_k, M_k $(k = 1, 2)$ such that for all $t \geq 0$,

$$f(t) \leq M_1 \exp(c_1 t), \quad g(t) \leq M_2 \exp(c_2 t). \tag{1.3.3}$$

Proof This proof is due to Greene [235]. Suppose $\mu > 0$. Let P be an upper bound for h_i, then

$$\begin{cases} f(t) \leq K_1 + P\int_0^t f(s)ds + P\int_0^t e^{\mu s}g(s)ds, & (1.3.4) \\ g(t) \leq K_2 + P\int_0^t f(s)e^{-\mu s}ds + P\int_0^t g(s)ds. & (1.3.5) \end{cases}$$

Define

$$f_T \equiv \max_{t\in[0,T]} f(t), \qquad g_T \equiv \max_{t\in[0,T]} g(t).$$

Since f and g are continuous, f_T and g_T are attained on $[0, T]$. Therefore,

$$\begin{cases} (1 - PT)f_T \leq K_1 + (P/\mu)(e^{\mu T} - 1)g_T, & (1.3.6) \\ (1 - PT)g_T \leq K_2 + (P/\mu)(1 - e^{-\mu T})f_T. & (1.3.7) \end{cases}$$

Using (1.3.4) in (1.3.2), we have for $T < 1/P$,

$$\begin{cases} \left(1 - PT - \dfrac{P^2}{\mu^2} \dfrac{(e^{\mu T} - 1)(1 - e^{-\mu T})}{1 - PT}\right) f_T \leq K_1 + \dfrac{P}{\mu}\left(\dfrac{e^{\mu T} - 1}{1 - PT}\right) K_2, & (1.3.8) \\ \left(1 - PT - \dfrac{P^2}{\mu^2} \dfrac{(e^{\mu T} - 1)(1 - e^{-\mu T})}{1 - PT}\right)]g_T \leq K_2 + \dfrac{P}{\mu}\left(\dfrac{1 - e^{-\mu T}}{1 - PT}\right) K_1. & (1.3.9) \end{cases}$$

We examine the function contained in (1.3.8) and (1.3.9),

$$H(T) \equiv P + \frac{P^2}{\mu^2} \frac{(e^{\mu T} - 1)(1 - e^{-\mu T})}{T(1 - PT)}.$$

$H(T)$ is continuous for $0 < T < 1/P, \lim_{T \to 0^+} H(T) = P$ and $H(T) \geq P$. Therefore, on the interval $0 < T \leq \delta < 1/P$, $H(T)$ has a minimum P and a maximum α. Then for $T < T_1 = \min(\delta, 1/\alpha)$,

$$\frac{1}{1 - PT} \leq \frac{1}{1 - H(T)T} \leq \frac{1}{1 - \alpha T}. \tag{1.3.10}$$

Furthermore, there exists a T_2 such that when $0 < T < T_2$,

$$\frac{1}{1 - H(T)T} \frac{P}{\mu} \left(\frac{e^{\mu T} - 1}{1 - PT}\right) \leq 1 \tag{1.3.11}$$

and

$$\frac{1}{1 - H(T)T} \frac{P}{\mu} \left(\frac{1 - e^{-\mu T}}{1 - PT}\right) \leq 1. \tag{1.3.12}$$

Let $T < \min(T_1, T_2)$ be fixed. Since $0 < 1 - \alpha/T < 1$, there exist constants β_i and $\gamma > 0$ such that

$$1 + 1/(1 - \alpha T) \leq 1/(1 - \alpha T)^{\beta_1}, \tag{1.3.13}$$

$$P/\mu + 1/(1 - \alpha T) \leq 1/(1 - \alpha T)^{\beta_2}, \tag{1.3.14}$$

$$1 + P/\mu(1 - \alpha T) \leq 1/(1 - \alpha T)^{\beta_3}, \tag{1.3.15}$$

$$1 + 2/(1 - \alpha T)^{\beta} \leq 1/(1 - \alpha T)^{\gamma} \tag{1.3.16}$$

where $\beta \equiv \max(\beta_1, \beta_2, \beta_3)$. Thus using inequalities (1.3.13)–(1.3.16) in (1.3.6) and (1.3.7) implies

$$\begin{cases} f_T \le \dfrac{K_1}{1-H(T)T} + \dfrac{K_2}{1-H(T)T}\dfrac{P}{\mu}\left(\dfrac{e^{\mu T}-1}{1-PT}\right) \le \dfrac{K}{(1-\alpha T)^{\beta}}, & (1.3.17) \\ g_T \le \dfrac{K_2}{1-H(T)T} + \dfrac{K_1}{1-H(T)T}\dfrac{P}{\mu}\left(\dfrac{1-e^{-\mu T}}{1-PT}\right) \le \dfrac{K}{(1-\alpha T)^{\beta}} & (1.3.18) \end{cases}$$

where $K \equiv \max(K_1, K_2)$.

We consider the interval $[0, 2T]$ and obtain from (1.3.1) and (1.3.2)

$$\begin{cases} (1-PT)f_{2T} \le K_1 + PTf_T + \dfrac{P}{\mu}(e^{\mu T}-1)g_T + \dfrac{P}{\mu}e^{\mu T}(e^{\mu T}-1)g_{2T}, & (1.3.19) \\ (1-PT)g_{2T} \le K_2 + PTg_T + \dfrac{P}{\mu}(1-e^{-\mu T})f_T + \dfrac{P}{\mu}e^{-\mu T}(1-e^{-\mu T})f_{2T}. & (1.3.20) \end{cases}$$

When (1.3.17) is used in (1.3.16), the function $H(T)$ reappears as in (1.3.5) and (1.3.6). The inequalities (1.3.7)–(1.3.13) then imply

$$f_{2T} \le Ke^{\mu T}/(1-\alpha T)^{\delta}, \quad g_{2T} \le K/(1-\alpha T)^{\delta}, \tag{1.3.21}$$

where $\delta = \beta + \gamma$.

Proceeding in the same manner, we obtain for all integers $n \ge 1$,

$$f_{nT} \le Ke^{\mu nT}/(1-\alpha T)^{n\delta}, \quad g_{nT} \le K/(1-\alpha T)^{n\delta}. \tag{1.3.22}$$

For every $t \ge 0$, there exists an integer n such that $(n-1)T < t \le nT$ and thus for all $t \ge 0$,

$$\begin{aligned} f(t) &\le f_{nT} \le Ke^{\mu nT}/(1-\alpha T)^{n\delta} \\ &\le \frac{K}{(1-\alpha T)^{\delta}} \exp\left(\mu T + (\mu + \frac{\alpha\delta}{1-\alpha T})t\right), \end{aligned} \tag{1.3.23}$$

$$g(t) \le g_{nT} \le K/(1-\alpha T)^{n\delta} \le \frac{K}{(1-\alpha T)^{\delta}} \exp\left(\frac{\alpha\delta t}{1-\alpha T}\right). \tag{1.3.24}$$

This result uses the identity $(1-\alpha T)^{-1} = 1 + \alpha T(1-\alpha T)^{-1}$ and the inequality $(1 + 1/n)^n < e$. A similar argument may be carried out starting with (1.3.1) and (1.3.2) to show that these bounds also hold for $\mu = 0$ when α, δ and T are properly chosen. Estimates (1.3.23) and (1.3.24) provide the constants c_i and M_i. □

Example 1.3.1 The system in Eqs. (1.3.1) and (1.3.2)

$$\begin{cases} f(t) \leq K_1 + P\int_0^t f(s)ds + P\int_0^t e^{\mu s} g(s)ds, & (1.3.25) \\ g(t) \leq K_2 + P\int_0^t e^{-\mu s} f(s)ds + P\int_0^t g(s)ds & (1.3.26) \end{cases}$$

arises in the study of kinetic model equations [234]. We would like to establish that the functions f and g are at most of exponential order so that the Laplace transform may be applied. Both the Gronwall inequality [574] and comparison theorem of Nohel [405, 431] provide bounds on f and g of exponential functions raised to exponential functions. The inequalities (1.3.25) and (1.3.26), on the other hand, establish that these functions are of exponential order. The sharp inequalities of this theorem are produced by the novel iteration carried out in the proof.

The proof due to Greene has been simplified by many authors, see, e.g., Wang [635] and Das [163]. The next proof is due to Wang [635], where the Jones inequality (see, Theorem 1.2.1) will be used to show the inequalities.

Proof Let P be an upper bound for h_i (the assumption $\mu > 0$ is not necessarily required here), then

$$\begin{cases} f(t) \leq K_1 + P\int_0^t f(s)ds + P\int_0^t e^{\mu s} g(s)ds, & (1.3.27) \\ g(t) \leq K_2 + P\int_0^t e^{-\mu s} f(s)ds + P\int_0^t g(s)ds. & (1.3.28) \end{cases}$$

Multiplying (1.3.27) by $e^{-\mu t}$ and then adding to (1.3.28),

$$\begin{aligned} e^{-\mu t} f(t) + g(t) &\leq K_1 e^{-\mu t} + K_2 + \int_0^t P[e^{\mu(s-t)} + 1][e^{-\mu s} f(s) + g(s)]ds \\ &\leq K_1 e^{-\mu t} + K_2 + 2\int_0^t P[e^{-\mu s} f(s) + g(s)]ds. \end{aligned} \tag{1.3.29}$$

Applying the Jones inequality (i.e.,Theorem 1.2.1) to (1.3.29),

$$\begin{aligned} e^{-\mu t} f(t) + g(t) &\leq \int_0^t [-K_1 \mu e^{-\mu s}] e^{\int_s^t 2P dr} ds + (K_1 + K_2) e^{\int_0^t 2P ds} \\ &= \frac{K_1 \mu}{2P + \mu} e^{-\mu t} + \frac{2P(K_1 + K_2) + K_2 \mu}{2P + \mu} e^{2Pt}. \end{aligned}$$

Thus the conclusion follows. □

The next is the proof given by Das [163], in which the following inequalities (1.3.30) are proved instead of (1.3.3):

$$f(t) \le Me^{\mu t+\int_0^t h(s)ds}, \quad g(t) \le Me^{\int_0^t h(s)ds} \tag{1.3.30}$$

where

$$h(t) = \max\left(|h_1 + h_3|(t), |h_2 + h_4|(t)\right)$$

and the h_i ($i = 1, 2, 3, 4$) are not necessarily bounded on $[0, +\infty)$. It is immediate that the bounds in (1.3.3) follow in view of the additional assumption of boundedness on the h_i ($i = 1, 2, 3, 4$).

Proof We note that (1.3.1) implies

$$e^{-\mu t}f(t) \le K_1 + \int_0^t e^{-\mu s}h_1(s)f(s)ds + \int_0^t h_2(s)g(s)ds. \tag{1.3.31}$$

Now we define

$$F(t) = e^{-\mu t}f(t) + g(t). \tag{1.3.32}$$

Thus (1.3.31), (1.3.2) and (1.3.32) lead to

$$F(t) \le M + \int_0^t h(s)F(s)ds \tag{1.3.33}$$

where $M = K_1 + K_2$.

Applying the Bellman inequality (Theorem 1.1.2) to (1.3.33) yields

$$F(t) \le M \exp\Big(\int_0^t h(s)ds\Big). \tag{1.3.34}$$

Inserting (1.3.34) into (1.3.32), gives us the bounds in (1.3.3). The proof is now complete. □

As a further example, we now consider a generalization of an inequality of Greene (in $\mathbb{R}$) which is due to Conlan and Wang [145].

Theorem 1.3.2 (Conlan-Wang [145]) *Let*

$$\begin{cases} u(x) \leq k_1(x) + f_1(x) \int_0^x h_1(s)\{u(s) + f_2(s) \int_0^s h_2(t)u(t)dt\}ds \\ \qquad + f_3(x) \int_0^x e^{\mu s} h_3(s)\{v(s) + f_4(s) \int_0^s h_4(t)v(t)dt\}ds, & (1.3.35) \\ v(x) \leq k_2(x) + f_5(x) \int_0^x e^{-\mu s} h_5(s)\{u(s) + f_2(s) \int_0^s h_2(t)u(t)dt\}ds \\ \qquad + f_6(x) \int_0^x h_6(s)\{v(s) + f_4(s) \int_0^s h_4(t)v(t)dt\}ds, & (1.3.36) \end{cases}$$

where μ is a non-negative constant, and the functions k_i, f_i, h_i, u, v are non-negative continuous functions ($\mathbb{R} \to \mathbb{R}_+$). Then there exist constants c_i, M_i such that for all $x \geq 0$,

$$u(x) \leq M_1 \exp(c_1 x),\ \ v(x) \leq M_2 \exp(c_2 x). \tag{1.3.37}$$

Proof Set

$$\begin{cases} z_1(x) = u(x) + f_2(x) \int_0^x h_2(t)u(t)dt, \\ z_2(x) = v(x) + f_4(x) \int_0^x h_4(t)v(t)dt. \end{cases}$$

Then,

$$\begin{cases} u(x) \leq z_1(x) \leq k_1(x) + f_1(x) \int_0^x h_1(s)z_1(s)ds \\ \qquad + f_3(x) \int_0^x e^{\mu s} h_3(s)z_2(s)ds + f_2(x) \int_0^x h_2(s)z_1(s)ds, \\ v(x) \leq z_2(x) \leq k_2(x) + f_5(x) \int_0^x e^{-\mu s} h_5(s)z_1(s)ds \\ \qquad + f_6(x) \int_0^x h_6(s)z_2(s)ds + f_4(x) \int_0^x h_4(s)z_2(s)ds. \end{cases}$$

Let $f_{1,2}(x) = \max\{f_1(x), f_2(x)\}$, etc. Then

$$\begin{cases} z_1(x) \leq k_1(x) + f_{1,2}(x) \int_0^x h_{12}(s)z_1(s)ds + f_3(x) \int_0^x e^{\mu s} h_3(s)z_2(s)ds, \\ z_2(x) \leq k_2(x) + f_5(x) \int_0^x e^{-\mu s} h_5(s)z_1(s)ds + \int_0^x f_{4,6}(s)z_2(s)ds. \end{cases}$$

This is now of the same form as that treated by Greene, and the proof follows exactly as in Greene [235] (see also Wang [639]). □

Note that similar results have been obtained by Shinde and Pachpatte [589] by more complicated methods.

The following system of inequalities can be considered as a simultaneously singular Gronwall-Bellman inequality (see, e.g., Dickstein and Loayza [185]).

Theorem 1.3.3 (Dickstein-Loayza [185]) *Let* $A > 0, B > 0, k > 0, T > 0, 0 < \alpha_1, \alpha_2, \beta_1, \beta_2 < 1$. *Consider* $\phi, \psi : [0, T) \to \mathbb{R}_+$ *continuous functions satisfying for any* $t \in [0, T)$*,*

$$\begin{cases} \phi(t) \leq A + kt^{\alpha_1} \displaystyle\int_0^t (t-s)^{-\alpha_1} s^{-\beta_1} \psi(s)ds, & (1.3.38) \\ \psi(t) \leq B + kt^{\alpha_2} \displaystyle\int_0^t (t-s)^{-\alpha_2} s^{-\beta_2} \phi(s)ds. & (1.3.39) \end{cases}$$

Then there exists a constant $C = C(\alpha_1, \alpha_2, \beta_1, \beta_2, k, T) > 0$ *such that for all* $t \in [0, T)$*,*

$$\phi(t) \leq C(A + Bt^{1-\beta_1}), \quad \psi(t) \leq C(B + At^{1-\beta_2}). \tag{1.3.40}$$

Proof Consider $\tilde{\phi}(t) = \sup_{s \leq t} \phi(s)$, $\tilde{\psi}(t) = \sup_{s \leq t} \psi(s)$. Then (1.3.38)–(1.3.39) hold for $\tilde{\phi}$ and $\tilde{\psi}$. Indeed, if $\tau < t$, then

$$\begin{aligned} \tau^{\alpha_1} \int_0^\tau (t-s)^{-\alpha_1} s^{-\beta_1} \tilde{\psi}(s)ds &= \tau^{1-\beta_1} \int_0^1 (1-z)^{-\alpha_1} z^{-\beta_1} \tilde{\psi}(\tau z)dz \\ &\leq t^{1-\beta} \int_0^1 (1-z)^{-\alpha_1} z^{-\beta_1} \psi(\tilde{tz})dz \\ &= t^{\alpha_1} \int_0^t (t-s)^{-\alpha_1} s^{-\beta_1} \tilde{\psi}(s)ds, \end{aligned}$$

so that,

$$\phi(\tau) \leq A + k\tau^{\alpha_1} \int_0^\tau (\tau-s)^{-\alpha_1} s^{-\beta_1} \tilde{\psi}(s)ds \leq A + kt^{\alpha_1} \int_0^t (t-s)^{-\alpha_1} s^{-\beta_1} \tilde{\psi}(s)ds$$

whence

$$\tilde{\phi}(t) \leq A + k\tau^{\alpha_1} \int_0^\tau (\tau-s)^{-\alpha_1} s^{-\beta_1} \tilde{\psi}(s)ds \leq A + kt^{\alpha_1} \int_0^t (t-s)^{-\alpha_1} s^{-\beta_1} \tilde{\psi}(s)ds.$$

A similar estimate holds for $\tilde{\psi}$. It suffices then to prove (1.3.40) for $\tilde{\phi}, \tilde{\psi}$. This is why we assume that ϕ and ψ are non-decreasing functions.

First we prove (1.3.40) for t small. To this end, define

$$M = k \max\left[\int_0^1 (1-s)^{-\alpha_1} s^{-\beta_1} ds, \int_0^1 (1-s)^{-\alpha_2} s^{-\beta_2} ds\right].$$

Then from (1.3.38)–(1.3.39), it follows

$$\phi(t) \leq A + Mt^{1-\beta_1}\psi(t), \quad \psi(t) \leq B + Mt^{1-\beta_2}\phi(t) \tag{1.3.41}$$

which gives us

$$\phi(t) \leq A + BMt^{1-\beta_1} + M^2 t^{2-\beta_1-\beta_2}\phi(t). \tag{1.3.42}$$

Fix $\tau > 0$ such that $M^2\tau^{2-\beta_1-\beta_2} < 1/2$. If $t \leq \tau$, then we derive from (1.3.42)

$$\phi(t) \leq 2(A + BMt^{1-\beta_1}). \tag{1.3.43}$$

Analogously,

$$\psi(t) \leq 2(B + AMt^{1-\beta_2})$$

which, along with (1.3.43), proves (1.3.40) for $t \leq \tau$ with $C = 2$.

Consider now $t > \tau$, we choose a, b such that

$$\begin{cases} \left(\int_0^a + \int_b^1\right)(1-s)^{-\alpha_1}s^{-\beta_1}ds \leq \dfrac{1}{2k}T^{-(1-\beta_1)}, & (1.3.44)\\ \left(\int_0^a + \int_b^1\right)(1-s)^{-\alpha_2}s^{-\beta_2}ds \leq \dfrac{1}{2k}T^{-(1-\beta_2)}. & (1.3.45)\end{cases}$$

Then by virtue of (1.3.44)–(1.3.45), we deduce from (1.3.38)

$$\begin{aligned}\phi(t) &\leq A + kt^{\alpha_1}\left(\int_0^{at} + \int_{at}^{bt} + \int_{bt}^t\right)(t-s)^{-\alpha_1}s^{-\beta_1}\psi(s)ds\\ &\leq A + \frac{1}{2}(tT^{-1})^{1-\beta_1}\psi(t) + k(1-b)^{-\alpha_1}(a\tau)^{-\beta_1}\int_0^t \psi(s)ds\\ &\leq A + \frac{1}{2}\psi(t) + k(1-b)^{-\alpha_1}(a\tau)^{-\beta_1}\int_0^t \psi(s)ds. \end{aligned}\tag{1.3.46}$$

Similarly,

$$\psi(t) \leq B + \frac{1}{2}\phi(t) + k(1-b)^{-\alpha_2}(a\tau)^{-\beta_2}\int_0^t \phi(s)ds. \tag{1.3.47}$$

Set

$$J = \frac{4}{3}\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}, \tag{1.3.48}$$

$$Q = \begin{pmatrix} 0 & k(1-b)^{-\alpha_1}(a\tau)^{-\beta_1} \\ k(1-b)^{-\alpha_2}(a\tau)^{-\beta_2} & 0 \end{pmatrix}, \tag{1.3.49}$$

$$P = JQ, \quad v = J\begin{pmatrix} A \\ B \end{pmatrix}, \tag{1.3.50}$$

and

$$f(t) = \begin{pmatrix} \int_0^t \phi(s)ds \\ \int_0^t \psi(s)ds \end{pmatrix}. \tag{1.3.51}$$

Then we derive from (1.3.46)–(1.3.47) that

$$f'(t) \le v + Pf(t)$$

which gives us

$$f'(t) \le e^{Pt}v.$$

This shows (1.3.40) for $C > 0$ and for all $t > \tau$. Thus the proof is complete. □

1.4 Linear One-Dimensional Henry Type Integral Inequalities and Their Bihari Type Versions

In this section, we shall introduce some linear one-dimensional Henry type integral inequalities and their Bihari type version.

1.4.1 One-Dimensional Henry-Gronwall-Bihari Integral Inequalities

First let us define a special class of nonlinear functions.

Definition 1.4.1 (Henry [272]) Let $q > 0$ be a real number and $0 < T \le +\infty$. We say that a function $\omega : \mathbb{R}_+ \to \mathbb{R}$ satisfies a condition (q), if for all $u \in \mathbb{R}_+, t \in [0, T)$,

$$e^{-qt}[\omega(u)]^q \le R(t)\omega(e^{-qt}u^q), \tag{1.4.1}$$

where $R(t)$ is a continuous, non-negative function.

Remark 1.4.1 If $\omega(u) = u^m, m > 0$, then

$$e^{-qt}[\omega(u)]^q = e^{(m-1)qt}\omega(e^{-qt}u^q) \tag{1.4.2}$$

for any $q > 1$, i.e., the condition (q) is satisfied with $R(t) = e^{(m-1)qt}$.

Let $\omega(u) = u + au^m$, where $0 \leq a \leq 1, \quad m \geq 1$. We shall show that ω satisfies the condition (q).

We need now the well-known inequality

$$(A_1 + A_2 + \cdots + A_n)^r \leq n^{r-1}(A_1^r + A_2^r + \cdots + A_n^r) \tag{1.4.3}$$

for any non-negative real numbers $A_1, A_2, \cdots, A_n$, where $r > 1$ is a real number and n is a natural number. This inequality is a consequence of Jensen's inequality. Using (1.4.3) with $r = q$ and $n = 2$, we have

$$e^{-qt}[\omega(u)]^q = e^{-qt}(u + au^m)^q \leq 2^{q-1}e^{-qt}(u^q + a^q u^{qm}), \tag{1.4.4}$$

$$\begin{aligned} 2^{q-1}e^{qmt}\omega(e^{-qt}u^q) &= 2^{q-1}e^{qmt}[e^{-qt}u^q + ae^{-qmt}u^{qm}] \\ &= 2^{q-1}e^{-qt}[e^{qmt}u^q + au^{qm}] \geq 2^{q-1}e^{-qt}[u^q + a^q u^{qm}] \end{aligned} \tag{1.4.5}$$

and thus the inequality (1.4.4) yields the condition (q), i.e., (1.4.1), with $R(t) = 2^{q-1}e^{qmt}$.

A new approach to an analysis of nonlinear integral inequalities with weakly singular kernels is used in the proof of Theorem 1.4.1 concerning a nonlinear integral inequality. Linear inequalities investigated by Henry [272] are special cases of this nonlinear one.

Theorem 1.4.1 (Medved' [396]) *Let $a(t)$ be a non-decreasing, non-negative C^1-function on $[0, T)$, and $F(t)$ a continuous, non-negative function on $[0, T), \omega : \mathbb{R}_+ \to \mathbb{R}$ a continuous, non-decreasing function, $\omega(0) = 0, \omega(u) > 0$ on $(0, T)$, and $u(t)$ a continuous, non-negative function on $[0, T)$ satisfying for all $t \in [0, T)$ and for a constant $\beta > 0$,*

$$u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1}F(s)\omega(u(s))ds. \tag{1.4.6}$$

Then the following assertions hold:

(1) Assume $\beta > \frac{1}{2}$, and let ω satisfy the condition (q) with $q = 2$. Then for all $t \in [0, T_1]$,

$$u(t) \leq e^t\left\{\Omega^{-1}\left[\Omega\left(2a^2(t)\right) + g_1(t)\right]\right\}^{\frac{1}{2}}, \tag{1.4.7}$$

where

$$g_1(t) = \frac{\Gamma(2\beta - 1)}{4^{\beta-1}} \int_0^t R(s)F^2(s)ds,$$

with Γ *being the gamma function,* $\Omega(v) = \int_{v_0}^{v} \left(\frac{dy}{\omega(y)}\right), v \geq v_0 > 0, \Omega^{-1}$ *the inverse of* Ω, *and* $T_1 \in \mathbb{R}_0 \equiv (0, +\infty)$ *such that* $\Omega\left(2a^2(t)\right) + g_1(t) \in$ *Dom* (Ω^{-1}) *for all* $t \in [0, T_1]$.

(2) *Let* $\beta \in (0, \frac{1}{2}]$, *and let* ω *satisfy the condition* (q) *with* $q = z + 2$, *where* $z = \frac{1-\beta}{\beta}$ (*i.e.,* $\beta = \frac{1}{z+1}$). *Then for all* $t \in [0, T_1]$,

$$u(t) \leq e^t \left\{\Omega^{-1}\left[\Omega\left(2^{q-1}a^q(t)\right) + g_2(t)\right]\right\}^{\frac{1}{q}}, \tag{1.4.8}$$

where

$$\begin{cases} g_2(t) = 2^{q-1}K_z^q \displaystyle\int_0^t F^q(s)R(s)ds, \\ K_z = \left[\dfrac{\Gamma(1-ap)}{p^{1-ap}}\right]^{\frac{1}{p}}, \quad \alpha = \dfrac{z}{z+1}, \ p = \dfrac{z+2}{z+1}, \end{cases} \tag{1.4.9}$$

where $T_1 \in \mathbb{R}_0$ *is such that* $\Omega\left(2^{q-1}a^q(t)\right) + g_2(t) \in$ *Dom* (Ω^{-1}) *for all* $t \in [0, T_1]$.

Proof First we shall prove the assertion (1.4.7). Using the Cauchy-Schwarz inequality, we obtain from (1.4.6)

$$\begin{aligned} u(t) &\leq a(t) + \int_0^t (t-s)^{\beta-1} e^s F(s) e^{-s} \omega(u(s)) ds \\ &\leq a(t) + \left[\int_0^t (t-s)^{2\beta-2} e^{2s} ds\right]^{\frac{1}{2}} \left[\int_0^t F^2(s) e^{-2s} \omega^2(u(s)) ds\right]^{\frac{1}{2}}. \end{aligned} \tag{1.4.10}$$

For the first integral in (1.4.10), we have

$$\begin{aligned} \int_0^t (t-s)^{2\beta-2} e^{2s} ds &= \int_0^t \tau^{2\beta-2} e^{2(t-\tau)} d\tau \\ &= e^{2t} \int_0^t \tau^{2\beta-2} e^{-2\tau} d\tau = \frac{2e^{2t}}{4^\beta} \int_0^t \sigma^{2\beta-2} e^{-\sigma} d\sigma \\ &\leq \frac{2e^{2t}}{4^\beta} \Gamma(2\beta - 1). \end{aligned} \tag{1.4.11}$$

Therefore, from (1.4.10)–(1.4.11) it follows

$$u(t) \le a(t) + \left[\frac{2e^{2t}}{4^{\beta}}\Gamma(2\beta-1)\right]^{\frac{1}{2}} \left[\int_0^t F^2(s)e^{-2s}\omega(u(s))^2 ds\right]^{\frac{1}{2}}. \tag{1.4.12}$$

Using (1.4.3) with $n = 2, r = 2$, we obtain

$$u^2(t) \le 2a^2(t) + \frac{e^{2t}\Gamma(2\beta-1)}{4^{\beta-1}} \int_0^t F^2(s)e^{-2s}\omega^2(u(s))ds, \tag{1.4.13}$$

and applying the condition (q) for $q = 2$, we have

$$v(t) \le \alpha(t) + K\int_0^t F^2(s)R(s)\omega(u(s))ds, \tag{1.4.14}$$

where

$$v(t) = \left(e^{-t}u(t)\right)^2, \quad \alpha(t) = 2a^2(t), \quad K = \frac{\Gamma(2\beta-1)}{4^{\beta-1}}. \tag{1.4.15}$$

Now we shall proceed in a standard way. Let $V(t)$ be the right-hand side of (1.4.14). Then

$$\omega(v(t))[\omega(V(t))]^{-1} \le 1,$$

which yields

$$\left[\alpha'(t) + KF^2(t)R(t)\omega(v(t))\right][\omega(V(t))]^{-1} \le \alpha'(t)\,[\omega(\alpha(t))]^{-1} + KF^2(t)R(t), \tag{1.4.16}$$

i.e.,

$$\frac{V'(t)}{\omega(V(t))} \le \frac{\alpha'(t)}{\omega(\alpha(t))} + KF^2(t)R(t) \tag{1.4.17}$$

or

$$\frac{d}{dt}\Omega(V(t)) \le \frac{d}{dt}\Omega(a(t)) + KF^2(t)R(t). \tag{1.4.18}$$

Integrating (1.4.18) from 0 to t, we obtain

$$\Omega(V(t)) \le \Omega(\alpha(t)) + g_1(t), \tag{1.4.19}$$

where

$$g_1(t) = K \int_0^t F^2(s)R(s)ds$$

and

$$v(t) \leq V(t) \leq \Omega^{-1}\Big(\Omega(\alpha(t)) + g_1(t)\Big).$$

Using (1.4.15), we obtain (1.4.7). Now let us prove the assertion (1.4.8). Obviously, $\beta - 1 = -\alpha = \frac{-z}{(z+1)}$. Let p, q be as in the theorem. Noting that $1/p + 1/q = 1$ and using Hölder's inequality, we obtain

$$\begin{aligned} u(t) &\leq a(t) + \int_0^t (t-s)^{\beta-1}F(s)\omega(u(s))ds \\ &= a(t) + \int_0^t (t-s)^{-\alpha}e^sF(s)e^{-s}\omega(u(s))ds \\ &\leq a(t) + \left[\int_0^t (t-s)^{-ap}e^{ps}ds\right]^{\frac{1}{p}}\left[\int_0^t F(s)^q e^{-qs}\omega(u(s))^q ds\right]^{\frac{1}{q}}. \end{aligned} \tag{1.4.20}$$

For the first integral in (1.4.20), we have

$$\begin{aligned} \int_0^t (t-s)^{-\alpha p}e^{ps}ds = e^{pt}\int_0^t \tau^{-\alpha p}e^{-p\tau}d\tau = \frac{e^{pt}}{p^{1-\alpha p}}\int_0^{pt}\sigma^{-\alpha p}e^{-\sigma}d\sigma \\ \leq \frac{e^{pt}}{p^{1-\alpha p}}\Gamma(1-\alpha p). \end{aligned} \tag{1.4.21}$$

Obviously, $1-\alpha p = \frac{1}{(z+1)^2} > 0$ and so $\Gamma(1-\alpha p) \in \mathbb{R}$. Thus it follows from (1.4.10) and the condition (q) that

$$u(t) \leq a(t) + e^t K_z\left[\int_0^t F^q(s)R(s)\omega(e^{-qs}u^q(s))ds\right]^{\frac{1}{q}}, \tag{1.4.22}$$

where K_z is defined by (1.4.9). Now using (1.4.3) for $n = 2, r = q$, we obtain

$$u^q(t) \leq 2^{q-1}a^q(t) + 2^{q-1}e^{qt}K_z^q \int_0^t F^q(s)R(s)\omega(e^{-qs}u^q(s))ds, \tag{1.4.23}$$

which yields

$$v(t) \leq \phi(t) + 2^{q-1}K_z^q \int_0^t F^q(s)R(s)\omega(v(s))ds, \tag{1.4.24}$$

where

$$v(t) = \left(e^{-t}u(t)\right)^q, \quad \phi(t) = 2^{q-1}a^q(t). \tag{1.4.25}$$

Now let $V(t)$ be the right-hand side of (1.4.24). Then $\omega(V(t))[\omega(V(t))]^{-1} \leq 1$, which yields

$$\begin{aligned}\left[\phi'(t) + 2^{q-1}K_z^q F^q(t)R(t)\omega(v(t))\right][\omega(V(t))]^{-1} \\ \leq \phi'(t)\,[\omega(\phi(t))]^{-1} + 2^{q-1}K_z^q F^q(t)R(t),\end{aligned} \tag{1.4.26}$$

i.e.,

$$\frac{V'(t)}{\omega(V(t))} \leq \frac{\phi'(t)}{\omega(\phi(t))} + 2^{q-1}K_z^q F^q(t)R(t), \tag{1.4.27}$$

or

$$\frac{d}{dt}\Omega(V(t)) \leq \frac{d}{dt}\Omega(\phi(t)) + 2^{q-1}K_z^q F^q(t)R(t). \tag{1.4.28}$$

Integrating (1.4.28) from 0 to t, we obtain

$$\Omega(V(t)) \leq \Omega(\phi(t)) + g_2(t), \tag{1.4.29}$$

where

$$g_2(t) = 2^{q-1}K_z^q \int_0^t F^q(s)R(s)ds$$

which gives us

$$v(t) \leq V(t) \leq \Omega^{-1}\Big(\Omega(\phi(t)) + g_2(t)\Big). \tag{1.4.30}$$

Using (1.4.27), we can obtain (1.4.8). □

As a consequence of Theorem 1.4.1, we have the following result of the linear case.

Theorem 1.4.2 (Medveď [396]) *Let* $0 < T \leq +\infty, a(t), F(t)$ *be as in Theorem 1.4.1, and let* $u(t)$ *be a continuous, non-negative function on* $[0, T)$ *such that for a constant* $\beta > 0$,

$$u(t) \leq a(t) + \int_0^t (t-s)^{\beta-1}F(s)u(s)ds. \tag{1.4.31}$$

Then the following conclusions hold:

(1) *If* $\beta > \frac{1}{2}$, *then for all* $t \in [0, T)$,

$$u(t) \leq \sqrt{2}a(t) \exp\left[\frac{2\Gamma(2\beta - 1)}{4^{\beta}} \int_0^t F^2(s)ds + t\right]. \tag{1.4.32}$$

(2) *If* $\beta = \frac{1}{z+1}$ *for some* $z \geq 1$, *then for all* $t \in [0, T)$,

$$u(t) \leq (2^{q-1})^{\frac{1}{q}} a(t) \exp\left[\frac{2^{q-1}}{q} K_z^q \int_0^t F^q(s)ds + t\right], \tag{1.4.33}$$

where K_z *is defined by* (1.4.9), $q = z + 2$.

The method used in the proof of Theorem 1.4.1 enables us to prove the following theorem concerning the inequality (1.4.31), where $a(t), F(t)$, and $u(t)$ are integrable on $[0, T)$.

Theorem 1.4.3 (Medveď [396]) *Let* $a(t), b(t)$ *be non-negative, integrable functions on* $[0, T)$ *for* $0 < T \leq +\infty$, *and let* $F(t), u(t)$ *be integrable, non-negative functions on* $[0, T)$ *such that, for a. e.* $t \in [0, T)$,

$$u(t) \leq a(t) + b(t) \int_0^t (t - s)^{\beta - 1} F(s)u(s)ds. \tag{1.4.34}$$

Then the following assertions hold: for a. e. $t \in [0, T)$,

(1) *If* $\beta > 1/2$, *then*

$$u(t) \leq e^t \Phi(t)^{1/2}, \tag{1.4.35}$$

where

$$\Phi(t) = 2a^2(t) + 2Kb^2(t) \int_0^t a^2(s)F^2(s) \exp\left[K \int_s^t b^2(r)F^2(r)dr\right] ds,$$

$$K = \frac{\Gamma(2\beta - 1)}{4^{\beta - 1}}.$$

(2) *If* $\beta = 1/(z + 1)$ *for some* $z \geq 1$, *then for a. e.* $t \in [0, T)$,

$$u(t) \leq e^t \Big(\Psi(t)\Big)^{1/q}, \tag{1.4.36}$$

where

$$\Psi(t) = 2^{q-1}a^q(t) + 2^{q-1}K_z^q b^q(t)\int_0^t a^q(s)F^q(s)\exp\left[2^{q-1}K_z^q\int_s^t b^q(r)F^q(r)dr\right]ds,$$

and $q = z + 2$, *and* K_z *is defined by* (1.4.9).

Proof First we shall prove the assertion (1). Using the same procedure as in the proof of the assertion (1) of Theorem 1.4.1, we can show that

$$v(t) \le 2a^2(t) + \frac{\Gamma(2\beta-1)}{4^{\beta-1}}b^2(t)\int_0^t F^2(s)v(s)ds, \tag{1.4.37}$$

where $v(t) = (e^{-t}u(t))^2$. From [388] (Theorem 1.4), we obtain the inequality (1.4.35). Using the procedure from the proof of the assertion (2) of Theorem 1.4.1, we can show that

$$v(t) \le 2^{q-1}a^q(t) + 2^{q-1}K_z^q B^q(t)\int_0^t F^q(s)v(s)ds, \tag{1.4.38}$$

where $v(t) = (e^{-t}u(t))^q$ and the inequality (1.4.36) is a direct consequence of [388], Theorem 1.4. □

The following result is an analogue of Theorem 1.4.2 (see, e.g., [111]).

Theorem 1.4.4 (Brauer [111]) *Assume that a non-negative and locally bounded function* $h = h(t)$ *satisfies that for all* $t \ge 0$,

$$h(t) \le C_1(1+t) + C_2\int_0^t (t-\tau)^{-a}(1+\tau)^{-b}h(\tau)d\tau \tag{1.4.39}$$

for some $a \in (0,1)$, $b > 0$, *positive constants* C_1 *and* C_2.

If $a + b > 1$, *then for all* $t \ge 0$,

$$h(t) \le C(1+t) \tag{1.4.40}$$

for a constant $C > 0$ *independent of* t. *The same conclusion* (1.4.40) *holds true in the limit case* $a + b = 1$ *under the weaker assumption*

$$h(t) \le C_1(1+t) + C_2\int_0^t (t-\tau)^{-a}\tau^{-b}h(\tau)d\tau \tag{1.4.41}$$

provided that $C_2 > 0$ *is sufficiently small.*

Proof If $a + b = 1$, then we deduce from (1.4.41) that

$$h(t) \le C_1(1+t) + C_2K(a,b)\sup_{0\le\tau\le t} h(\tau),$$

where

$$K(a,b) = \int_0^t (t-\tau)^{-a}\tau^{-b}d\tau = \int_0^1 (1-s)^{-a}s^{-b}ds.$$

Consequently,

$$\sup_{0\le\tau\le t} h(\tau) \le \frac{C_1}{1-C_2K(a,b)}(1+t)$$

provided that $C_2 < 1/K(a,b)$, which gives us (1.4.40).

In the case $a+b>1$, using (1.4.39), we write $b = b_1+\eta$ for $a+b_1 = 1$ and $\eta>0$, and we fix $t_1>0$ such that

$$C_2(1+t_1)^{-\eta} < \frac{1}{K(a,b_1)}.$$

Now splitting the integral in (1.4.39) at $t_1>0$ yields

$$h(t) \le C(1+t) + C_2K(a,b_1)(1+t_1)^{-\eta}\sup_{0\le\tau\le t} h(\tau)$$

for some constant $C>0$ independent of t. Hence the conclusion follows. □

Now we shall prove a result which is a modification of Theorem 1.4.3 (Henry [272], Lemma 7.1.2).

Theorem 1.4.5 (Medveď [396]) *Let $a(t)$ be a non-negative, non-decreasing C^1 function on $[0,T]$ for $0<T\le+\infty$, and $F(t)$ a continuous, non-negative function on $[0,T)$. Moreover, let $u(t)$ be a non-negative, continuous function on $[0,T)$ such that for all $t\in[0,T)$ and for constants $\beta>0, \gamma>0$,*

$$u(t) \le a(t) + \int_0^t (t-s)^{\beta-1}s^{\gamma-1}F(s)u(s)ds. \tag{1.4.42}$$

Then the following assertions hold:

(1) *If $\beta>\frac{1}{2}$ and $\gamma>1-\frac{1}{2p}$, where $p>1$ is a real number, then for all $t\in[0,T)$,*

$$u(t) \le 2^{1-\frac{1}{2q}}a(t)\exp\left[\frac{4^q}{2q}K^qL^q\int_0^t F^{2q}(s)e^{qs}ds + t\right], \tag{1.4.43}$$

where

$$K = \frac{\Gamma(2\beta-1)}{4^{\beta-1}},\quad L = \left[\frac{\Gamma((2\gamma-2)p+1)}{p^{(2\gamma-2)p}}\right]^{\frac{1}{p}},\ \frac{1}{p}+\frac{1}{q} = 1.$$

(2) *Let* $\beta = \frac{1}{m+1}$ *for some real number* $m \geq 1, \gamma > 1 - \frac{1}{\kappa q}$, *where* $\kappa > 1$ *is a real number,* $p = \frac{m+2}{m+1}, q = m + 2$. *Then for all* $t \in [0, T)$,

$$u(t) \leq 2^{\frac{\kappa r - 1}{qr}} a(t) \exp\left[\frac{p}{\kappa r} \int_0^t e^{rs} F^{rq}(s) ds + t\right], \tag{1.4.44}$$

where $r > 1$ *is such that* $1/\kappa + 1/r = 1$,

$$P = \left[\frac{\Gamma(1 - \alpha p)}{p^{1-\alpha p}}\right]^{\frac{rq}{p}} \left[\frac{\Gamma(\kappa q(\gamma - 1) + 1)}{\kappa^{\kappa q(\gamma-1)}}\right]^{\frac{r}{\kappa}}$$

and $-\alpha = \beta - 1 = \frac{-m}{m+1}$.

Proof Let us first prove the conclusion (1). In fact, from (1.4.42) it follows

$$\begin{aligned} u(t) &\leq a(t) + \left[\int_0^t (t-s)^{2\beta-2} e^{2s} ds\right]^{\frac{1}{2}} \left[\int_0^t s^{2\gamma-2} F^2(s) e^{-2s} u^2(s) ds\right]^{\frac{1}{2}} \\ &\leq a(t) + e^t K^{\frac{1}{2}} \left[\int_0^t s^{2\gamma-2} F^2(s) (e^{-s} u(s))^2 ds\right]^{1/2} \end{aligned} \tag{1.4.45}$$

where $K = \Gamma(2\beta - 1)/4^{\beta-1}$, which yields

$$u^2(t) \leq 2a^2(t) + 2e^{2t} K \int_0^t s^{2\gamma-2} F^2(s) (e^{-s} u(s))^2 ds \tag{1.4.46}$$

whence

$$v(t) \leq c(t) + 2K \int_0^t s^{2\gamma-2} F^2(s) v(s) ds, \tag{1.4.47}$$

where

$$c(t) = 2a^2(t), \quad v(t) = (e^{-t} u(t))^2. \tag{1.4.48}$$

Thus from (1.4.48), it follows

$$\begin{aligned} v(t) &\leq c(t) + 2K \int_0^t s^{2\gamma-2} e^{-s} F^2(s) e^s v(s) ds \\ &\leq c(t) + 2K \left[\int_0^t s^{(2\gamma-2)p} e^{-ps} ds\right]^{\frac{1}{p}} \left[\int_0^t F^{2q}(s) e^q (v(s))^q ds\right]^{\frac{1}{q}}, \end{aligned} \tag{1.4.49}$$

where $q > 1, \frac{1}{q} + \frac{1}{p} = 1$. For the first integral in (1.4.49), we have

$$\int_0^t s^{(2\gamma-2)p} e^{-ps} ds \leq \frac{e^{pt}}{p^{(2\gamma-2)p}} \Gamma((2\gamma-2)p+1).$$

Obviously the assumptions on p and γ yield

$$(2\gamma-2)p + 1 > \left[2\left(1 - \frac{1}{2p}\right) - 2\right]p + 1 = 0$$

whence

$$\Gamma((2\gamma-2)p+1) \in \mathbb{R}.$$

Noting that the definition of L be as in Theorem 1.4.5, from (1.4.49) we derive

$$v^q(t) \leq 2^{q-1} c^q(t) + \frac{4^q}{2} K^q L^q \int_0^t F^{2q}(s) e^{qs} v^q(s) ds \tag{1.4.50}$$

which yields

$$v^q(t) \leq 2^{q-1} c^q(t) \exp\left[\frac{4^q}{2} K^q L^q \int_0^t F^{2q}(s) e^{qs} ds\right]. \tag{1.4.51}$$

Therefore, (1.4.43) follows from (1.4.48) and (1.4.51). Now let us prove the assertion (2). From the inequality (1.4.42), we obtain

$$\begin{aligned} u(t) &\leq a(t) + \left[\int_0^t (t-s)^{-p\alpha} e^{ps} ds\right]^{\frac{1}{p}} \left[\int_0^t s^{q(\gamma-1)} e^{-qs} F^q(s) u^q(s) ds\right]^{\frac{1}{q}} \\ &\leq a(t) + e^t \left[\frac{\Gamma(1-\alpha p}{p^{(1-\alpha p}}\right]^{\frac{1}{p}} \left[\int_0^t s^{\kappa q(\gamma-1)} e^{-\kappa s} ds\right]^{\frac{1}{\kappa q}} \\ &\quad \times \left[\int_0^t e^{rs} F^{rq}(s) (e^{-s} u(s))^{rq} ds\right]^{\frac{1}{rq}} \\ &\leq a(t) + e^t \left[\frac{\Gamma(1-\alpha p}{p^{(1-\alpha p}}\right]^{\frac{1}{p}} \frac{\Gamma(\kappa q(\gamma-1)+1)^{\frac{1}{\kappa q}}}{\kappa^{\kappa q(\gamma-1)-1}} \\ &\quad \times \left[\int_0^t e^{rs} F^{rq}(s) (e^{-s} u(s))^{rq} ds\right]^{\frac{1}{rq}}, \end{aligned} \tag{1.4.52}$$

where r is as in the theorem. We assume that $\gamma > 1 - \frac{1}{\kappa q}$ and thus we have $\kappa q(\gamma - 1) + 1 > \kappa q(\frac{-1}{\kappa q}) + 1 = 0$, i.e., $\Gamma(\kappa q(\gamma - 1) + 1) \in \mathbb{R}$, this yields

$$v(t) \leq 2^{rq-1}\left[a^{qr}(t) + P\int_0^t e^{rs}F^{rq}(s)v(s)ds\right], \tag{1.4.53}$$

where $v(t) = (e^{-t}u(t))^{rq}$ and P is defined as in the theorem. Therefore, we obtain

$$v(t) \leq 2^{rq-1}a^{rq}(t)\exp\left[P\int_0^t e^{rs}F^{rq}(s)ds\right] \tag{1.4.54}$$

which yields (1.4.44). □

For the special case when $a(t) = t^{-\alpha}$ ($\alpha > 0$, *a constant*), $\beta = 1/2$, $\gamma = 1/2$, $F = constant > 0$, we have the following Theorem 1.4.6 whose proof needs the following lemma.

Lemma 1.4.1 *Let* $a < 1$, $b > 0$, $d < 1$. *If* $b + d < 1$, *then*

$$\int_0^t (t-s)^{-a}(s+1)^{-b}s^{-d}ds \leq Ct^{1-a-d}(1+t)^{-b}. \tag{1.4.55}$$

If $b + d = 1$, *then*

$$\int_0^t (t-s)^{-a}(s+1)^{-b}s^{-d}ds \leq Ct^{-a}\ln(1+t). \tag{1.4.56}$$

If $b + d > 1$, *then*

$$\int_0^t (t-s)^{-a}(s+1)^{-b}s^{-d}ds \leq Ct^{-a}. \tag{1.4.57}$$

Proof If we set $I := \int_{t/2}^t (t-s)^{-a}(s+1)^{-b}s^{-d}ds$, $II := \int_0^{t/2}(t-s)^{-a}(s+1)^{-b}s^{-d}ds$, then

$$I \leq C(1+t)^{-b}t^{-d}\int_{t/2}^t (t-s)^{-a}ds = C(1+t)^{-b}t^{-d}t^{1-a}$$
$$= C(1+t)^{-b}t^{1-a-d}, \tag{1.4.58}$$

$$II \leq Ct^{-a}\int_0^{t/2}(s+1)^{-b}s^{-d}ds. \tag{1.4.59}$$

If $t \geq 2$, then

$$\begin{aligned}
\int_0^{t/2} (s+1)^{-b} s^{-d} ds &= \int_1^{t/2} (s+1)^{-b} s^{-d} ds + \int_0^1 (s+1)^{-b} s^{-d} ds \\
&\leq C \int_1^{t/2} (s+1)^{-b-d} ds + C \int_0^1 s^{-d} ds \\
&\leq \begin{cases} C(t+1)^{1-b-d}, & \textit{if} \ \ b+d<1, \\ C\ln(t+1), & \textit{if} \ \ b+d=1, \\ C, & \textit{if} \ \ b+d>1. \end{cases}
\end{aligned} \tag{1.4.60}$$

If $t \leq 2$, then

$$\int_0^{t/2} (s+1)^{-b} s^{-d} ds \leq C \int_0^1 (s+1)^{-b} s^{-d} ds \leq C. \tag{1.4.61}$$

Hence it follows from (1.4.59)–(1.4.61) that

$$II \leq \begin{cases} Ct^{-a}(t+1)^{1-b-d}, & \textit{if} \ \ b+d<1, \\ Ct^{-a}\ln(t+1), & \textit{if} \ \ b+d=1, \\ Ct^{-a}, & \textit{if} \ \ b+d>1. \end{cases} \tag{1.4.62}$$

Thus from (1.4.58) and (1.4.62) it follows that

$$\int_0^t (t-s)^{-a}(s+1)^{-b}s^{-d}ds \leq \begin{cases} Ct^{1-a-d}(t+1)^{-b} + Ct^{-a}(t+1)^{1-b-d}, \ \textit{if} \ b+d<1, \\ Ct^{1-a-d}(t+1)^{-b} + Ct^{-a}\ln(t+1), \ \textit{if} \ b+d=1, \\ Ct^{1-a-d}(t+1)^{-b} + Ct^{-a}, \ \textit{if} \ b+d>1. \end{cases} \tag{1.4.63}$$

The proof is thus complete. □

Theorem 1.4.6 (Bae-Jin [38]) *Assume that $x(t) \geq 0$ satisfies that the following inequality for all $t > 0$,*

$$x(t) \leq Ct^{-\alpha} + \varepsilon \int_0^t (t-s)^{-1/2} s^{-1/2} x(s) ds. \tag{1.4.64}$$

Then

$$x(t) \leq Ct^{-\alpha} + C\varepsilon t^{-1/2} \int_0^t s^{-1/2} x(s) ds \tag{1.4.65}$$

where $\alpha > 0, C > 0$ are constants independent of $t > 0$ and $\varepsilon > 0$.

Proof Let $I := \int_{t/2}^{t}(t-s)^{-1/2}s^{-1/2}x(s)ds, II := \int_0^{t/2}(t-s)^{-1/2}s^{-1/2}x(s)ds$. Then we derive that

$$I \le Ct^{-1/2}\int_{t/2}^{t}(t-s)^{-1/2}x(s)ds, \qquad II \le Ct^{-1/2}\int_0^{t/2}s^{-1/2}x(s)ds. \tag{1.4.66}$$

If we iterate (1.4.64) to I, then we obtain

$$\begin{aligned}
t^{1/2}I &\le C\int_{t/2}^{t}(t-s)^{-1/2}x(s)ds\\
&\le C\varepsilon\int_{t/2}^{t}(t-s)^{-1/2}\Big(s^{-\beta}+\varepsilon\int_0^s(s-\tau)^{-1/2}\tau^{-1/2}x(\tau)d\tau\Big)ds\\
&= C\int_{t/2}^{t}(t-s)^{-1/2}s^{-\beta}ds + C\varepsilon\int_{t/2}^{t}\int_0^s(t-s)^{-1/2}(s-\tau)^{-1/2}\tau^{-1/2}x(\tau)d\tau ds\\
&= I_1+I_2.
\end{aligned} \tag{1.4.67}$$

A straightforward computation yields

$$I_1 \le C^2t^{-\beta+1/2}. \tag{1.4.68}$$

On the other hand, Fubini's theorem gives us

$$\begin{aligned}
I_2 &= C\varepsilon\int_0^{t/2}\int_{t/2}^{t}(t-s)^{-1/2}(s-\tau)^{-1/2}\tau^{-1/2}x(\tau)dsd\tau\\
&\quad +C\varepsilon\int_{t/2}^{t}\int_{\tau}^{t}(t-s)^{-1/2}(s-\tau)^{-1/2}\tau^{-1/2}x(\tau)dsd\tau\\
&= C\varepsilon\int_0^{t/2}\tau^{-1/2}x(\tau)\Big(\int_{t/2}^{t}(t-s)^{-1/2}(s-\tau)^{-1/2}ds\Big)d\tau\\
&\quad +C\varepsilon\int_{t/2}^{t}\tau^{-1/2}x(\tau)\Big(\int_{\tau}^{t}(t-s)^{-1/2}(s-\tau)^{-1/2}ds\Big)d\tau\\
&\le C\varepsilon\int_0^{t}\tau^{-1/2}x(\tau)d\tau
\end{aligned} \tag{1.4.69}$$

where we have used the following estimates for $t/2 < \tau < t$,

$$\int_{\tau}^{t}(t-s)^{-1/2}(s-\tau)^{-1/2}ds = C\int_0^{t-\tau}z^{-1/2}(t-\tau-z)^{-1/2}dz \le C$$

and for $0 < \tau < t/2$,

$$\int_{t/2}^{t} (t-s)^{-1/2}(s-\tau)^{-1/2} ds \le C \int_{\tau}^{t} (t-s)^{-1/2}(s-\tau)^{-1/2} ds \le C$$

with some constant $C > 0$ independent of t owing to Lemma 1.4.1. Hence from (1.4.68)–(1.4.69), we conclude

$$I \le Ct^{-\beta} + C\varepsilon t^{-1/2} \int_0^t \tau^{-1/2} x(\tau) d\tau. \tag{1.4.70}$$

Combining (1.4.70) and the estimates of I and II, we can complete the proof. □

1.4.2 One-Dimensional Ou-Yang and Pachapatte Type Integral Inequalities

Now we begin to study the integral inequality of the form

$$u^r(t) \le a(t) + \int_0^t (t-s)^{\beta-1} F(s)\omega(u(s)) ds, \quad \beta > 0. \tag{1.4.71}$$

The following result concerns the case $r = 2$ which was studied by Medveď in [396].

Theorem 1.4.7 (Medveď [396]) *Assume that* $r = 2$. *Let* $a(t)$ *be a non-decreasing, non-negative* C^1 *function on* $[0, T)$ *for* $0 < T \le +\infty$, $F(t)$ *a continuous, non-negative function, and* $\omega : \mathbb{R}_0 \to \mathbb{R}$, $d\omega(u)/du$ *be continuous, non-decreasing functions such that* $\omega(0) = 0, \omega(u) > 0$ *on* $(0, T)$, *and let* $u(t)$ *be a continuous, non-negative function on* $[0, T)$ *satisfying the inequality* (1.4.71). *Then the following assertions hold:*

(i) *Assume* $\beta > \frac{1}{2}$ *and let* ω *satisfy the condition* (q) (i.e., (1.4.1)) *for* $q = 2$. *Then for all* $t \in [0, T_1)$,

$$u(t) \le e^t \left\{ \Lambda^{-1} \left[\Lambda(2a^2(t)) + K \int_0^t F^2(s) R(s) ds \right] \right\}^{\frac{1}{4}}, \tag{1.4.72}$$

where

$$K = \frac{\Gamma(2\beta - 1)}{4^{\beta-1}}, \quad \Lambda(v) = \int_{v_0}^{v} \frac{d\sigma}{\omega(\sqrt{\sigma})}, \quad v \ge v_0 > 0 \tag{1.4.73}$$

and $T_1 \in \mathbb{R}_0$ *is such that* $\Lambda(2a^2(t)) + K \int_0^t F^2(s) R(s) ds \in Dom\ (\Lambda^{-1})$ *for all* $t \in [0, T_1]$.

(ii) *Let* $\beta \in (0, \frac{1}{2}]$ *and let* ω *satisfy the condition* (q) (i.e., (1.4.1)) *for* $q = z + 2$, *where* $z = \frac{1-\beta}{\beta}$, *i.e.*, $\beta = \frac{1}{z+1}$. *Then for all* $t \in [0, T_1]$,

$$u(t) \leq e^t \left\{ \Lambda^{-1} \left[\Lambda(2^{q-1} a^q(t)) + 2^{q-1} K_z^q \int_0^t F^q(s) R(s) ds \right] \right\}^{\frac{1}{2q}}, \tag{1.4.74}$$

where

$$K_z = \left[\frac{\Gamma(1 - \beta p)}{p^{1-\beta p}} \right]^{\frac{1}{p}}, \quad \beta = \frac{1}{z+1}, \quad p = \frac{z+2}{z+1}, \tag{1.4.75}$$

and $T_1 \in \mathbb{R}_0$ *is such that* $\Lambda(2^{q-1} a^q(t)) + 2^{q-1} K_z^q \int_0^t F^q(s) R(s) ds \in Dom\ (\Lambda^{-1})$ *for all* $t \in [0, T_1]$.

Proof First let us prove the assertion (i). Following the proof of Theorem 1.4.1, we can show that

$$\begin{cases} v^2(t) \leq \alpha(t) + K \displaystyle\int_0^t F^2(s) R(s) \omega(v(s)) ds, & (1.4.76) \\ v(t) = (e^{-t} u(t))^2, \quad \alpha(t) = 2a^2(t), \quad K = \dfrac{\Gamma(2\beta - 1)}{4^{\beta - 1}}. & (1.4.77) \end{cases}$$

Indeed, let $V(t)$ be the right-hand side of (1.4.76). Then $v(t) \leq \sqrt{V(t)}$. This yields $\omega(v(t)) \leq \omega(\sqrt{V(t)})$ and

$$\begin{aligned} \frac{V'(t)}{\omega(\sqrt{V(t)})} &= \frac{\alpha'(t) + K F^2(t) R(t) \omega(v(t))}{\omega(\sqrt{V(t)})} \\ &\leq \frac{\alpha'(t)}{\omega(\sqrt{\alpha(t)})} + K F^2(t) R(t) \end{aligned} \tag{1.4.78}$$

which readily implies

$$\frac{d}{dt} \int_0^{V(t)} \frac{d\sigma}{\omega(\sqrt{V(\sigma)})} \leq \frac{d}{dt} \int_0^{\alpha(t)} \frac{d\sigma}{\omega(\sqrt{\alpha(\sigma)})} + K F^2(t) R(t). \tag{1.4.79}$$

Thus we have

$$\frac{d}{dt} \Lambda(V(t)) \leq \frac{d}{dt} \Lambda(\alpha(t)) + K F^2(t) R(t), \tag{1.4.80}$$

where Λ is defined by (1.4.73). This yields

$$V(t) \leq \Lambda^{-1}\left[\Lambda(\alpha(t)) + K\int_0^t F^2(s)R(s)ds\right] \tag{1.4.81}$$

whence

$$v(t) \leq \sqrt{V(t)} \leq \left\{\Lambda^{-1}\left[\Lambda(\alpha(t)) + K\int_0^t F^2(s)R(s)ds\right]\right\}^{\frac{1}{2}}. \tag{1.4.82}$$

Using (1.4.77), we may obtain (1.4.76). Now we shall prove the assertion (ii). Following the proof of the assertion (2) of Theorem 1.4.1, we can show that

$$v^2(t) \leq \phi(t) + 2^{q-1}K_z^q\int_0^t F^q(s)R(s)\omega(v(s))ds, \tag{1.4.83}$$

where

$$v(t) = (e^{-t}u(t))^q, \quad \phi(t) = 2^{q-1}a^q(t). \tag{1.4.84}$$

Following the proof of the assertion (i), we obtain

$$v(t) \leq \left\{\Lambda^{-1}(\Lambda(\phi(t))) + 2^{q-1}K_z^q\int_0^t F^q(s)R(s)ds\right\}^{\frac{1}{2}} \tag{1.4.85}$$

which, together with (1.4.77), yields (1.4.74). □

Remark 1.4.2 It is possible to prove a result of a type similarly as in Theorem *1.4.7* for an inequality which is an analogue of the inequality *(1.4.71)* with multiple integrals. We do not formulate such results here because their formulation would be technically very complicated.

The nonsingular version of this inequality for $r = 2, \beta = 1$ was studied by Pachpatte in [449], where a result published by Ou-Yang [442] was generalized. Applying the method developed in [396], we can prove the following theorem.

Theorem 1.4.8 (Medveď [396]) *Let $a(t)$ be a non-negative, non-decreasing C^1-function on the interval $[0, T]$ $(0 < T < +\infty)$, let $F(t)$ be a non-negative, continuous function on $[0, T]$, $0 < \beta < 1, r \geq 1$, and let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous, non-decreasing, positive function. Assume that $u(t)$ is a continuous, non-negative function on $[0, T]$ satisfying the inequality* (1.4.71). *Then*

$$\Lambda_{qr}(u(t)^{qr}) \leq \Lambda_{qr}(2^{q-1}a^q) + K_q\int_0^t e^{-qs}F^q(s)ds, \tag{1.4.86}$$

or

$$u(t) \leq \left\{ \Lambda_{qr}^{-1} \left[\Lambda_{qr}(2^{q-1}a^q) + K_q \int_0^t e^{-qs} F(s)^q ds \right] \right\}^{1/qr}, \tag{1.4.87}$$

where $\beta = \frac{1}{1+z},\ z > 0,\ q = \frac{1}{\beta} + \varepsilon = 1 + z + \varepsilon,\ p = \frac{1+z+\varepsilon}{z+\varepsilon},\ \varepsilon > 0,$

$$\Lambda_{qr}(v) = \int_{v_0}^{v} \frac{d\sigma}{\omega(\sigma^{1/rq})^q}, \tag{1.4.88}$$

with $2^{q-1}a^q(0) \geq v_0 > 0$, *and* Λ_{qr}^{-1} *is the inverse of* $\Lambda_{qr},\ a = a(t)$,

$$K_q = \frac{2^{q-1}e^{pT}}{p^{1-\alpha p}} \Gamma(1 - \alpha p),$$

with $\alpha = 1 - \beta = \frac{z}{1+z}$, *and* Γ *is Euler's Gamma function, and* $T_1 > 0$ *is such that for all* $t \in [0, T_1]$,

$$\Lambda_{qr}(2^{q-1}a^q) + K_q \int_0^t e^{-qs} F^q(s) ds \in Dom\ (\Lambda_{qr}^{-1}).$$

Proof Obviously, $\frac{1}{p} + \frac{1}{q} = 1$. Using the Hölder inequality, we obtain from (1.4.71)

$$\begin{aligned} u^r(t) &\leq a(t) + \int_0^t (t-s)^{-\alpha} e^s e^{-s} F(s)\omega(u(s)) ds \\ &\leq a(t) + \left[\int_0^t (t-s)^{-\alpha p} e^{ps} ds \right]^{1/p} \left[\int_0^t e^{-qs} F^q(s) \omega^q(u(s)) ds \right]^{1/q}. \end{aligned} \tag{1.4.89}$$

Since $(A + B)^q \leq 2^{q-1}(A^q + B^q)$ holds for any $A \geq 0, B \geq 0$ and for $1 - \alpha p = \frac{\varepsilon}{(1+z)(z+\varepsilon)} > 0$,

$$\int_0^t (t-s)^{-\alpha p} e^{ps} ds = e^{pt} \int_0^t \tau^{-\alpha p} e^{-p\tau} d\tau \leq \frac{e^{pt}}{p^{1-\alpha p}} \Gamma(1 - \alpha p), \tag{1.4.90}$$

we derive from (1.4.89) that for all $t \in [0, T]$,

$$u^{rq}(t) \leq 2^{q-1}a^q + K_q \int_0^t e^{-qs} F(s) \omega^q(u(s)) ds. \tag{1.4.91}$$

Let $W(t)$ be the right-hand side of the inequality (1.4.91). Then $u(t) \leq (W(t))^{1/rq}$, which yields $\omega^q(u(t)) \leq (\omega(W(t)^{1/rq}))^q$. Therefore, from (1.4.91) it follows

$$\frac{W'(t)}{(\omega(W(t)^{1/rq}))^q} \leq \frac{K_q e^{-qt} F^q(t)\omega^q(u(t))}{(\omega(W^{1/rq}(t)))^q} + \frac{\alpha'(t)}{(\omega(\alpha^{1/rq}(t)))^q},$$

i.e.,

$$\frac{d}{dt}\int_0^{W(t)} \frac{d\sigma}{\omega(\sigma^{1/rq})^q} \leq K_q e^{-qt} F^q(t) + \frac{d}{dt}\int_0^{\alpha(t)} \frac{d\sigma}{\omega(\sigma^{1/rq})^q}, \tag{1.4.92}$$

or

$$\frac{d}{dt}\Lambda_{qr}(W(t)) \leq K_q e^{-qt} F^q(t) + \frac{d}{dt}\Lambda_{qr}(d(t)), \tag{1.4.93}$$

where Λ_{qr} is defined by (1.4.88) and $\alpha(t) = 2^{q-1}a^q(t)$. Integrating (1.4.93) from 0 to t, we can obtain (1.4.86). □

1.4.3 One-Dimensional Henry Type Inequalities with Multiple Integrals

In this section, we shall introduce some one-dimensional Henry type inequalities with multiple integrals.

Lemma 1.4.2 *If $H(t)$ is a C^1-function on $[0,T), H(t) \geq 0$ for all $t \in [0,T)$, and $H(0) = 0$, then for all $t \in [0,T)$,*

$$\int_0^t \frac{H'(s)}{\omega(V(s))}ds \geq \frac{H(t)}{\omega(V(t))}. \tag{1.4.94}$$

Proof Integrating by parts on the left hand-side of (1.4.94), we obtain

$$\int_0^t \frac{H'(s)}{\omega(V(s))}ds = \frac{H(t)}{\omega(V(t))} + \int_0^t H(s)\frac{\omega'(V(s))}{[\omega(V(s))]^2}V'(s)ds \geq \frac{H(t)}{\omega(V(t))}.$$

□

The following theorem is a modification of Theorem 1.4.1.

Theorem 1.4.9 (Medveď [396]) *Let $a(t), a'(t), \cdots, a^{(m-1)}(t)$ ($a^{(i)} = \frac{d^i a}{dt^i}$) be non-negative, continuous function on $[0,T)$ ($0 < T \leq +\infty$), $F_i(t)$ ($i = 1, 2, \cdots, m$) non-negative, continuous functions on $[0,T)$, ω as in Theorem 1.4.7, and let $u(t)$ be*

a continuous, non-negative function on $[0, T)$ *satisfying*

$$\begin{aligned} u(t) &\le a(t) + \int_0^t (t-s)^{\beta_1 - 1} F_1(s)\omega(u(s))ds \\ &\quad + \int_0^t \int_0^{t_1} (t_1 - s)^{\beta_2 - 1} F_2(s)\omega(u(s))dsdt_1 + \cdots \\ &\quad + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} (t_{m-1} - s)^{\beta_m - 1} F_m(s)\omega(u(s))ds \cdots dt_1, \end{aligned} \tag{1.4.95}$$

where $\beta_i > 1/2 \quad (i = 1, 2, \cdots, m)$ *and* ω *satisfies the condition* (q) (i.e., (1.4.1)) *for* $q = 2$*. Then for all* $t \in [0, T_1]$,

$$u(t) \le e^t \chi^{1/2}(t), \tag{1.4.96}$$

where

$$\begin{cases} \chi(t) = \omega^{-1}\big(\omega((m+1)a^2(t)) + G(t)\big), & (1.4.97) \\ G(t) = h_1(t) + \displaystyle\int_0^t h_2(s)ds + \cdots + \int_0^t \int_0^{t_1} \int_0^{t_{m-1}} h_m(s)ds \cdots dt_1, & (1.4.98) \\ h_i(t) = \eta_i (m+1) F_i^2(t) R(t), \quad \eta_i = \dfrac{\Gamma(2\beta_i - 1)}{2^{2\beta_i + m - 1}}, \ i = 1, 2, \cdots, m & (1.4.99) \end{cases}$$

and $T_1 \in \mathbb{R}_0$ *is such that* $\omega((m+1)a^2(t)) + G(t) \in Dom\ (\omega^{-1})$ *for all* $t \in [0, T_1]$.

Proof Indeed, the inequality (1.4.95) yields

$$\begin{aligned} u(t) &\le a(t) + \left[\int_0^t (t-s)^{2\beta_1 - 2} e^{2s} ds\right]^{\frac{1}{2}} \left[\int_0^t F_1^2(s) e^{-2s} \omega(u(s))^2 ds\right]^{\frac{1}{2}} \\ &\quad + \cdots + \left[\int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} (t_{m-1} - s)^{2\beta_m - 2} e^{2s} ds \cdots dt_1\right]^{\frac{1}{2}} \\ &\quad \times \left[\int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} F_1^2(s) e^{-2s} \omega(u(s))^2 ds \cdots dt_1\right]^{\frac{1}{2}} \\ &\le a(t) + e^t \eta_1^{\frac{1}{2}} \left[\int_0^t F_1^2(s) e^{-2s} \omega(u(s))^2 ds\right]^{\frac{1}{2}} + \cdots \\ &\quad + e^t \eta_m^{\frac{1}{2}} \left[\int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} F_m^2(s) e^{-2s} \omega(u(s))^2 ds \cdots dt_1\right]^{\frac{1}{2}}, \end{aligned} \tag{1.4.100}$$

where η_i $(i = 1, 2, \cdots, m)$ are defined by (1.4.99) and we have used the following estimate

$$\begin{aligned}
&\int_0^t \int_0^{t_1} \cdots \int_0^{t_{i-1}} (t_{i-1} - s)^{2\beta_i - 1} e^{2s} ds \cdots dt_1 \\
&= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{i-2}} e^{2t_{i-1}} \int_0^{t_{i-1}} \sigma^{2\beta_i - 1} e^{-2\sigma} d\sigma \cdots dt_1 \\
&\le \frac{e^{2t}}{2^{2\beta_i}} \Gamma(2\beta_i - 1) \int_0^t \int_0^{t_1} \cdots \int_0^{t_{i-2}} e^{2t_{i-2}} dt_{i-1} \cdots dt_1 \\
&\le \frac{e^{2t}\Gamma(2\beta_i - 1)}{2^{2\beta_i + i - 1}}, \quad i = 1, 2, \cdots, m.
\end{aligned} \tag{1.4.101}$$

Thus we derive from (1.4.100) and (1.4.2)

$$\begin{aligned}
u^2(t) \le (m+1)\Big[& a^2(t) + e^{2t}\eta_1 \int_0^t F_1^2(s) e^{-2s} \omega^2(u(s)) ds + \cdots \\
& + e^{2t}\eta_m \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} F_m^2(s) e^{-2s} \omega^2(u(s)) ds \cdots dt_1 \Big]
\end{aligned} \tag{1.4.102}$$

which, by the property (q) (i.e., (1.4.1)) for $q = 2$, implies

$$\begin{aligned}
v(t) \le (m+1)\Big[& a^2(t) + \eta_1 \int_0^t F_1^2(s) R(s) \omega(u(s)) ds + \cdots \\
& + \eta_m \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} F_m(s) R(s) \omega(u(s)) ds \cdots dt_1 \Big],
\end{aligned} \tag{1.4.103}$$

where

$$v(t) = (e^{-t} u(t))^2. \tag{1.4.104}$$

Let $V(t)$ be the right-hand side of (1.4.103) and

$$\alpha(t) = (m+1) a^2(t), \quad h_i(t) = c_i^2 \eta_i (m+1) F_i^2(t) R(t). \tag{1.4.105}$$

Then for all $t \in [0, t)$,

$$\begin{cases}
V'(t) - \alpha'(t) - h_1(t)\omega(v(t)) = V_1(t), & (1.4.106) \\
V_1'(t) - h_2(t)\omega(v(t)) = V_2(t), \cdots, & (1.4.107) \\
V_{m-2}'(t) - h_{m-1}\omega(v(t)) = V_{m-1}(t), & (1.4.108) \\
V_{m-1}'(t) = h_m(t)\omega(v(t)) \le h_m(t)\omega(V(t)). & (1.4.109)
\end{cases}$$

Using (1.4.94) and (1.4.109), we have

$$\frac{V_{m-1}(t)}{\omega(V(t))} \le \int_0^t \frac{V'_{m-1}(s)}{\omega(V(t))}ds \le \int_0^t h_m(s)ds. \tag{1.4.110}$$

By (1.4.107), (1.4.94) and (1.4.110), we get

$$\begin{aligned}\frac{V_{m-2}(t)}{\omega(V(t))} &\le \int_0^t \frac{V'_{m-2}(s)}{\omega(V(t))}ds \le \int_0^t h_{m-1}(s)ds + \int_0^t \frac{V_{m-1}(s)}{\omega(V(s))}ds \\ &\le \int_0^t h_{m-1}(s)ds + \int_0^t \int_0^{t_1} h_m(s)dsdt_1. \end{aligned}\tag{1.4.111}$$

Proceeding in this way, we can prove

$$\begin{aligned}\frac{V_1(t)}{\omega(V(t))} \le{} & \int_0^t h_2(s)ds + \int_0^t \int_0^{t_1} h_3 dsdt_1 + \cdots \\ & + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} h_m(s)dsdt_{m-1}\cdots dt_1, \end{aligned}\tag{1.4.112}$$

which implies

$$\begin{aligned}\frac{V'(t)}{\omega(V(t))} - \frac{\alpha'(t)}{\omega(\alpha(t))} &\le \frac{V'(t)-\alpha'(t)}{\omega(V(t))} \le h_1(t) + \frac{V_1(t)}{\omega(V(t))} \\ &\le h_1(t) + \int_0^t h_2(s)ds + \int_0^t \int_0^{t_1} h_3 dsdt_1 + \cdots \\ &\quad + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} h_m(s)dsdt_{m-1}\cdots dt_1 := G(t) \end{aligned}\tag{1.4.113}$$

whence for all $t \in [0, T_1)$,

$$v(t) \le \Omega^{-1}\Big(\omega(\alpha(t)) + G(t)\Big) \tag{1.4.114}$$

where $T_1 \in \mathbb{R}_0$ is as in Theorem 1.4.9. Therefore, using (1.4.104), we obtain (1.4.96). □

Remark 1.4.3 The assertion for the case $\beta_j = \frac{1}{z+1}, z \ge 1$ for all j and its proof is similar to the assertion *(1.4.2)* of Theorem *1.4.1*. We do not formulate it here. The case $\beta_i > \frac{1}{z+1}$ for a real number $z \ge 1$ is more complicated and we also do not formulate any result concerning this case.

Theorem 1.4.10 (Dixon-Mckee [188]) *Let the function x be continuous and non-negative on the interval $[0, T]$. If for all $0 \le t \le T$,*

$$x(t) \le \phi(t) + M \int_0^t \int_0^{t_m} \cdots \int_0^{t_1} \frac{x(s)}{(t_1 - s)^\alpha} ds dt_1 \cdots dt_m, \tag{1.4.115}$$

where $\alpha < 1,\ m \ge 1,\ M > 0$ is constant, and $\phi(t)$ is a non-negative, non-decreasing continuous function in $t \in [0, T]$, then for all $0 \le t \le T$,

$$x(t) \le \phi(t) E_{1-(\alpha-m)}(M\Gamma(1-\alpha)t^{1-(\alpha-m)}), \tag{1.4.116}$$

where $E_{1-\beta}(z)$ is the Mittag-Leffler function defined for any β by

$$E_{1-\beta}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(n(1-\beta)+1)}$$

and $\Gamma(a)$ is the Gamma function defined for $Re a > 0$ by $\Gamma(a) = \int_0^{+\infty} w^{a-1} e^{-w} dw$. The exponential function, which is obtained when $\beta = 0$, is a special case of the Mittag-Leffler function.

The Mittag-Leffler function has been studied in some detail in the literature, for references, see Erdelyi [205].

Proof For $m \ge 1$, by interchanging the order of integration,

$$\int_0^t \int_0^{t_m} \cdots \int_0^{t_1} \frac{x(s)}{(t_1 - s)^\alpha} ds dt_1 \cdots dt_m = \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} \int_0^t (t-s)^{m-\alpha} x(s) ds.$$

Consequently, the inequality (1.4.115) is equivalent to

$$x(t) \le \phi(t) + \int_0^t k(t,s) x(s) ds$$

where the kernel $k(t, s)$ given by

$$k(t,s) = \frac{\Gamma(1-\alpha)(t-s)^{m-\alpha}}{\Gamma(1-\alpha+m)}, \quad 0 \le s \le t \le T, \tag{1.4.117}$$

and $\alpha < 1,\ m \ge 1$, is continuous and non-negative.
Thus using Theorem 1.2.38,

$$x(t) \le y(t)$$

where $y(t)$ is the solution of (1.2.286) is given by for all $0 \le t \le T$,

$$y(t) = \phi(t) + \int_0^t \Gamma(t,s)y(s)ds,$$

where for all $0 \le t \le T$,

$$\Gamma(t,s) = \sum_{n=1}^{+\infty} k^{(n)}(t,s),$$

is a the resolvent kernel of $k(t,s)$ and $k^{(n)}(t,s)$ are the iterated kernels of $k(t,s)$ defined by

$$k^{(1)}(t,s) = k(t,s), \quad k^{(n)}(t,s) = \int_s^t k(t,u)k^{(n-1)}(t,u)du, \ n \ge 2.$$

Using mathematical induction it can be shown that the iterated kernels satisfy

$$k^{(n)}(t,s) = \frac{M^n \Gamma(1-\alpha)^n (t-s)^{n(m+1-\alpha)-1}}{\Gamma(n(1-\alpha+m))}, \ n = 1, 2, \ldots \tag{1.4.118}$$

Hence, we obtain for all $0 \le t \le T$,

$$\begin{aligned} x(t) &\le \phi(t) + \sum_{n=1}^{+\infty} \frac{(M\Gamma(1-\alpha))^n}{\Gamma(n(1-\alpha+m))} \int_0^t (t-s)^{n(m+1-\alpha)-1}\phi(s)ds \\ &\le \phi(t)E_{1-(\alpha-m)}(M\Gamma(1-\alpha)t^{1-(\alpha-m)}). \end{aligned} \tag{1.4.119}$$

The proof is now complete. □

Remark 1.4.4 If $\alpha = 0$, then (1.4.116) reduces to

$$x(t) \le \phi(t) \cos h(M^{\frac{1}{2}}t).$$

Note that in the case $\phi(t) = \phi, \ 0 \le t \le T$, (1.4.116) is the best possible result since equality in (1.4.115) implies equality in (1.4.116). For a more general $\phi(t)$, the best possible result is given by

$$x(t) \le \frac{d}{dt}\int_0^t E_{1-(\alpha-m)}(M\Gamma(1-\alpha)(t-s)^{1-(\alpha-m)})\phi(s)ds, \tag{1.4.120}$$

where the right-hand side of (1.4.116) is the solution of the integral equation (1.2.286) with kernel (1.4.117). We also remark that if $\alpha \leq 0$ Theorem 1.2.38 remains valid if $m = 0$, that is, if (1.4.115) involves a single, rather that repeated, integral and in this case Theorem 1.4.10 is an example of Theorem 1.2.39. If $0 < \alpha < 1$ and $m = 0$, then the kernel $k(t, s) = M/(t - s)^{\alpha}$ is weakly singular; Gronwall inequalities where the kernel of the associated integral equation is weakly singular can be found in Dixon and McKee [187].

Chapter 2
Linear One-Dimensional Discrete (Difference) Inequalities

2.1 Linear One-Dimensional Discrete Gronwall-Bellman Inequalities and Their Generalizations

It is well-known that discrete inequalities play a vital role in the continuing development of the theory of difference equations. It appears in the literature that none of the results deals directly with discrete inequalities that involves higher order differences.

In this section, we shall introduce linear discrete Gronwall-Bellman inequalities. Recurrent inequalities involving sequences of real numbers, which may be considered as discrete analogous of Theorem 1.1.1, have been extensively used in the analysis of finite difference equations. For an elementary introduction to application of such results to numerical solutions of ordinary differential equations, we refer to the book by Henrici [271].

Discrete analogues of Theorem 1.1.1 have also proved to be very useful in the numerical solutions of partial differential equations. Before we mention some of the typical results in this direction, we prove the following basic result which can be found in [299].

Theorem 2.1.1 (Hull-Luxemburg [299]) *Let m be a positive integer, $u_0, u_1, \ldots, u_m$ a sequence of $(m+1)$ non-negative numbers, and $z_0, z_1, \ldots, z_m$ a non-decreasing sequence of $(m+1)$ real numbers.*

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Furthermore, let $\{f_m\}$ *be a non-negative non-decreasing sequence and* $L \geq 0$. *Suppose that the following inequality is valid for* $l = 1, 2, \ldots, m$,

$$\begin{aligned} u_l &\leq f_l + L\sum_{j=0}^{l-1} u_j(z_{j+1} - z_j) \\ &= \{f_l + Lu_0(z_1 - z_0)\} + L\sum_{j=1}^{l-1} u_j(z_{j+1} - z_j). \end{aligned} \tag{2.1.1}$$

Then the next inequality holds for $l = 1, 2, \ldots, m$,

$$u_l \leq \{f_l + Lu_0(z_1 - z_0)\}\prod_{j=1}^{l}[1 + L(z_j - z_{j-1})]. \tag{2.1.2}$$

Proof Set $h_j = (z_{j+1} - z_j), j = 0, 1, \ldots, m - 1$. By hypothesis, we get

$$u_l \leq f_l + Lu_0h_0 + L\sum_{j=1}^{l-1} u_jh_j.$$

Since $1 + Lh_0 \geq 1$, the inequality (2.1.2) certainly holds for $l = 1$. Suppose that it is true for $l \leq n - 1$. Then we shall show that it is true for $l = n$. Now since $\{f_n\}$ is non-decreasing, we derive

$$\begin{aligned} u_n &\leq (f_n + Lu_0h_0) + L\sum_{j=1}^{n-1} u_jh_j \\ &\leq (f_n + Lu_0h_0) + L\sum_{j=1}^{n-1} h_j(f_j + Lu_0h_0)\prod_{i=1}^{j}(1 + Lh_{i-1}) \\ &\leq (f_n + Lu_0h_0)\Big\{1 + L\sum_{j=1}^{n-1} h_j\prod_{i=1}^{j}(1 + Lh_{i-1})\Big\} \\ &\leq (f_n + Lu_0h_0)\prod_{j=1}^{n-1}(1 + Lh_{j-1}) \end{aligned}$$

and

$$\begin{aligned} &1 + L\sum_{j=1}^{n-1} h_j\prod_{i=1}^{j}(1 + Lh_{i-1}) \\ &= 1 + Lh_1(1 + Lh_0) + Lh_2(1 + Lh_0)(1 + Lh_1) + \cdots \end{aligned}$$

$$
\begin{aligned}
&+Lh_{n-1}(1+Lh_0)\cdots(1+Lh_{n-2})\\
&\le (1+Lh_0)\{1+Lh_1+Lh_2(1+Lh_1)+\cdots\\
&\quad +Lh_{n-1}(1+Lh_1)\cdots(1+Lh_{n-2})\}\\
&=(1+Lh_0)(1+Lh_1)\cdots(1+Lh_{n-1})\\
&=\prod_{j=1}^{n}(1+Lh_{j-1}).
\end{aligned}
$$

This hence completes the proof. □

By setting $f_i = \varepsilon$ in Theorem 2.1.1, we arrive at the "convergence inequality" which Diaz [184] employed in developing an analogue of the classical Euler-Cauchy polygon method for the solutions of characteristic boundary value problems for a class of nonlinear hyperbolic equations. Similarly, in the investigation of convergence properties of several finite difference schemes for nonlinear parabolic equations, Lees [357] has used the following theorem.

Theorem 2.1.2 (Lees [357]) *Let u and f be non-negative functions defined on the integers* $1, 2, \ldots, m$. *Let f be non-decreasing. If there holds that*

$$u_l \le f_l + Lk\sum_{i=1}^{l-1} u_i, \quad l = 1, 2, .., m, \tag{2.1.3}$$

where $u_i = u(i), f_i = f(i)$ and L and k are positive constants, then

$$u_l \le f_l \exp(Lkl), \quad l = 1, 2, .., m. \tag{2.1.4}$$

Proof The theorem is readily derived by setting $u_0 = 0$ and $(z_j - z_{j-1}) \equiv k, k > 0$, for $j = 1, 2, \ldots, m$. For, under these assumptions, by Theorem 2.1.1, (2.1.3) implies

$$u_l \le f_l \prod_{i=1}^{l}(1 + Lk) \le f_l \exp(Lkl).$$

The proof is thus complete. □

Note that this theorem can also be considered as a corollary of Theorem 2.1.4, see Corollary 2.1.5 below.

For other useful inequalities which may be considered as discrete analogues of Theorem 1.1.1, we refer to Hull and Luxemburg [299], Jones [305], Li [360], and Willett and Wong [648].

In the sequels, we shall introduce some notations. Let $\alpha \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N}_\alpha = \{\alpha + n : n \in \mathbb{N}_0\}$, $\Delta u(n) = u(n+1) - u(n)$, $n \in \mathbb{N}_0$.

One of the simplest discrete inequalities is stated in the following theorem.

Theorem 2.1.3 (Agarwal [10]) *Let u_n, f_n and $b_n \geq -1$ be sequences defined for all $n \in \mathbb{N}_\alpha$ satisfying the inequality for all $n \in \mathbb{N}_\alpha$,*

$$\Delta u_n \leq b_n u_n + f_n. \tag{2.1.5}$$

Then for all $n \in \mathbb{N}_\alpha$,

$$u_n \leq u_\alpha \prod_{s=\alpha}^{n-1}(1+b_s) + \sum_{s=\alpha}^{n-1} f_s \prod_{i=s+1}^{n-1}(1+b_i). \tag{2.1.6}$$

Proof In fact, we may rewrite (2.1.5) as

$$u_{s+1} - (1+b_s)u_s \leq f_s. \tag{2.1.7}$$

Multiplying (2.1.7) by $\prod_{i=s+1}^{n-1}(1+b_i)$, we have

$$u_{s+1} \prod_{i=s+1}^{n-1}(1+b_i) - u_s \prod_{i=s}^{n-1}(1+b_i) \leq f_s \prod_{i=s+1}^{n-1}(1+b_i). \tag{2.1.8}$$

Thus summation from α to $n-1$ yields

$$u_n \prod_{i=n}^{n-1}(1+b_i) - u_\alpha \prod_{i=\alpha}^{n-1}(1+b_i) \leq \sum_{s=\alpha}^{n-1} f_s \prod_{i=s+1}^{n-1}(1+b_i)$$

which implies (2.1.6). □

Corollary 2.1.1 *Let $\triangle u_n \leq f_n$, then*

$$u_n \leq u_\alpha + \sum_{s=\alpha}^{n-1} f_s.$$

Corollary 2.1.2 *Let $\triangle u_n \leq b_n u_n$, then*

$$u_n \leq u_\alpha \prod_{s=\alpha}^{n-1}(1+b_s).$$

Theorem 2.1.4 (Pachpatte [449]) *Let $u_n, a_n, b_n \geq 0, q_n \geq 0$ be sequences defined satisfying the inequality for all $n \in \mathbb{N}_\alpha$,*

$$u_n \leq a_n + q_n \sum_{s=\alpha}^{n-1} b_s u_s. \tag{2.1.9}$$

Then for all $n \in \mathbb{N}_\alpha$,

$$u_n \leq a_n + q_n \sum_{s=\alpha}^{n-1} b_s a_s \prod_{i=s+1}^{n-1} (1 + b_i q_i). \tag{2.1.10}$$

Proof Setting

$$w_n = \sum_{s=\alpha}^{n-1} b_s u_s, \quad w_\alpha = 0,$$

we can obtain

$$\begin{cases} u_n \leq a_n + q_n w_n, \quad \triangle w_n = b_n u_n, & (2.1.11)\\ \triangle w_n \leq b_n a_n + b_n q_n w_n. & (2.1.12)\end{cases}$$

Applying Theorem 2.1.3 to (2.1.12), we can obtain

$$w_n \leq \Big(\sum_{s=\alpha}^{n-1} b_s a_s \Big) \prod_{i=s+1}^{n-1} (1 + b_i q_i)$$

which, along with (2.1.11), implies (2.1.10). □

Corollary 2.1.3 (Sugiyama [612]) *If, for all* $n \in \mathbb{N}_\alpha$,

$$u_n \leq a_n + \sum_{s=\alpha}^{n-1} b_s u_s, \tag{2.1.13}$$

where $b_s \geq 0, s \in \mathbb{N}_\alpha$, *then for all* $n \in \mathbb{N}_\alpha$,

$$u_n \leq a_n + \sum_{s=\alpha}^{n-1} b_s a_s \Pi_{i=s+1}^{n-1} (1 + b_i). \tag{2.1.14}$$

Corollary 2.1.4 (Sugiyama [611]) *Let* $x(n)$ *and* $f(n)$ *be real-valued functions defined for all* $n \in \mathbb{N}$, *and suppose that* $f(n) \leq 0$ *for every* $n \in \mathbb{N}$. *If, for all* $n \in \mathbb{N}$,

$$x(n) \leq x_0 + \sum_{s=n_0}^{n-1} f(s) x(s),$$

where $\mathbb{N}$ *is the set of points* $n_0 + k \quad (k = 0, 1, 2, \cdots)$, $n_0 \leq 0$ *is a given integer and* x *is a non-negative constant, then for all* $n \in \mathbb{N}$,

$$x(n) \leq x_0 \prod_{s=n_0}^{n-1} [1 + f(s)].$$

Corollary 2.1.5 (Lees [357]) *If* $b_n \geq 0$ *and* a_n *is non-decreasing for all* $n \in \mathbb{N}_\alpha$, *then (2.1.13) implies that for all* $n \in \mathbb{N}_\alpha$,

$$u_n \leq a_n \prod_{s=\alpha}^{n-1}(1 + b_s) \leq a_n \exp\left(\sum_{s=\alpha}^{n-1} b_s\right). \tag{2.1.15}$$

Corollary 2.1.6 (Beesack [51]) *If* $b_n \geq 0$ *and* a_n *is non-decreasing for all* $n \in \mathbb{N}_\alpha$, *then (2.1.13) implies that for all* $n \in \mathbb{N}_\alpha$,

$$u_n \leq a_n \prod_{s=\alpha}^{n-1}(1 + b_s) \leq a_n \exp\left(\sum_{s=\alpha}^{n-1} b_s\right). \tag{2.1.16}$$

Corollary 2.1.7 (Gronwall [239]) *Under assumptions of Theorem 2.1.4, inequalities (2.1.9)–(2.1.10) implies that for all* $n \in \mathbb{N}_\alpha$,

$$\sum_{s=\alpha}^{n-1} b_s u_s \leq \sum_{s=\alpha}^{n-1} b_s a_s \prod_{i=s+1}^{n-1} (1 + b_i q_i). \tag{2.1.17}$$

Proof Estimate (2.1.17) follows from (2.1.15) with $q_i = 1$, and (2.1.9)–(2.1.10). Estimate (2.1.17) has been obtained in the process of proving Theorem 2.1.4. □

The next result is a consequence of Corollaries 2.1.3 and 2.1.4.

Corollary 2.1.8 (Sugiyama [611]) *If a non-negative sequence* $y_n, n = 0, \ldots, N$, *satisfies*

$$y_0 = 0, \qquad y_n \leq A + Bh \sum_{j=0}^{n-1} y_j, \qquad 1 \leq n \leq N, \ \ h = 1/N, \tag{2.1.18}$$

then

$$\max_{0 \leq i \leq N} y_i \leq Ae^B, \tag{2.1.19}$$

where A *and* B *are positive constants independent of* h.

Note that Corollary 2.1.8 plays an important role in proving convergence of numerical solutions of Volterra integral equations with a continuous kernel

[168, 174, 385]. However, it is ineffective to prove convergence of numerical solutions of Volterra integral equations of the second kind with a weakly singular kernel. For example, we consider the following nonlinear Volterra equation of the second kind

$$u(s) = y(s) + \int_a^s k^*(s,t,u(t))dt, \qquad a \le s \le b, \tag{2.1.20}$$

where the kernel

$$k^*(s,t,u(t)) = (s-t)^\alpha (\ln|s-t|)^\beta k(s,t,u(t)), \qquad -1 < \alpha \le 0, \quad \beta = 0,1, \tag{2.1.21}$$

is weakly singular and $k(s,t,u(t))$ is a continuous function on variables s,t,u, especially, there exists a positive constant L satisfying, for all $t,s \in [a,b]$,

$$|k(s,t,u) - k(s,t,v)| \le L|u-v|. \tag{2.1.22}$$

Moreover, for fixed s and t, we assume that $k(s,t,u(t))$ has high order derivatives on u, and let $k_u(s,t,u(t)) = \frac{\partial}{\partial u}k(s,t,u(t))$. In order to get a discrete version of (2.1.20), we can apply the quadrature formula in Navot [426] and Lyness [372] of computing integrals with the end point singularity.

Consider the integral

$$I(G) = \int_a^b G(x)dx = \int_a^b (b-a)^\alpha (\ln|b-x|)^\beta g(x)dx, \tag{2.1.23}$$

where $-1 < \alpha < 0, \beta = 0,1$, and $G(x) = (b-a)^\alpha(\ln|b-x|)^\beta g(x)$, $g(x)$ is smooth on $[a,b]$. Take the step width $h = (b-a)/N$, and $x_i = a+ih, i = 0,\cdots,N$. If $g(x) \in C^{2m}[a,b]$, then Navot [426] and Lyness [372] proved that the quadrature formula

$$Q_N(G) = \frac{h}{2}G(x_0) + h\sum_{i=1}^{N-1} G(x_i) - [-\beta\zeta'(-\alpha) + \zeta(-\alpha)(\ln h)^\beta]g(b)h^{1+\alpha} \tag{2.1.24}$$

possesses the following Euler-Maclaurin asymptotic expansion

$$\begin{aligned} E_N(G) &= Q_N(G) - I(G) \\ &= \sum_{j=1}^{m-1} \frac{B_{2j}}{(2j)!}G^{(2j-1)}(a)h^{2j} + \sum_{j=1}^{2m-1}(-1)^j[-\beta\zeta'(-\alpha-j) + \zeta(-\alpha-j)(\ln h)^\beta] \\ &\quad \times g^{(j)}(b)h^{j+\alpha+1} + O(h^{2m}), \end{aligned} \tag{2.1.25}$$

where B_{2j} are Bernoulli numbers, and $\zeta(x)$ is the Riemann Zeta function. From (2.1.26) it follows that if $g(x) \in C^2[a, b]$, then

$$E_N(G) = O(h^{2+\alpha}|\ln h|^\beta). \tag{2.1.26}$$

Taking $s = x_i$ in (2.1.20) and using the quadrature formula (2.1.24) for

$$u(x_i) = y(x_i) + \int_{x_0}^{x_i} (x_i - t)^\alpha (\ln |x_i - t|)^\beta k(x, t, u(t))dt, \tag{2.1.27}$$

we obtain the following nonlinear discrete equations: find $u_i, i = 0, 1, \cdots, N$, satisfying

$$\begin{cases} u_0 = y(x_0), \\ u_i = y(x_i) + \dfrac{h}{2}(x_i - x_0)^\alpha (\ln |x_i - x_0|)^\beta k(x_i, x_0, u_0) \\ \qquad + h\displaystyle\sum_{j=1}^{i-1} (x_i - x_j)^\alpha (\ln |x_i - x_j|)^\beta k(x_i, x_j, u_j) \\ \qquad - [-\beta\zeta'(-\alpha) + \zeta(-\alpha)(\ln h)^\beta]k(x_i, x_i, u_i)h^{1+\alpha}, \qquad i = 1, \ldots, N. \end{cases} \tag{2.1.28}$$

But by (2.1.25), the integral equation (2.1.20) can be expressed as

$$\begin{aligned} u(x_i) = y(x_i) + h\sum_{j=0}^{i-1} \omega_{ij}(x_i - x_j)^\alpha (\ln |x_i - x_j|)^\beta k(x_i, x_j, u(x_j)) \\ +h\omega_{ii}k(x_i, x_i, u(x_i)) + E_{i,t}((x_i - t)^\alpha (\ln |x_i - t|)^\beta k(x_i, t, u(t)), \\ i = 0, 1, \cdots N, \end{aligned} \tag{2.1.29}$$

where

$$\omega_{i0} = \frac{1}{2}, \quad \omega_{ii} = h^\alpha[\beta\zeta'(-\alpha) - \zeta(-\alpha)(\ln h)^\beta], \quad \omega_{ij} = 1, \quad for \quad 1 \le j < i, \tag{2.1.30}$$

and the remainder satisfies an estimate

$$|E_{i,t}((x_i - t)^\alpha (\ln |x_i - t|)^\beta k(x_i, t, u(t))| = O(h^{2+\alpha}|\ln h|^\beta). \tag{2.1.31}$$

Letting $e_i = u(x_i) - u_i$ and subtracting (2.1.28) from (2.1.29), we obtain the error e_i satisfies the equation

$$\begin{cases} e_0 = 0, \\ e_i = h\sum_{j=0}^{i-1}\omega_{ij}(x_i - x_j)^{\alpha}(\ln|x_i - x_j|)^{\beta}[k(x_i, x_j, u(x_j)) - k(x_i, x_j, u_j)] \\ \qquad + h\omega_{ii}[k(x_i, x_i, u(x_i)) - k(x_i, x_i, u_i)] + E_{i,t}(x_i, t, u(t)), \qquad 1 \le i \le N. \end{cases} \tag{2.1.32}$$

Thus it follows from (2.1.22) that for all $1 \le i \le N$,

$$|e_i| \le Lh\sum_{j=1}^{i-1}\omega_{ij}(x_i - x_j)^{\alpha}(\ln|x_i - x_j|)^{\beta}|e_j| + Lh\omega_{ii}|e_i| + |E_{i,t}(x_i, t, u(t))|. \tag{2.1.33}$$

Let h be so small that $Lh\omega_{ii} \le \frac{1}{2}$, then we easily derive that

$$\begin{cases} e_0 = 0, \\ |e_i| \le 2Lh\sum_{j=1}^{i-1}(x_i - x_j)^{\alpha}(\ln|x_i - x_j|)^{\beta}|e_j| + 2|E_{i,t}(x_i, t, u(t))|, \quad 1 \le i \le N. \end{cases} \tag{2.1.34}$$

Let

$$\begin{cases} A = \max_{1\le i\le N}\max_{a\le t\le b}|2E_{i,t}(x_i, t, u(t))|, \\ B_{ij} = 2Lh(x_i - x_j)^{\alpha}(\ln|x_i - x_j|)^{\beta}, \ \ for \ \ i > j, \ -1 < \alpha < 0, \ \beta = 0 \ or \ 1, \end{cases} \tag{2.1.35}$$

then (2.1.34) can be simplified as

$$\begin{cases} |e_0| = 0, \\ |e_i| \le A + \sum_{j=1}^{i-1}B_{ij}|e_j|, \qquad 1 \le i \le N. \end{cases} \tag{2.1.36}$$

Thus the convergence and error estimate of the approximation equation (2.1.28) reduce to estimate $\{|e_i|\}$ satisfying (2.1.36). Unfortunately, if A and B_{ij} in (2.1.36) are defined by (2.1.35), then the discrete Gronwall inequality (2.1.19) in Corollary 2.1.8 does not hold.

However, instead of (2.1.19), we shall prove a new generalization of the discrete Gronwall inequality.

Theorem 2.1.5 (Lü-Huang [368]) *If A and B_{ij} are defined by (2.1.35) and e_i satisfies the inequality (2.1.36), then there is a positive constant c, independent of h, such that*

$$|e_i| \le ch^{2+\alpha}(\ln h)^{\beta}. \tag{2.1.37}$$

Proof Since $e_0 = 0$, successively substituting the right-hand side of (2.1.36) when $i = n-1, \cdots, 1$ into

$$|e_n| \le A + \sum_{j_1=1}^{n-1} B_{nj}|e_j|,$$

we derive

$$\begin{aligned} |e_n| \le A + A\sum_{j_1=1}^{n-1} B_{nj_1} + A\sum_{j_2=1}^{n-1} B_{nj_2}\sum_{j_1=1}^{j_2-1} B_{j_2j_1} \\ + \cdots + A\sum_{j_{n-1}=1}^{n-1} B_{nj_{n-1}}\sum_{j_{n-2}=1}^{j_{n-1}-1} B_{j_{n-1}j_{n-2}}\cdots\sum_{j_1=1}^{j_2-1} B_{j_2j_1}. \end{aligned} \tag{2.1.38}$$

In order to estimate $\{|e_n|\}$, we shall use the following simple inequality: if a non-negative function $f(x)$ is monotone on $[0, n]$, then

$$\sum_{i=1}^{n-1} f(i) \le \int_0^n f(x)dx. \tag{2.1.39}$$

We divide two cases to discuss.

Case 1: $\beta = 0$, i.e., $B_{ij} = 2Lh(x_i - x_j)^{\alpha}$.
Since x^{α} is monotone and non-negative, using (2.1.39) and setting $x = j_2y$, we get

$$\begin{aligned} \sum_{j_1=1}^{j_2-1} B_{j_2j_1} &= 2Lh^{1+\alpha}\sum_{j_1=1}^{j_2-1}(j_2 - j_i)^{\alpha} \\ &\le 2Lh^{1+\alpha}\int_0^{j_2}(j_2 - x)^{\alpha}dx = 2Lh^{1+\alpha}B(1+\alpha, 1)j_2^{1+\alpha}, \end{aligned} \tag{2.1.40}$$

where $B(r, s)$ denote Beta function. Similarly,

$$
\begin{aligned}
F_{n,k} &= A \sum_{j_k=1}^{n-1} B_{nj_k} \sum_{j_{k-1}=1}^{j_k-1} B_{j_k j_{k-1}} \cdots \sum_{j_1=1}^{j_2-1} B_{j_2 j_1} \\
&\le A(2Lh^{1+\alpha})^k n^{k(1+\alpha)} B(1+\alpha, 1) \cdots B(1+\alpha, (k-1)(1+\alpha)+1).
\end{aligned}
\tag{2.1.41}
$$

However,

$$
\begin{aligned}
B(1+\alpha, m(1+\alpha)+1) &= \frac{\Gamma(1+\alpha)\Gamma(m(1+\alpha)+1)}{\Gamma((m+1)(1+\alpha)+1)} \\
&= \frac{m}{m+1}\Gamma(1+\alpha)\frac{\Gamma(m(1+\alpha))}{\Gamma((m+1)(1+\alpha))}.
\end{aligned}
\tag{2.1.42}
$$

By the Stirling formula, there exists $\theta_z \in (0, 1)$ such that

$$\Gamma(z) = \sqrt{2}\pi z^{z-\frac{1}{2}} e^{-z} e^{\frac{\theta_z}{12z}}.$$

Letting

$$s = 1 + \alpha,$$

we derive that

$$
\begin{aligned}
\frac{\Gamma(ms)}{\Gamma((m+1)s)} &= (\frac{m}{m+1})^{ms-\frac{1}{2}}[(m+1)s]^{-s} e^s \exp\left(\frac{\theta_m}{12ms} - \frac{\theta_{m+1}}{12(m+1)s}\right) \\
&\le e^{\frac{1}{12s}} \left(\frac{e}{s}\right)^s (m+1)^{-s}.
\end{aligned}
\tag{2.1.43}
$$

Substituting (2.1.43) into (2.1.42), we obtain

$$B(s, ms+1) \le M(m+1)^{-s}, \tag{2.1.44}$$

where $M = \Gamma(s) e^{\frac{1}{12s}} (\frac{e}{s})^s$. Therefore, inserting (2.1.44) into (2.1.41), we have

$$F_{n,k} \le \frac{A(2L(b-a)^s M)^k}{(k!)^s} \le A\frac{R^k}{(k!)^s}, \tag{2.1.45}$$

where $R = 2L(b-a)sM$. Thus substituting (2.1.45) into (2.1.38), we obtain

$$|e_n| \leq A + A\sum_{k=1}^{n-1} \frac{R^k}{(k!)^s} < A\sum_{k=0}^{+\infty} \frac{R^k}{(k!)^s} = HA, \tag{2.1.46}$$

where

$$H = \sum_{k=0}^{+\infty} \frac{R^k}{(k!)^s} < +\infty,$$

and H is a positive constant independent of h. Now it follows from (2.1.26) that $A = O(h^{2+\alpha})$, which readily yields that there is a positive constant c, independent of h, satisfying,

$$\max_{0 \leq i \leq N} |e_i| \leq ch^{2+\alpha}. \tag{2.1.47}$$

Case 2: $\beta = 1$, i.e., $B_{ij} = 2Lh(x_i - x_j)^\alpha(\ln|x_i - x_j|)$.

Note that since $\alpha > -1$, we can find such an $\varepsilon > 0$, that $\alpha - \varepsilon > -1$. However, by an inequality in [410],

$$|\ln((i-j)h)| = -\ln((i-j)h) \leq \frac{((i-j)h)^{-\varepsilon}}{\varepsilon e},$$

we may derive

$$|B_{ij}| \leq 2Lh[(i-j)h]^\alpha \frac{[(i-j)h]^{-\varepsilon}}{\varepsilon e} = \frac{2L}{\varepsilon e}h[(i-j)h]^{\alpha-\varepsilon}. \tag{2.1.48}$$

Setting

$$\alpha_1 = \alpha - \varepsilon, \qquad L_1 = \frac{L}{\varepsilon e},$$

and using the results of **Case 1**, we can prove that

$$\max_{0 \leq i \leq N} |e_i| \leq ch^{2+\alpha}(\ln h)^\beta.$$

Thus the proof is complete. □

From Theorem 2.1.5, we easily prove the following corollary.

Corollary 2.1.9 *If $k(s,t,u)$ satisfies (2.1.22), then the solutions u_i of (2.1.28) converges to $\{u(x_i)\}$ as $h \to 0$, and there exists a constant c, independent of h,*

such that when h is sufficiently small, the error satisfies the estimate

$$\max_{0\leq i\leq N} |e_i| \leq ch^{2+\alpha}(\ln h)^{\beta}. \tag{2.1.49}$$

Remark 2.1.1 After Theorem 1.1.1 [239], many authors generalized Gronwall's inequality and its discrete analogue (see, e.g., [47, 49, 410, 491, 648]). The discrete analogues of the generalizations of Gronwall's inequality are often applied to the numerical treatment of differential equations and integral equations (see, e.g., [385]), however the numerical treatment of weakly singular integral equations seems to be difficult.

From Corollary 2.1.8, we can prove the following theorem which can be regarded as a corollary of Theorem 2.1.4.

Theorem 2.1.6 (McKee [393]) *Let u_n, p_n, and $b_n \geq 0$ be sequences defined for all $n \in \mathbb{N}_\alpha$ such that for all $n \in \mathbb{N}_\alpha$,*

$$u_n \leq u_\alpha + \sum_{s=\alpha}^{n-1}(b_s u_s + p_s). \tag{2.1.50}$$

Then for all $n \in \mathbb{N}_\alpha$,

$$u_n \leq u_\alpha \prod_{s=\alpha}^{n-1}(1+b_s) + \sum_{s=\alpha}^{n-1} p_s \prod_{i=s+1}^{n-1}(1+b_i). \tag{2.1.51}$$

Gronwall [239] in 1919 introduced Theorem 1.1.1. However, in numerical analysis literature, we can frequently find the corresponding discrete form of Theorem 1.1.2.

Theorem 2.1.7 (Mckee [393]) *If x_j, $j = 0, 1, \cdots, N$, is a sequence of real numbers with*

$$|x_i| \leq hM \sum_{j=0}^{i-1} |x_j| + \delta, \quad i = 1, 2, \cdots, N, \tag{2.1.52}$$

where $M > 0$ is usually independent of h $(= T/N)$ and $\delta > 0$, then

$$|x_i| \leq (hM|x_0| + \delta)e^{Mih}, \quad i = 1, 2, \cdots, N. \tag{2.1.53}$$

Proof We need only to show inductively that (2.1.52) implies

$$|x_i| \leq (hM|x_0| + \delta)(1 + hM)^{i-1} \tag{2.1.54}$$

and then observe that

$$(1 + hM)^{i-1} \le (1 + (ih)\frac{M}{i})^i \le \exp\Big(Mih\Big).$$

□

Indeed, in the numerical analysis literature, see, e.g., Henrici [271] or Linz [363], it is more common to find (2.1.53) replaced by the less sharp result

$$||x||_\infty \le (hM|x_0| + \delta)e^{MNh} = (hM|x_0| + \delta)e^{MT},$$

where $||x||_\infty = \max_{1\le i\le N} |x_i|$.

The importance of this theorem is that it is invariably employed to demonstrate the convergence of the discrete solution of some discretisation algorithm to that of its corresponding operator equation. For example, we also refer to Henrici [271] who considered ordinary differential equations or Holyhead, McKee and Taylor [288] who considered first kind Volterra integral equations.

We also note that recurrent inequalities involving sequences of real numbers, which may be considered to be discrete Gronwall inequalities, have been widely used in the analysis of finite difference equations. The book by Henrici [271] provides an elementary introduction to the application of such results to the numerical solution of ordinary differential equations.

The following lemma, which is encountered frequently in numerical analysis, may be regarded as the discrete analogue, and improves Theorems 2.1.2 and 2.1.4.

Theorem 2.1.8 (Dixon-Mckee [188]) *Let x_i, $i = 0, 1, \ldots, N$, be a sequence of non-negative real numbers satisfying*

$$x_0 \le \delta, \quad x_i \le \delta + Mh\sum_{j=0}^{i-1} x_j, \quad i = 1, 2, \ldots, N, \tag{2.1.55}$$

where δ, M are non-negative constants with M bounded independently of h $(= T/N)$, then

$$x_i \le \delta \exp(Mih), \quad i = 0, 1, \ldots, N. \tag{2.1.56}$$

Proof The proof is similar to that of Theorem 2.1.4. □

Now we present different and rather special generalizations of Theorem 2.1.1. We shall consider essentially equations with an Abel's type singularity, for example,

$$y(t) = \int_0^t \frac{k(t,s)y(s)}{(t-s)^\epsilon} ds + f(t), \quad 0 \le \epsilon < 1, \tag{2.1.57}$$

where $k(t,s)$ and $f(t)$ have sufficient continuity on their respective domains $\{0 \le s \le t \le T\}$ and $\{0 \le t \le T\}$. Because the case when the exponent ϵ is equal to $\frac{1}{2}$ is the one of most practical interest and partly because this case is helpful in understanding the general result in the following theorem.

Theorem 2.1.9 (Mckee [393]) *If* $x_j,\ j = 0, 1, \cdots, N$, *is a sequence of real numbers with*

$$|x_0| < \delta, \quad |x_i| \le h^{\frac{1}{2}} M \sum_{j=0}^{i-1} \frac{1}{(i-j)^{\frac{1}{2}}} |x_j| + \delta, \quad i = 1, 2, \cdots, N, \tag{2.1.58}$$

where $M > 0$ *is independent of* $h,\ \delta > 0$ *and* $T = Nh$, *then*

$$||x||_\infty \le \delta\left(1 + h^{\frac{1}{2}} M + hM^2\pi + 2MT^{\frac{1}{2}}\right) e^{M^2\pi T}. \tag{2.1.59}$$

Proof Multiplying (2.1.58) by $h^{\frac{1}{2}} M \sum_{j=1}^{k-1} \frac{1}{(k-i)^{\frac{1}{2}}},\quad k > i$ and summing from 1 to $k-1$, we obtain

$$h^{\frac{1}{2}} M \sum_{i=1}^{k-1} \frac{1}{(k-i)^{\frac{1}{2}}} |x_i| \le hM^2 \sum_{i=1}^{k-1} \sum_{j=0}^{i-1} \frac{1}{(k-i)^{\frac{1}{2}} (i-j)^{\frac{1}{2}}} |x_j| + h^{\frac{1}{2}} M \sum_{i=1}^{k-1} \frac{1}{(k-i)^{\frac{1}{2}}} \delta. \tag{2.1.60}$$

Thus we have, by (2.1.58)

$$\begin{aligned} |x_k| &\le h^{\frac{1}{2}} M \sum_{i=0}^{k-1} \frac{1}{(k-i)^{\frac{1}{2}}} |x_i| + \delta = h^{\frac{1}{2}} M \sum_{i=1}^{k-1} \frac{1}{(k-i)^{\frac{1}{2}}} |x_i| + h^{\frac{1}{2}} M \frac{1}{(k)^{\frac{1}{2}}} |x_0| + \delta \\ &\le h^{\frac{1}{2}} M \sum_{i=1}^{k-1} \frac{1}{(k-i)^{\frac{1}{2}}} |x_i| + \delta(1 + h^{\frac{1}{2}} M), \end{aligned}$$

since $|x_0| \le \delta$ and $h^{\frac{1}{2}} M \frac{1}{(k)^{\frac{1}{2}}} \delta \le h^{\frac{1}{2}} M\delta$, and so using (2.1.60), we obtain

$$|x_k| \le hM^2 \sum_{i=1}^{k-1} \sum_{j=0}^{i-1} \frac{1}{(k-i)^{\frac{1}{2}} (i-j)^{\frac{1}{2}}} |x_j| + h^{\frac{1}{2}} M\delta \sum_{i=1}^{k-1} \frac{1}{(k-i)^{\frac{1}{2}}} + \delta(1 + h^{\frac{1}{2}} M). \tag{2.1.61}$$

Changing the order of the summation, then we derive from (2.1.61) that

$$|x_k| \le hM^2 \sum_{j=0}^{k-2} \Big\{ \sum_{i=j+1}^{k-1} \frac{1}{(k-i)^{\frac{1}{2}}(i-j)^{\frac{1}{2}}} \Big\} |x_j| + h^{\frac{1}{2}} M\delta \sum_{i=1}^{k-1} \frac{1}{(k-i)^{\frac{1}{2}}} + \delta(1 + h^{\frac{1}{2}} M). \tag{2.1.62}$$

On the one hand, we know that by considering the summation as a Riemann sum, we have

$$h^{\frac{1}{2}} \sum_{i=1}^{k-1} \frac{1}{(k-i)^{\frac{1}{2}}} = h \sum_{i=1}^{k-1} \frac{1}{(t_k - t_i)^{\frac{1}{2}}} \le \int_0^{t_k} \frac{ds}{(t_k - s)^{\frac{1}{2}}} = et_k^{1/2}, \tag{2.1.63}$$

where $t_j = jh$. On the other hand, we also get

$$\begin{aligned} \sum_{i=j+1}^{k-1} \frac{1}{(k-i)^{\frac{1}{2}}(i-j)^{\frac{1}{2}}} &= h \sum_{i=j+1}^{k-1} \frac{1}{(t_k - t_i)^{\frac{1}{2}}(t_i - t_j)^{\frac{1}{2}}} \\ &= h \sum_{l=1}^{k-j-1} \frac{1}{(t_{k-j} - t_l)^{\frac{1}{2}} t_l^{\frac{1}{2}}} \le \int_0^{t_{k-j}} \frac{1}{(t_{k-j} - s)^{\frac{1}{2}} s^{\frac{1}{2}}} ds \\ &= \int_0^1 \frac{1}{(1-u)^{\frac{1}{2}} u^{\frac{1}{2}}} du = B(\frac{1}{2}, \frac{1}{2}) = \pi. \end{aligned} \tag{2.1.64}$$

Thus it follows from (2.1.62)–(2.1.64) that since $t_k \le T$,

$$\begin{aligned} |x_k| &\le hM^2 \pi \sum_{j=0}^{k-2} |x_j| + 2M\delta t_k^{\frac{1}{2}} + \delta(1 + h^{\frac{1}{2}} M) \\ &\le hM^2 \pi \sum_{j=0}^{k-2} |x_j| + 2M\delta T^{\frac{1}{2}} + \delta(1 + h^{\frac{1}{2}} M). \end{aligned}$$

We now observe that Theorem 2.1.7 with (2.1.53) replaced by (2.1.54) can be used to give

$$||x||_\infty \le \Big\{ hM^2 \pi |x_0| + \delta(1 + h^{\frac{1}{2}} M + 2MT^{\frac{1}{2}}) \Big\} e^{M^2 \pi T}.$$

Since $|x_0| < \delta$, we obtain

$$||x||_\infty \le \delta(1 + h^{\frac{1}{2}} M + hM^2 \pi + 2MT^{\frac{1}{2}}) e^{M^2 \pi T}.$$

□

Next, we shall see how the ideas associated with the proof of Theorem 2.1.9 can be generalized to deal with the exponential ϵ.

Theorem 2.1.10 (Mckee [393]) *If $x_j,\ j = 0, 1, \cdots, N$, is a sequence of real numbers with*

$$|x_0| < \delta, \quad |x_i| \le h^{1-\epsilon} M \sum_{j=0}^{i-1} \frac{1}{(i-j)^{\epsilon}} |x_j| + \delta, \quad i = 1, 2, \cdots, N, \tag{2.1.65}$$

where $M > 0$ is independent of h, $\delta > 0$ and $0 \le \epsilon < 1$ (independent of h), then

$$||x||_{\infty} \le (\delta' + hM'T^{n-1-n\epsilon}\delta)e^{M'T^{n-n\epsilon}}, \tag{2.1.66}$$

where $M' = M$, $n = 1$ and

$$\begin{cases} M' = M^n \displaystyle\prod_{k=1}^{n-1} B(k(1-\epsilon), 1-\epsilon), & n \ge 2, \\ \delta' = \delta \left\{ (h^{1-\epsilon}M) \displaystyle\sum_{j=0}^{n-2} \gamma^j + \gamma^{n-1} \right\}, & n \ge 2, \end{cases}$$

and $\gamma = \frac{MT^{1-\epsilon}}{1-\epsilon}$, and n is the smallest positive integer such that $\epsilon \le (n-1)/n$, $B(\cdot, \cdot)$ is the beta function.

Proof Let $\{i_j,\ j = 1, 2, \cdots, n\}$ be a finite set of integer variables such that $0 \le i_1 \le \cdots \le i_n \le N$ and n is chosen to be the smallest positive integer such that

$$n - 1 - \epsilon \ge 0. \tag{2.1.67}$$

These integer variables will play the analogous role of the "dummy" continuous variables $\{t_j,\ j = 1, 2, \cdots, n\}$ in the simple integration

$$\int_0^t \int_0^{t_n} \cdots \int_0^{t_2} F(t_1) dt_1 dt_2 \cdots dt_n$$

such that for any $\epsilon \in [1, 2]$, a finite n can always be chosen such that (2.1.67) is satisfied.

We shall first prove by induction on m that for any fixed ϵ such that $\frac{n-2}{n-1} < \epsilon \le \frac{n-1}{n}$, (2.1.65) implies, for all $m = 1, 2, \ldots, n$, that

$$|x_{i_m}| \le h^{m-m\epsilon} M_m \sum_{i=0}^{i_m-1} (i_m - i)^{m-1-m\epsilon} |x_i| + \delta_M, \quad i_m = 1, 2, \ldots, N, \tag{2.1.68}$$

where

$$\begin{cases} M_m = MB((m-1)(1-\epsilon), 1-\epsilon)M_{m-1}, \quad M-1 = M, \\ \delta_m = \delta(1+h^{1-\epsilon}M) + M\dfrac{(t_{i_n})^{1-\epsilon}}{1-\epsilon}\delta_{m-1}, \quad \delta_1 = \delta. \end{cases}$$

We note that for $m = 1$, (2.1.68) is simply (2.1.65). Assume then that (2.1.68) holds for any $m \in \{1, 2, \ldots, N\}$.

Now using the inductive hypothesis, we derive from (2.1.65) that,

$$\begin{aligned} |x_{i_{m+1}}| &\le h^{1-\epsilon}M \sum_{i_m=0}^{i_{m+1}-1} \frac{1}{(i_{m+1}-i_m)^\epsilon}|x_{i_m}| + \delta \\ &= h^{1-\epsilon}M \sum_{i_m=1}^{i_{m+1}-1} \frac{1}{(i_{m+1}-i_m)^\epsilon}|x_{i_m}| + h^{1-\epsilon}M\frac{1}{(i_{m+1})^\epsilon}|x_0| + \delta \\ &\le h^{1-\epsilon}M \sum_{i_m=1}^{i_{m+1}-1} \frac{1}{(i_{m+1}-i_m)^\epsilon}|x_{i_m}| + \delta(1+h^{1-\epsilon}M) \\ &\le h^{1-\epsilon}M \sum_{i_m=1}^{i_{m+1}-1} \frac{1}{(i_{m+1}-i_m)^\epsilon}\left\{h^{m-m\epsilon}M_m\sum_{i=0}^{i_m-1}(i_m-i)^{m-1-m\epsilon}|x_i| + \delta_M\right\} \\ &\quad +\delta(1+h^{1-\epsilon}M), \end{aligned}$$

whence

$$\begin{aligned} |x_{i_{m+1}}| \le\ & h^{m+1-(m+1)\epsilon}MM_m \sum_{i=0}^{i_{m+1}-2}\sum_{i_m=i+1}^{i_{m+1}-1} \frac{1}{i_{m+1}-i_m)^\epsilon(i_m-i)^{-(m-1)+m\epsilon}}|x_i| \\ & +\delta(l+h^{1-\epsilon}M) + h^{1-\epsilon}M\sum_{i_m=1}^{i_{m+1}-1}\frac{1}{(i_{m+1}-i_m)^\epsilon}\delta_m \end{aligned}$$

where we have changed the order of the summation.

Now as in the proof of Theorem 2.1.9, we can derive

$$\begin{aligned} h^{1-\epsilon}\sum_{i_m=1}^{i_{m+1}-1}\frac{1}{(i_{m+1}-i_m)^\epsilon} &= h^{1-\epsilon}\sum_{i_m=1}^{i_{m+1}-1}\frac{1}{(t_{i_{m+1}}-t_{i_m})^\epsilon} \le \int_0^{t_{i_{m+1}}}\frac{ds}{(t_{i_{m+1}}-s)^\epsilon} \\ &= \frac{1}{1-\epsilon}(t_{i_{m+1}})^{1-\epsilon} \end{aligned}$$

whence

$$\sum_{i_m=i+1}^{i_{m+1}-1} \frac{1}{(i_{m+1}-i_m)^{\epsilon}(i_m-i)^{-(m-1)+m\epsilon}} = \sum_{l=1}^{i_{m+1}-i-1} \frac{1}{(i_{m+1}-i-l)^{\epsilon} l^{-(m-1)+m\epsilon}}$$

$$= h \sum_{l=1}^{i_{m+1}-i-1} \frac{h^{-m+(m+1)\epsilon}}{(t_{i_{m+1}-i}-t_l)^{\epsilon} t_l^{-(m-1)+m\epsilon}} \leq h^{-m+(m+1)\epsilon} \int_0^{t_{i_{m+1}-i}} \frac{ds}{(t_{i_{m+1}}-s)^{\epsilon} s^{-(m-1)+m\epsilon}}$$

$$= h^{-m+(m+1)\epsilon}[(i_{m+1}-i)h]^{m-(m+1)\epsilon} \int_0^1 \frac{du}{(1-u)^{\epsilon} u^{-(m-1)+m\epsilon}}$$

$$= (i_{m+1}-i)^{m-(m+1)\epsilon} B(m(1-\epsilon), 1-\epsilon)$$

where we have, as in Theorem 2.1.9, compared the Riemann sum with the area under the curve.

Thus we conclude that

$$|x_{i_{m+1}}| \leq h^{m+1-(m+1)\epsilon} M M_m \sum_{i=0}^{i_{m+1}-2} (i_{m+1}-i)^{m-(m+1)\epsilon} B(m(1-\epsilon), 1-\epsilon)|x_i|$$
$$+\delta(l+h^{1-\epsilon}M) + \frac{M(t_{i_{m+1}})^{1-\epsilon}}{1-\epsilon}\delta_m$$

which implies

$$|x_{i_{m+1}}| \leq h^{m+1-(m+1)\epsilon} M M_m \sum_{i=0}^{i_{m+1}-1} (i_{m+1}-i)^{m-(m+1)\epsilon} B(m(1-\epsilon), 1-\epsilon)|x_i|$$
$$+\delta(l+h^{1-\epsilon}M) + \frac{M(t_{i_{m+1}})^{1-\epsilon}}{1-\epsilon}\delta_m$$
$$= h^{m+1-(m+1)\epsilon} M B(m(1-\epsilon), 1-\epsilon) M_m \sum_{i=0}^{i_{m+1}-1} (i_{m+1}-i)^{m-(m+1)\epsilon}|x_i|$$
$$+\delta(l+h^{1-\epsilon}M) + \frac{M(t_{i_{m+1}})^{1-\epsilon}}{1-\epsilon}\delta_m$$
$$= h^{m+1-(m+1)\epsilon} M_{m+1} \sum_{i=0}^{i_{m+1}-1} (i_{m+1}-i)^{m-(m+1)\epsilon}|x_i| + \delta_{m+1}.$$

The induction is then complete and in particular, we have

$$|x_{i_n}| \leq h^{n-n\epsilon} M_n \sum_{i=0}^{i_n-1} (i_n-i)^{n-1-n\epsilon}|x_i| + \delta_n,$$

and since

$$M_m = MB((m-1)(1-\epsilon), 1-\epsilon)M_{m-1}, \quad \delta_m = (l + h^{1-\epsilon}M)\delta + \frac{M(t_{i_{m+1}})^{1-\epsilon}}{1-\epsilon}\delta_{m-1},$$

$m = 2, 3, \ldots, n$, with $M_1 = M$ and $\delta_1 = \delta$, we can prove trivially by induction that

$$M_n = M_n \prod_{k=1}^{n-1} B(k(1-\epsilon), 1-\epsilon) = M', \quad n \geq 2 \quad \textit{with} \quad M_1 = M,$$

$$\delta_n = \delta\{(l + h^{1-\epsilon}M)\sum_{j=0}^{n-2}\gamma^j + \gamma^{n-1} = \delta'\}, \quad n \geq 1, \; \textit{where } \gamma = \frac{M(t_{i_{m+1}})^{1-\epsilon}}{1-\epsilon}.$$

Furthermore, since $t_i = ih \leq T$, we have

$$|x_i| \leq hM'T^{n-1-n\epsilon}\sum_{j=0}^{i=1}(\frac{i-j}{i})^{n-1-n\epsilon}|x_j| + \delta'$$

which yields, since $(\frac{i-j}{i})^{n-1-n\epsilon} \leq 1$ as $n - 1 - n\epsilon \geq 0$,

$$|x_i| \leq hM'T^{n-1-n\epsilon}\sum_{j=0}^{i=1}|x_j| + \delta'.$$

Now applying Theorem 2.1.7 with (2.1.52) replaced by (2.1.53) to (2.1.66), we get

$$\begin{aligned} ||x||_\infty &\leq (hM'T^{n-1-n\epsilon}|x_0| + \delta')\exp M'T^{n-1-n\epsilon(Nh)} \\ &= (hM'T^{n-1-n\epsilon}\delta + \delta')\exp M'T^{n-n\epsilon}. \end{aligned}$$

The proof is thus complete. □

In what follows, we shall study the inequality with another Abel's type singularity, for example, for all $n \in \mathbb{N}$,

$$\int_0^t \frac{k(t,s)y(s)}{(t^{\frac{n+1}{n}} - s^{\frac{n+1}{n}})^{\frac{n}{n+1}}}ds, \tag{2.1.69}$$

where $k(t,s)$ has sufficient continuity on the domain $\{0 \leq s \leq t \leq T\}$. This type of Eq. (2.1.69) is quite comparatione. We refer, for instance, to Atkinson [35] where $n = 1$ and Lighthill [361] and Noble [430] where $n = 2$.

We shall present the next general result.

Theorem 2.1.11 (Mckee [393]) *If* $x_j,\ j = 0, 1, \ldots, N$*, is a sequence of real numbers with*

$$|x_0| < \delta, \quad |x_i| \le hM \sum_{j=0}^{i-1} \frac{j}{(i^{\frac{n+1}{n}} - j^{\frac{n+1}{n}})^{\frac{n}{n+1}}} |x_j| + \delta, \quad i = 1, 2, \ldots, N, \tag{2.1.70}$$

where $M > 0$ *is independent of* h, $\delta > 0$, $n \in \mathbb{N}$ *(independent of* h*), then*

$$||x||_\infty \le \delta \Big\{ \sum_{j=0}^{n} (nMT)^j + h\Big(\frac{n+1}{n}\Big)^n T^n M^{n+1} \prod_{j=1}^{n} B(\frac{1}{n+1}, \frac{j}{n+1}) \Big\}$$
$$\times \exp\Big\{ (\frac{n+1}{n})^n M^{n+1} T^{n+1} \prod_{j=1}^{n} B(\frac{1}{n+1}, \frac{j}{n+1}) \Big\}. \tag{2.1.71}$$

Proof Let $\{i_j,\ j = 0, 1, \ldots, n\}$ be a (finite) set of integer variables such that $1 \le i_0 \le \ldots \le i_n \le N$. We shall first prove by induction on m that (2.1.70) implies, for all $m = 0, 1, \ldots, n$, that

$$|x_{i_m}| \le h\Big\{(\frac{n+1}{n})^m M^{m+1} (t_{i_m}^{\frac{n-1}{n}})^m \prod_{j=1}^{m} B(\frac{1}{n+1}, \frac{j}{n+1})\Big\} \sum_{j=0}^{i_m-1} \frac{t_j}{(t_{i_m}^{\frac{n+1}{n}} - t_j^{\frac{n+1}{n}})^{\frac{n-m}{n+1}}} |x_j|$$
$$+\delta\Big\{1 + (nmt_{i_m}) + \ldots + (nmt_{i_m})^m|\Big\}. \tag{2.1.72}$$

We note that for $m = 0$, (2.1.72) is simply (2.1.71). Assume then that (2.1.72) holds for any $m \in \{1, \ldots, n\}$. We derive from (2.1.70) that

$$|x_{i_{m+1}}| \le hM \sum_{i_m=0}^{i_{m+1}-1} \frac{t_j}{(i_{m+1}^{\frac{n+1}{n}} - i_m^{\frac{n+1}{n}})^{\frac{n}{n+1}}} |x_{i_m}| + \delta$$
$$\le hM \sum_{i_m=0}^{i_{m+1}-1} \frac{t_j}{(i_{m+1}^{\frac{n+1}{n}} - i_m^{\frac{n+1}{n}})^{\frac{n}{n+1}}} \{h\Big\{(\frac{n+1}{n})^m M^{m+1} (t_{i_m}^{\frac{n-1}{n}})^m \prod_{j=1}^{m} B(\frac{1}{n+1}, \frac{j}{n+1})$$
$$\times \sum_{j=0}^{i_m-1} \frac{t_j}{(t_{i_m}^{\frac{n+1}{n}} - t_j^{\frac{n+1}{n}})^{\frac{n-m}{n+1}}} |x_j| + \delta(1 + (nmt_{i_m}) + \ldots + (nmt_{i_m})^m)\Big\} + \delta$$

arising the inductive hypothesis (2.1.72).

Since $t_{i_m} \le t_{i_{m+1}}$ and $t_{i_m} = i_m h$, we obtain

$$|x_{i_{m+1}}| \le h^2 M(\frac{n+1}{n})^m M^{m+1}\Big(t_{i_m}^{\frac{n-1}{n}}\Big)^m \prod_{j=1}^{m} B(\frac{1}{n+1}, \frac{j}{n+1})$$

$$\times \sum_{i_m=0}^{i_{m+1}-1} \sum_{j=0}^{i_m-1} \frac{t_{i_m} t_j}{(t_{i_{m+1}}^{\frac{n+1}{n}} - t_m^{\frac{n+1}{n}})^{\frac{n}{n+1}} (t_{i_m}^{\frac{n+1}{n}} - t_j^{\frac{n+1}{n}})^{\frac{n-m}{n+1}}} |x_j| + \delta$$

$$+\delta\Big(1 + (nmt_{i_m}) + \ldots + (nmt_{i_m})^m\Big) hM \sum_{i_m=0}^{i_{m+1}-1} \frac{t_{im}}{(i_{m+1}^{\frac{n+1}{n}} - i_m^{\frac{n+1}{n}})^{\frac{n}{n+1}}}$$

$$\le h(\frac{n+1}{n})^m M^{m+2} (t_{i_m}^{\frac{n-1}{n}})^m \prod_{j=1}^{m} B(\frac{1}{n+1}, \frac{j}{n+1}) \sum_{j=0}^{i_{m+1}-2} (t_j |x_j|) h$$

$$\times \sum_{i_m=j+1}^{i_{m+1}-1} \frac{(t_{i_m})^{\frac{n-1}{n}} (t_j)^{\frac{1}{n}}}{(t_{i_{m+1}}^{\frac{n+1}{n}} - t_m^{\frac{n+1}{n}})^{\frac{n}{n+1}} (t_{i_m}^{\frac{n+1}{n}} - t_j^{\frac{n+1}{n}})^{\frac{n-m}{n+1}}} + \delta$$

$$+\delta\Big(1 + (nmt_{i_m}) + \ldots + (nmt_{i_m})^m\Big) hM \sum_{i_m=0}^{i_{m+1}-1} \frac{t_{i_m}}{(i_{m+1}^{\frac{n+1}{n}} - i_m^{\frac{n+1}{n}})^{\frac{n}{n+1}}}.$$

But we now know that

$$h \sum_{i_m=j+1}^{i_{m+1}-1} \frac{(t_j)^{\frac{1}{n}}}{(t_{i_{m+1}}^{\frac{n+1}{n}} - t_m^{\frac{n+1}{n}})^{\frac{n}{n+1}} (t_{i_m}^{\frac{n+1}{n}} - t_j^{\frac{n+1}{n}})^{\frac{n-m}{n+1}}}$$

$$\le \int_{t_j}^{t_{i_{m+1}}} \frac{s^{\frac{1}{n}} ds}{(t_{i_{m+1}}^{\frac{n+1}{n}} - s^{\frac{n+1}{n}})^{\frac{n}{n+1}} (s^{\frac{n+1}{n}} - t_j^{\frac{n+1}{n}})^{\frac{n-m}{n+1}}}$$

$$\le \frac{n}{n+1} \int_0^1 \frac{du}{(1-u)^{\frac{n}{n+1}} u^{\frac{n-m}{n+1}}} \cdot \frac{1}{(t_{i_{m+1}}^{\frac{n+1}{n}} - t_j^{\frac{n+1}{n}})^{\frac{n-(m+1)}{n+1}}}$$

$$= (\frac{n}{n+1}) B(\frac{1}{n+1}, \frac{m+1}{n+1}) \cdot \frac{1}{(t_{i_{m+1}}^{\frac{n+1}{n}} - t_j^{\frac{n+1}{n}})^{\frac{n-(m+1)}{n+1}}}$$

and

$$h \sum_{i_m=0}^{i_{m+1}-1} \frac{t_{i_m}}{(t_{i_{m+1}}^{\frac{n+1}{n}} - t_m^{\frac{n+1}{n}})^{\frac{n}{n+1}}} \le t_{i_{m+1}}^{\frac{n-1}{n}} \int_0^{t_{i_{m+1}}} \frac{s^{\frac{1}{n}} ds}{(t_{i_{m+1}}^{\frac{n+1}{n}} - s^{\frac{n+1}{n}})^{\frac{n}{n+1}}} = n t_{i_{m+1}}$$

whence

$$|x_{i_{m+1}}| \leq h(\frac{n+1}{n})^m M^{m+2}(t_{i_m}^{\frac{n-1}{n}})^m \prod_{j=1}^{m+1} B(\frac{1}{n+1}, \frac{j}{n+1}) \sum_{j=0}^{i_{m+1}-1} \frac{t_j}{(t_{i_{m+1}}^{\frac{n+1}{n}} - t_j^{\frac{n+1}{n}})^{\frac{n-(m+1)}{n+1}}} |x_j|$$
$$+\delta + \delta\Big(1 + (nmt_{i_m}) + \ldots + (nmt_{i_m})^m\Big)$$

which implies that the induction is complete. In particular, when $m = n - 1$, we obtain

$$|x_{i_n}| \leq h\Big(\frac{n+1}{n}\Big)^n M^{n+1}\Big(t_{i_n}^{\frac{n-1}{n}}\Big)^n \prod_{j=1}^{n} B(\frac{1}{n+1}, \frac{j}{n+1}) \sum_{j=0}^{i_n-1} t_j|x_j| + \delta \sum_{j=0}^{n} \Big(nMT\Big)^j$$

But $t_j \leq t_{i_n} \leq T$ and so we have

$$|x_{i_n}| \leq h(\frac{n+1}{n})^n M^{n+1} T^n \prod_{j=1}^{n} B(\frac{1}{n+1}, \frac{j}{n+1}) \sum_{j=0}^{i_n-1} |x_j| + \delta \sum_{j=0}^{n} (nMT)^j.$$

Thus using Theorem 2.1.7 with (2.1.53) replaced by (2.1.54), we finally obtain the required result

$$|x_{i_n}| \leq \delta \left\{ h(\frac{n+1}{n})^n M^{n+1} T^n \prod_{j=1}^{n} B(\frac{1}{n+1}, \frac{j}{n+1}) + \delta \sum_{j=0}^{n} (nMT)^j \right\}$$
$$\times \exp \left\{ (\frac{n+1}{n})^n M^{n+1} T^{n+1} \prod_{j=1}^{n} B(\frac{1}{n+1}, \frac{j}{n+1}) \right\}.$$

The proof is hence complete. □

The next theorem is the discrete form, due to Chu and Metcalf [135], of the linear generalization of Theorem 1.1.2.

Theorem 2.1.12 (Dixon-Mckee [188]) *Let x_i, $i = 0, 1, \ldots, N$, be a sequence of non-negative real numbers satisfying*

$$x_i \leq \phi_i + h \sum_{j=0}^{i-1} k_{ij} x_j, \quad i = 0, 1, \ldots, N, \tag{2.1.73}$$

where ϕ_i $(i = 0, 1, \ldots, N)$ is a sequence of non-negative finite real numbers, and $0 \leq k_{ij}$, $0 \leq j < i \leq N$, for some M bounded independently of h $(= T/N)$. If there exists a continuous, non-negative function $k(t, s)$ defined on the triangle

$0 \le s \le t \le T$ such that

$$k(t,s) \ge k_{ij},$$

for

$$ih \le t < (i+1)h, \quad jh \le s < (j+1)h, \quad 0 \le j < i < N,$$

and

$$k(t,s) \ge k_{Nj},$$

for

$$t = Nh, \quad jh \le s < (j+1)h, \quad 0 \le j < N,$$

then

$$x_i \le y(ih), \quad i = 0, 1, \ldots, N, \tag{2.1.74}$$

where y is the unique solution of the integral equation

$$y(t) = \phi(t) + \int_0^t k(t,s)y(s)ds, \quad 0 \le t \le T,$$

and $\phi(t)$ is the step function defined on $[0,T]$ by

$$\begin{cases} \phi(t) = \phi_i, \quad ih \le t < (i+1)h, \quad i = 0, 1, \ldots, N-1 \\ \phi(t) = \phi_N, \quad t = Nh = T. \end{cases}$$

(Here and elsewhere we assume that

$$\sum_{j \in \varnothing} D_j = 0 \quad \textit{and} \quad \prod_{j \in \theta} D_j = 1$$

if $\varnothing$ is the empty set; thus $x_0 \le \phi_0$ is assumed in (2.1.73).)

Proof Since $\phi_i, i = 0, 1, \ldots, N$, is a sequence of finite real numbers and $0 \le k_{ij} \le M$ for some constant M,

$$x_i \le \Phi + Mh \sum_{j=0}^{i-1} x_j, \quad i = 0, 1, \ldots, N, \tag{2.1.75}$$

where

$$\Phi = \max_{0 \le i \le N} \phi_i.$$

Applying Theorem 2.1.8 to (2.1.75), we get,

$$x_i \le \Phi \exp(Mih), \quad i = 0, 1, \ldots, N,$$

and consequently $x_i,\ i = 0, 1, \ldots, N$, is bounded.

We may now define a step function $x(t)$ on $[0, T]$ as follows:

$$\begin{cases} x(t) = x_i,\ ih \le t < (i+1)h,\ i = 0, 1, \ldots, N-1, \\ x(t) = x_N,\ t = Nh = T. \end{cases}$$

For any $t \in [0, T]$, there exists a unique $i,\ 0 \le i \le N-1$, such that $ih \le t < (i+1)h$. With this t, we have

$$x(t) = x_i \le \phi_i + h\sum_{j=0}^{i-1} k_{ij}x_j$$

and

$$\begin{aligned} \phi(t) + \int_0^t k(t,s)x(s)ds &= \phi_i + \sum_{j=0}^{i-1}\int_{jh}^{(j+1)h} k(t,s)x(s)ds + \int_{ih}^t k(t,s)x(s)ds \\ &= \phi_i + \sum_{j=0}^{i-1} x_j \int_{jh}^{(j+1)h} k(t,s)ds + x_i\int_{ih}^t k(t,s)ds \\ &\ge \phi_i + \sum_{j=0}^{i-1} k_{ij}x_j \ge x_i = x(t). \end{aligned}$$

Similarly, if $t = T$,

$$x(t) = x_N \le \phi_N + h\sum_{j=0}^{i-1} k_{Nj}x_j$$

and

$$\phi(t) + \int_0^t k(t,s)x(s)ds \le x_N = x(t).$$

Therefore, for every $t \in [0, T]$, using Theorem 1.2.38 (with $x,\ \phi$ bounded and continuous almost everywhere on $[0, T]$), we obtain, for all $0 \le t \le T$,

$$x(t) \le y(t),$$

where $0 \le t \le T$,

$$y(t) = \phi(t) + \int_0^t k(t,s)y(s)ds.$$

Letting $t = ih$, we conclude that

$$x_i \le y(ih), \quad i = 0, 1, \ldots, N. \tag{2.1.76}$$

□

Mate and Nevai [392] proved the following finite difference inequality which is a discrete analogue of Theorem 1.1.2 and a generation of Corollary 1.2.7.

Theorem 2.1.13 (Mate-Nevai [392]) *Let $f,\ g \ge 0$ be functions defined on $\mathbb{Z}$ and let $c \ge 0$ be a constant.*

(i) Suppose that for all integers $x \ge 1$,

$$f(x) \le c + \sum_{t=1}^{x-1} f(t)g(t), \tag{2.1.77}$$

then for all integers $x \ge 1$,

$$f(x) \le c \exp\Big(\sum_{t=1}^{x-1} g(t)\Big). \tag{2.1.78}$$

(ii) Suppose that for every integer $x \in \mathbb{Z}$,

$$f(x) \le c + \sum_{t=x+1}^{+\infty} f(t)g(t) < +\infty, \tag{2.1.79}$$

then for all integers $x \in \mathbb{Z}$,

$$f(x) \le c \exp\Big(\sum_{t=x+1}^{+\infty} g(t)\Big). \tag{2.1.80}$$

Proof To prove (i), we may write $F(x) = c + \sum_{x=1}^{x-1} f(t)g(t)$. Then it holds for all $x \ge 1$,

$$F(x+1) - F(x) = f(x)g(x) \le F(x)g(x).$$

That is,we have for all $x \geq 1$,

$$F(x+1) \leq F(x)(1+g(x)) \leq F(x)e^{g(x)}$$

whence the result follows by induction if we note that $F(1) = C$.

To establish (ii), we may write $F(x) = c + \sum_{t=x+1}^{+\infty} f(t)g(t)$. Then for every integer $x \in \mathbb{Z}$,

$$F(x) - F(x+1) = f(x+1)g(x+1) \leq F(x+1)g(x+1),$$

whence

$$F(x) \leq F(x+1)(1+g(x+1)) \leq F(x+1)e^{g(x+1)},$$

that is, for all $s > x$,

$$F(x) \leq F(s)\exp\Big(\sum_{t=x+1}^{s} g(t)\Big).$$

Noting that $\lim_{s\to+\infty} F(s) = c$, the desired result follows by making $s \to +\infty$ □

The next theorem is a slightly modified form of the above theorem.

Theorem 2.1.14 (Mate-Nevai [392]) *Let $f, g \geq 0$ be functions defined on integers, let $C \geq 0, 0 < \varepsilon < 1$, and suppose that $g(x) \leq \varepsilon$ for all $x \geq 1$.*

(i) Suppose that for all integers $x \geq 1$,

$$f(x) \leq C + \sum_{t=1}^{x} f(t)g(t). \tag{2.1.81}$$

Then for all integers $x \geq 1$,

$$f(x) \leq \frac{C}{1-\varepsilon}\exp\Big(\frac{1}{1-\varepsilon}\sum_{t=1}^{x-1} g(t)\Big). \tag{2.1.82}$$

(ii) Suppose that for all integers $x \geq 1$,

$$f(x) \leq C + \sum_{t=x}^{+\infty} f(t)g(t) < +\infty.$$

Then for all integers $x \geq 1$,

$$f(x) \leq \frac{C}{1-\varepsilon}\exp\Big(\frac{1}{1-\varepsilon}\sum_{t=x+1}^{+\infty} g(t)\Big). \tag{2.1.83}$$

Proof The result is an easy consequence of Theorem 2.1.13. Indeed, (2.1.81) implies that

$$f(x)(1-g(x)) \leq C + \sum_{t=1}^{x-1} f(t)g(t),$$

that is,

$$f(x) \leq C/(1-\varepsilon) + \sum_{t=1}^{x-1} f(t)g(t)/(1-\varepsilon).$$

Therefore, inequality (2.1.82) follows immediately from part (i) of Theorem 2.1.13. Inequality (2.1.83) can be established similarly. □

The next result is due to Lees [358].

Theorem 2.1.15 (Lees [358]) *Let $\omega(t)$ and $\rho(t)$ be non-negative functions defined on the discrete set $\Lambda = \{2k, 3k, \ldots, Mk\}$, $(k > 0)$. If $C \geq 0$, $\rho(t)$ is non-decreasing and*

$$\omega(t) \leq \rho(t) + Ck \sum_{s=2k}^{t-k} \omega(s), \tag{2.1.84}$$

then

$$\omega(t) \leq \rho(t) \exp[C(t-2k)]. \tag{2.1.85}$$

Proof Let t_1 be an arbitrary point of $\Lambda, t_1 \neq 2k$. Let $\eta(t)$ be that function on Λ defined by the formula

$$\omega(t) = \eta(t) \exp\left(C(t-2k)\right),$$

and set

$$\eta(t_2) = \max_{2k \leq t \leq t_1} \eta(t).$$

Then it follows from (2.1.84)

$$\eta(t_2) \exp\Big(C(t_2-2k)\Big) \leq \rho(t_2) + Ck\eta(t_2) \sum_{s=2k}^{t_2-k} \exp\Big(C(s-2k)\Big). \tag{2.1.86}$$

Comparing areas, we see that

$$k \sum_{s=2k}^{t_2-k} \exp\Big(C(s-2k)\Big) \leq \int_{2k}^{t_2} \exp\left(C(s-2k)\right) ds.$$

with the initial condition $\psi(2k) = c_1$. By Taylor's theorem, there exists a τ, $0 < \tau < 1$, such that

$$\begin{aligned}\psi(t+k) - \psi(t) &= k\psi'(t) + (k^2/2)\psi''(t+\tau k)\\ &= k\psi'(t) + (k^2/2)c_2^2\delta[\psi(t+\tau k)]^{2\delta-1}.\end{aligned}$$

Hence, over any interval in which ψ is non-negative, we have

$$\psi(t+k) - \psi(t) \geq k\psi'(t).$$

It is readily verified that ψ is given by the right-hand side of (2.1.88).

Therefore,

$$\begin{aligned}\psi(t) - \psi(2k) &= \sum_{s=2k}^{t-k} \Big(\psi(s+k) - \psi(s)\Big) \geq k \sum_{s=2k}^{t-k} \psi'(s)\\ &= kc_2 \sum_{s=2k}^{t-k} \Big(\psi(s)\Big)^{\delta}\end{aligned}$$

which implies

$$\psi(t) \geq c_1 + kc_2 \sum_{s=2k}^{t-k} \Big(\psi(s)\Big)^{\delta}. \tag{2.1.90}$$

It suffices to prove that $\psi(t) \geq \omega(t)$. In the contrary case, there exists a value of $t > 2k$, say $t = t_1$, such that $\omega(t_1) \geq \psi(t_1)$ and $\omega(t) \leq \psi(t)$ for $2k \leq t \leq t_1$. Thus from (2.1.87) and (2.1.90), it follows

$$0 > \psi(t_1) - \omega(t_1) \geq c_2 k \sum_{s=2k}^{t_1-k} \left\{[\psi(s)]^{\delta} - [\omega(s)]^{\delta}\right\}.$$

This is impossible since $[\psi(s)]^{\delta} \geq [\omega(s)]^{\delta}$ in the range $2k \leq s \leq t_1 - k$. This hence completes the proof. □

Agarwal and Thandapani [17] proved the following theorem.

Theorem 2.1.17 (Agarwal-Thandapani [17]) *If u, f, g are non-negative functions defined on the non-negative integers, and if, for all x, s, t integers,*

$$u(x) \leq \eta_0 + \sum_{0\leq s<x} f(s)\Big\{u(s) + \sum_{0\leq t<s} g(t)u(t)\Big\}, \tag{2.1.91}$$

then for all x, s, t integers,

$$u(x) \leq \eta_0 \Big\{ 1 + \sum_{0 \leq s < x} f(s)[1 - \phi(s)] \prod_{0 \leq t < s} [1 + f(t) + g(t)] \Big\} \tag{2.1.92}$$

where

$$\phi(s) = \sum_{0 \leq t < s} g(t) \Big\{ \prod_{0 \leq r < t} [1 + f(r) + g(r)] \Big\}^{-1} \sum_{0 < r < t} g(s). \tag{2.1.93}$$

Proof We define $m(t)$ as the right-hand side of (2.1.91). Then

$$\begin{aligned} \Delta m(t) &= f(t)[u(t) + \sum_{\tau=0}^{t-1} g(\tau)u(\tau)] \\ &\leq f(t)[m(t) + \sum_{\tau=0}^{t-1} g(\tau)m(\tau)]. \end{aligned}$$

Define $n(t)$ as

$$n(t) = m(t) + \sum_{\tau=0}^{t-1} g(\tau)m(\tau),$$

then we have

$$\Delta n(t) = m(t) + \Delta g(t)m(t),$$

$$m(t) \leq n(t) - \eta_0 \sum_{\tau=0}^{t-1} g(\tau).$$

Thus we get

$$\Delta n(t) \leq f(t)n(t) + g(t)n(t) - \eta_0 g(t) \sum_{\tau=0}^{t-1} g(\tau)$$

or

$$n(t+1) - [1 + f(t) + g(t)]n(t) \leq -\eta_0 g(t) \sum_{\tau=0}^{t-1} g(\tau).$$

Multiplying the above inequality by $\prod_{s=0}^{t}(1+f(s)+g(s))^{-1}$ and summing over from 0 to $t-1$, we get

$$n(t) \leq u_\eta[1-\phi(t)]\prod_{s=0}^{t-1}(1+f(s)+g(s)).$$

On substituting the above estimate in $\Delta m(t)$ and summing over from 0 to $t-1$, we obtain (2.1.92). □

Theorem 2.1.18 (Jones [305]) *Let x, f, g, and z be real-valued function defined on an interval [a, b] with g and z non-negative and let* $\tau_0 < \tau_1 < \cdots < \tau_m$ *be a sequence of numbers in [a, b]. If for all* $t \in [a, b]$*,*

$$x(t) \leq f(t) + g(t)\sum_{\tau_i<t} z(\tau_i)x(\tau_i), \tag{2.1.94}$$

then for all $t \in [a, b]$*,*

$$x(t) \leq f(t) + g(t)\sum_{\tau_i<t}\left(\prod_{\tau_i<\tau_j<t}(1+g(\tau_j)z(\tau_j))z(\tau_i)f(\tau_i)\right). \tag{2.1.95}$$

Proof Let us define a function $y \in [a, b]$ satisfying the formula

$$y(t) = f(t) + g(t)\sum_{\tau_i<t} z(\tau_i)y(\tau_i). \tag{2.1.96}$$

Obviously, the function

$$y(t) = f(t) + g(t)\sum_{\tau_i<t}\left(\prod_{\tau_i<\tau_j<t}(1+g(\tau_j)z(\tau_j))z(\tau_i)f(\tau_i)\right) \tag{2.1.97}$$

will do for all $t \in [a, \tau_1]$ and at all points t where $g(t) = 0$. Let us assume (2.1.97) satisfies (2.1.96) for all $t \in [a, \tau_k]$. Then for all $t \in (\tau_k, min\{\tau_{h+1}, b\}]$, we have

$$\begin{aligned} y(t) = f(t) + g(t)\Bigg\{&\sum_{\tau_i<\tau_k} z(\tau_i)y(\tau_i) + z(\tau_k)\Bigg[f(\tau_k) \\ &+ g(\tau_k)\sum_{\tau_i<\tau_k}(\prod_{\tau_i<\tau_j<\tau_k}(1+g(\tau_j)z(\tau_j))z(\tau_i)f(\tau_i))\Bigg]\Bigg\}. \end{aligned} \tag{2.1.98}$$

if $g(\tau_k) \neq 0$, then our assumption implies

$$\sum_{\tau_i<\tau_k} z(\tau_i)y(\tau_i) = \sum_{\tau_i<\tau_k} \left(\prod_{\tau_i<\tau_j<\tau_k} (1 + g(\tau_j)z(\tau_j))z(\tau_i)f(\tau_i) \right),$$

and making the obvious substitution in (2.1.98), we observe that (2.1.97) is satisfied for all $t \in (\tau_k, min\{\tau_{k+1}, b\}]$. If $g(\tau_i) = 0$ for all $k \geq i > k-p$ with either $k-p = 0$ or $g(\tau_{k-p}) \neq 0$, then

$$\begin{aligned} y(t) &= f(t) + g(t) \Bigg[\sum_{\tau_{k-p}<\tau_i<t} \Big(\prod_{\tau_i<\tau_j<t} (1 + g(\tau_j)z(\tau_j))z(\tau_i)f(\tau_i) \Big) \\ &\quad + z(\tau_{k-p})y(\tau_{k-p}) + \sum_{\tau_i<\tau_{k-p}} \Big(\prod_{\tau_i<\tau_j<t} (1 + g(\tau_j)z(\tau_j))z(\tau_i)f(\tau_i) \Big) \Bigg] \\ &= f(t) + g(t) \sum_{\tau_i<t} \left(\prod_{\tau_i<\tau_j<t} (1 + g(\tau_j)z(\tau_j))z(\tau_i)f(\tau_i) \right). \end{aligned}$$

Hence we may conclude by induction that formula (2.1.97) is valid on $[a, b]$.

Now let us define the function $\omega = x - y$. Clearly $\omega(t) \geq 0$ for all $t \in [a, \tau_1]$. Assume $\omega(t) \geq 0$ on $[a, \tau_k]$ and let t be an arbitrary point in $(\tau_k, min\{\tau_{k+1}, b\}]$. Then

$$\omega(t) = g(t) \sum_{\tau_i \leq \tau_k} z(\tau_i)\omega(\tau_i) \geq 0,$$

and it is immediate from induction that $\omega \geq 0$ on $[a, b]$. This fact, together with the previously established validity of (2.1.97), completes the proof of the theorem. □

Corollary 2.1.10 (Jones [305]) *Let u, η, ϕ, ψ be non-negative functions defined on the non-negative integrals. If for all x, t non-negative integers,*

$$u(x) \leq \eta(x) + \phi(x) \sum_{0 \leq t<x} \psi(t)u(t),$$

then, for all x, t non-negative integers,

$$u(x) \leq \eta(x) + \phi(x) \sum_{0 \leq t<x} \Big\{ \psi(x)\eta(t) \prod_{t \leq s<x} [1 + \phi(s)\psi(s)] \Big\}.$$

We shall treat a slightly more general case (an exact analogue to Theorem 2.1.17).

Theorem 2.1.19 (Conlan-Wang [145]) *Assume that all functions are as in Theorem 2.1.17, and for all x, s, t integers,*

$$u(x) \leq h(x) + \sum_{0 \leq s < x} f(s)\Big(u(s) + \sum_{0 \leq t < s} g(s)u(t)\Big). \tag{2.1.99}$$

Then for x, s, t integers,

$$u(x) \leq h(x) + \sum_{0 \leq s < x} h(s)[f(s) + g(s)] \prod_{0 \leq t < x} \Big(1 + g(t)[f(t) + g(t)]\Big). \tag{2.1.100}$$

Proof Let

$$z(x) = u(x) + \sum_{0 \leq t < x} g(t)u(t).$$

Then by (2.1.99),

$$\begin{aligned} u(x) \leq z(x) &\leq h(x) + \sum_{0 \leq s < x} f(x)z(s) + \sum_{0 \leq s < x} g(s)u(s) \\ &\leq h(x) + \sum_{0 \leq s < x} [f(s) + g(s)]z(s), \end{aligned}$$

and therefore (2.1.100) follows immediately from Corollary 2.1.10. □

Since Theorem 2.1.19 can be extended from $\mathbb{R}$ to $\mathbb{R}^n$, it follows that Theorem 2.1.17 can also be true in $\mathbb{R}^n$. Also, a very similar line of reasoning can be used to obtain a discrete analogue of Theorem 2.1.18.

Theorem 2.1.20 (Agarwal [10]) *Let $u_n, a_n, b_n \geq 0$ be sequences defined for all $n \in \mathbb{N}_\alpha$ such that for all $n \in \mathbb{N}_\alpha$,*

$$u_n \leq a_n + \sum_{s=\alpha}^{n-1} \left[b_s u_s + \sum_{j=\alpha}^{s-1} c_{sj} u_j \right], \tag{2.1.101}$$

where $c_{sj} \geq 0, s, j \in \mathbb{N}_\alpha$. Then for all $n \in \mathbb{N}_\alpha$,

$$u_n \leq a_n + \sum_{s=\alpha}^{n-1} A_s \Pi_{j=s+1}^{n-1}(1 + B_j), \tag{2.1.102}$$

where $A_s = a_s b_s + \sum_{j=\alpha}^{s-1} c_{sj} a_j, B_s = b_s + \sum_{j=\alpha}^{s-1} c_{sj}$.

Proof Put

$$y_n = \sum_{s=\alpha}^{n-1} \left[b_s u_s + \sum_{j=\alpha} c_{sj} u_j \right].$$

Then $y_\alpha = 0$, and (2.1.101) successively implies

$$\begin{cases} u_n \le a_n + y_n, \\ \triangle y_n = b_n u_n + \sum_{j=\alpha}^{n-1} c_{nj} u_j \le A_n + B_n y_n. \end{cases} \tag{2.1.103}$$

By Theorem 2.1.3, (2.1.103) yields

$$y_n \le \sum_{s=\alpha}^{n-1} A_s \prod_{j=s+1}^{n-1} (1 + B_j)$$

which, together with (2.1.103), implies (2.1.102). □

Corollary 2.1.11 *If, in Theorem 2.1.20, $a_n \equiv a$, then (2.1.101) implies that for all $n \in \mathbb{N}_\alpha$,*

$$u_n \le a \prod_{s=\alpha}^{n-1} \left(1 + b_s + \sum_{j=\alpha}^{n-1} c_{sj} \right). \tag{2.1.104}$$

Theorem 2.1.21 (Pachpatte [451]) *Let $u_n \ge 0, a_n \ge 0, p_n \ge 0, f_n \ge, g_n \ge 0$ be sequences defined for all $n \in \mathbb{N}_\alpha$ such that for all $n \in \mathbb{N}_\alpha$,*

$$u_n \le a_n + p_n \left[\sum_{s=\alpha}^{n-1} f_s u_s + \sum_{s=\alpha}^{n-1} f_s p_s \sum_{i=\alpha}^{s-1} g_i u_i \right]. \tag{2.1.105}$$

Then for all $n \in \mathbb{N}_\alpha$,

$$u_n \le a_n + p_n \left[\sum_{s=\alpha}^{n-1} f_s \left(a_s + p_s \sum_{i=\alpha}^{s-1} a_i (f_i + g_i) \Pi_{j=i+1}^{s-1} (1 + p_j (f_j + g_j)) \right) \right]. \tag{2.1.106}$$

Proof Let m_n denote the expression of between square brackets in (2.1.105). Then we have

$$\begin{cases} u_n \le a_n + p_n m_n, \\ \triangle m_n = f_n u_n + f_n p_n \sum_{i=\alpha}^{n-1} g_i u_i, \quad m_\alpha = 0, \\ \triangle m_n \le f_n a_n + f_n p_n \left[m_n + \sum_{i=\alpha}^{n-1} g_i (a_i + p_i m_i) \right]. \end{cases} \tag{2.1.107}$$

Setting $v_n = m_n + \sum_{i=\alpha}^{n-1} g_i(a_i + p_i m_i)$, $v_\alpha = 0$, then (2.1.107) implies

$$\triangle v_n \le a_n(f_n + g_n) + p_n(f_n + g_n)v_n.$$

By Theorem 2.1.3, we get

$$v_n \le \sum_{s=\alpha}^{n-1} a_s(f_s + g_s) \prod_{i=s+1}^{n-1} (1 + p_i(f_i + g_i)),$$

and from (2.1.107) it follows

$$\triangle m_n \le f_n \left[a_n + p_n \sum_{s=\alpha}^{n-1} a_s(f_s + g_s) \prod_{i=s+1}^{n-1} (1 + p_i(f_i + g_i)) \right].$$

Thus Corollary 2.1.11 and Theorem 2.1.3 imply

$$m_n \le \sum_{s=\alpha}^{n-1} f_s \left[a_s + p_s \sum_{i=\alpha} a_i(f_i + g_i) \prod_{j=i+1}^{s-1} (1 + p_j(f_j + g_j)) \right]$$

which, together with (2.1.107), implies (2.1.106). □

Corollary 2.1.12 (Agarwal [10]) *Let $u_n \ge 0, a_n \ge 0, \triangle a_n \ge 0, f_n \ge 0, g_n \ge 0$ be sequences defined for all $n \in \mathbb{N}_\alpha$ such that for all $n \in \mathbb{N}_\alpha$,*

$$u_n \le a_n + \sum_{s=\alpha}^{n-1} f_s u_s + \sum_{s=\alpha}^{n-1} f_s \sum_{i=\alpha}^{s-1} g_i u_i. \tag{2.1.108}$$

Then for all $n \in \mathbb{N}_\alpha$,

$$u_n \le a_n \left[1 + \sum_{s=\alpha}^{n-1} f_s \prod_{i=\alpha}^{s-1} (1 + f_i + g_i) \right]. \tag{2.1.109}$$

Proof In fact, (2.1.109) follows from (2.1.105) by setting $p_n \equiv 1$, replacing a_s by a_n, which is larger, and applying (2.1.106). □

Theorem 2.1.22 (Agarwal-Thandapani [18]) *Let $u(t), a(t), b(t), f_i(t), i = 1, \cdots, r$, and $g_j, j = 1, \cdots, r-1$, be non-negative functions, defined for $t \in \mathbb{N}_\alpha$, such that for all $t \in \mathbb{N}_\alpha$,*

$$\begin{aligned} u(t) \leq a(t) + b(t)\Big[& \sum_{t_1=\alpha}^{t-1} f_1(t_1)u(t_1) + \sum_{t_1=\alpha}^{t-1} g_1(t_1) \sum_{t_2=\alpha}^{t_1-1} f_2(t_2)u(t_2) \\ & + \cdots + \sum_{t_1=\alpha}^{t-1} g_1(t_1) \sum_{t_2=\alpha}^{t_1-1} g_2(t_2) \cdots \sum_{t_{r-1}=\alpha}^{t_{r-2}-1} g_{r-1}(t_{r-1}) \sum_{t_r=\alpha}^{t_{r-1}-1} f_r(t_r)u(t_r)\Big]. \end{aligned} \tag{2.1.110}$$

Then for all $t \in \mathbb{N}_\alpha$,

$$u(t) \leq a(t) + b(t)Q_k(t), \ k = 1, \cdots, r, \tag{2.1.111}$$

where

$$\begin{cases} Q_k(t) = \displaystyle\sum_{s=\alpha}^{t-1} \left[a(s) \sum_{i=1}^{k} f_i(s) + g_k(s)Q_{k+1}(s) \right] \prod_{\tau=s+1}^{t-1} (1 + M_k(\tau) - g_k(\tau)), \\ M_k(t) = \max \left[b(t) \displaystyle\sum_{i=1}^{k} f_i(t), g_1(t), \cdots, g_{k-1}(t) \right], \ k = 2, \cdots, r, \\ M_1(t) = b(t)f_1(t), Q_{r+1}(t) \equiv 0, g_r(t) \equiv 0. \end{cases}$$

Proof This is a discrete analogue of Theorem 2.1.17. The proof is similar, so we omit it. □

Theorem 2.1.23 (Pachpatte [468]) *Suppose that the inequality for all $n \in \mathbb{N}_\alpha$, there holds*

$$\triangle u_n \leq f_n \left[u_n + \sum_{s=\alpha}^{n-1} g_s \triangle u_s \right], \tag{2.1.112}$$

where $u_n \geq 0, \triangle u_n \geq 0, f_n \geq 0, g_n \geq 0$ are sequences defined for all $n \in \mathbb{N}_\alpha$. Then for all $n \in \mathbb{N}_\alpha$,

$$u_n \leq u_\alpha \left[1 + \sum_{s=\alpha}^{n-1} f_s \prod_{i=\alpha}^{s-1} (1 + f_i + f_i g_i) \right]. \tag{2.1.113}$$

Proof Set $m_n = u_n + \sum_{s=\alpha}^{n-1} g_s \triangle u_s, m_\alpha = u_\alpha$. Then we get

$$\begin{cases} \triangle u_n \leq f_n m_n, \triangle m_n \leq \triangle u_n + g_n \triangle u_n, \\ \triangle m_n \leq (f_n + f_n g_n) m_n. \end{cases} \tag{2.1.114}$$

Thus (2.1.114) and Corollary 2.1.2 imply

$$m_n \leq u_\alpha \prod_{s=\alpha}^{n-1} (1 + f_s + f_s g_s),$$

which, together with (2.1.114), yields

$$\triangle u_n \leq u_\alpha f_n \prod_{s=\alpha}^{n-1} (1 + f_s + f_s g_s). \tag{2.1.115}$$

Applying Corollary 2.1.12, we can obtain (2.1.113). □

The next theorem can be shown by simple induction which is a discrete analogue of Theorem 1.2.38 of Chu and Metcalf.

Theorem 2.1.24 (Agarwal [10]) *Let u_n and a_n be sequences defined for all $n \in \mathbb{N}_\alpha$ such that for all $n \in \mathbb{N}_\alpha$,*

$$u_n \leq a_n + \sum_{s=\alpha}^{n-1} k_{ns} u_s, \tag{2.1.116}$$

where $k_{ns} \geq 0, n \in \mathbb{N}_\alpha, s \in \mathbb{N}_\alpha$. Then

(i) $u_n \leq w_n, n \in \mathbb{N}_\alpha$, where w_n is the solution of equation, for all $n \in \mathbb{N}_\alpha$,

$$w_n = a_n + \sum_{s=\alpha}^{n-1} k_{ns} w_s. \tag{2.1.117}$$

(ii) The solution w_n has the form, for all $n \in \mathbb{N}_\alpha$,

$$w_n = \sum_{m=0}^{n-\alpha} y_m(n), \tag{2.1.118}$$

where $y_0(n) = a_n$,

$$y_{m+1}(n) = \sum_{s=m+\alpha}^{n-1} k_{ns} y_m(s), \quad n \in \mathbb{N}_\alpha, \quad m \in \mathbb{N}_0. \tag{2.1.119}$$

Proof In fact, in some cases we may estimate the solution w_n of (2.1.117) as follows. Multiplying equalities

$$w_i = a_i + \sum_{j=\alpha}^{i-1} k_{ij} w_j, \quad i \in \mathbb{N}_\alpha,$$

by k_{ni}, summing over i from α to $n-1$, this gives us for all $n \in \mathbb{N}_\alpha$,

$$\sum_{i=\alpha}^{n-1} k_{ni} w_i = \sum_{i=\alpha}^{n-1} k_{ni} a_i + \sum_{i=\alpha}^{n-1} \sum_{j=\alpha}^{i-1} k_{ni} k_{ij} w_j. \tag{2.1.120}$$

Thus from (2.1.117) it follows

$$w_n = a_n + \sum_{i=\alpha}^{n-1} k_{ni} a_i + \sum_{i=\alpha}^{n-1} \sum_{j=\alpha}^{i-1} k_{ni} k_{ij} w_j, \tag{2.1.121}$$

and by changing the order of summation, we get

$$w_n = a_n + \sum_{i=\alpha}^{n-1} k_{ni} a_i + \sum_{j=\alpha}^{n-2} \left(\sum_{i=j+1}^{n-1} k_{ni} k_{ij} \right) w_j. \tag{2.1.122}$$

Let $A_n = a_n + \sum_{i=\alpha}^{n-1} k_{ni} a_i$, and assume that

$$\sum_{i=j+1}^{n-1} k_{ni} k_{ij} \le B_j, \quad j \in \mathbb{N}_{\alpha,n-1},$$

independently of $n \in \mathbb{N}_\alpha$. Since $B_{n-1} = 0$, (2.1.122) implies the inequality

$$w_n \le A_n + \sum_{j=\alpha}^{n-1} B_j w_j.$$

Therefore applying Theorem 2.1.7 to the above inequality yields the required estimate (2.1.117). $\square$

The next result, due to Dixon and Mckee [188], is the discrete analogue of Lemma 1.3.6.

Theorem 2.1.25 (Dixon-Mckee [188]) *Let $x_i, i = 0, 1, \ldots, N$, be a sequence of non-negative real numbers satisfying*

$$\begin{cases} x_i \leq \phi_i, \quad i = 0, 1, \ldots, m, \\ x_i \leq \phi_i + Mh^{1-(\alpha-m)} \sum_{i_m=0}^{i-1} \sum_{i_{m-1}=0}^{i_m-1} \cdots \sum_{j=0}^{i_1-1} \frac{x_j}{(i_1-j)^\alpha}, \quad i = m, m+1, \ldots, N, \end{cases} \tag{2.1.123}$$

where $\alpha < 1$, $m \geq 1$, $M > 0$ is bounded independently of h, and ϕ_i, $(i = 0, 1, \ldots, N)$, is non-decreasing sequence of non-negative finite real numbers, then

$$x_i \leq \phi_i E_{1-(\alpha-m)}(M\Gamma(1-\alpha)(ih)^{1-(\alpha-m)}), \quad i = 0, 1, \ldots, N. \tag{2.1.124}$$

Proof We first proceed by mathematical induction to show that for all $m \geq 1$, there holds that

$$\sum_{i_m=0}^{i-1} \sum_{i_{m-1}=0}^{i_m-1} \cdots \sum_{j=0}^{i_1-1} \frac{x_j}{(i_1-j)^\alpha} \leq \frac{\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} \sum_{j=0}^{i-1} (i-j-1)^{m-\alpha} x_j. \tag{2.1.125}$$

First, we consider the case $m = 1$. Interchanging the order of summation, we know

$$\sum_{i_1=0}^{i-1} \sum_{j=0}^{i_1-1} \frac{x_j}{(i_1-j)^\alpha} = \sum_{j=0}^{i-2} (\sum_{i_1=j+1}^{i-1} \frac{1}{(i_1-j)^\alpha}) x_j.$$

We bound the inner summation on the right-hand side of the above equality by an integral as follows,

$$\sum_{i_1=j+1}^{i-1} \frac{x_j}{(i_1-j)^\alpha} \leq \sum_{i_1=j+1}^{i-1} \int_{i_1-1}^{i_1} \frac{1}{(s-j)^\alpha} ds = \frac{(i-j-1)^{1-\alpha}}{1-\alpha},$$

which, since $x_i \geq 0,\ 0 \leq i \leq N$, yields

$$\sum_{i_1=0}^{i-1} \sum_{j=0}^{i_1-1} \frac{x_j}{(i_1-j)^\alpha} \leq \frac{\Gamma(1-\alpha)}{\Gamma(2-\alpha)} \sum_{j=0}^{i-1} (i-j-1)^{1-\alpha} x_j,$$

and (2.1.125) holds when $m = 1$.

Using similar arguments, we can also show that if (2.1.125) holds for $m = n$, then it also holds for $n+1$, and hence the induction is complete. Consequently, (2.1.123)

implies

$$x_i \leq \phi_i + \frac{M\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} h^{1-(\alpha-m)} \sum_{j=0}^{i-1} (i-(j+1))^{m-\alpha} x_j,$$

which is of the form

$$x_i \leq \phi_i + h \sum_{j=0}^{i-1} k_{ij} x_j,$$

where

$$0 \leq k_{ij} \leq \frac{M\Gamma(1-\alpha)T^{m-\alpha}}{\Gamma(1-\alpha+m)} = \hat{M}$$

where $\hat{M}$ is bounded independent of h.

Furthermore,

$$k_{ij} = \frac{M\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} \Big(h(i-(j+1))\Big)^{m-\alpha} \leq \frac{M\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} (t-s)^{m-\alpha} = k(t,s)$$

for

$$ih \leq t < (i+1)h, \quad jh \leq s < (j+1)h, \quad 0 \leq j < i < N$$

and

$$k_{Nj} \frac{M\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} \Big(h\Big(N-(j+1)\Big)\Big)^{m-\alpha} \frac{M\Gamma(1-\alpha)}{\Gamma(1-\alpha+m)} (T-s)^{m-\alpha} = k(T,s)$$

for

$$t = Nh, \quad jh \leq s < (j+1)h, \quad 0 \leq j < N.$$

Applying Theorem 2.1.12 and employing Theorem 1.4.10, we conclude (2.1.124). □

Remark 2.1.2 If $\alpha = 0, \ m = 1$, then (2.1.124) reduces to

$$x_i \leq \phi_i \cos h(M^{\frac{1}{2}}(ih)).$$

We also note that Theorem 2.1.24 remains valid for $m = 0$ if $\alpha \leq 0$. For the case $m = 0$ and $0 < \alpha < 1$, we also refer to Dixon and McKee [188].

The next inequality is the most used discrete inequality, which is the analogue of the celebrated Gronwall-Bellman-Ried inequality established by Jones [305] and Sugiyama [611]. On the basis of various motivations, this inequality has been extended and used in various contexts. The discrete analogue of Bihari's inequality is due to Hull and Luxemburg [299] (see, Theorem 2.1.1).

The following notations, the expression $\sum_{s=0}^{t-1} b(s)$ represents a solution of the linear difference equation $\Delta x(t) = b(t)$ for all $t \in \mathbb{N}_0$ under the initial condition $x(0) = 0$, where Δ is the operator defined by $\Delta x(t) = x(t+1) - x(t)$. It is supposed that $\sum_{s=0}^{-1} b(s)$. The expression $\prod_{s=0}^{t-1} c(s)$ represents the solution of the linear difference equation $x(t+1) = c(t)x(t)$ for all $t \in \mathbb{N}_0$ under the initial condition $x(0) = 1$. It is supposed that $\prod_{s=0}^{-1} c(s) = 1$.

In what follows, we shall assume that all the functions and their differences appearing in the inequalities are real-valued, non-negative and defined on $\mathbb{N}_0$.

We shall denote $(m)^{(n)} = m(m-1)(m-2)\cdots(m-n+1)$.

Theorem 2.1.26 (Agarwal-Thandapani [17]) *Assume that the following inequality holds for all $t \in \mathbb{N}_0$,*

$$u(t) \le p(t) + q(t)\sum_{\Omega=1}^{n} E_\Omega(t,u), \tag{2.1.126}$$

where for all $t \in \mathbb{N}_0$,

$$E_\Omega(t,u) = \sum_{t_1=0}^{t-1} f_{\Omega_1}(t_1)\sum_{t_2=0}^{t_1-1} f_{\Omega_2}(t_2)\cdots\sum_{t=0}^{t_{\Omega-1}-1} f_{\Omega\Omega}(t_\Omega)u(t_\Omega). \tag{2.1.127}$$

Then for all $t \in \mathbb{N}_0$,

$$u(t) \le p(t) + q(t)\sum_{s=0}^{t-1}\Big(\sum_{\Omega=1}^{n}\Delta E_\Omega(s,p)\Big)\prod_{\tau=s+1}^{t-1}\Big[1+\sum_{\Omega=1}^{n}\Delta E_\Omega(\tau,q)\Big]. \tag{2.1.128}$$

Proof Define

$$m(t) = \sum_{\Omega=1}^{n} E_\Omega(t,u), \quad m(0) = 0,$$

whence

$$\Delta m(t) = \sum_{\Omega=1}^{n} \Delta E_\Omega(t,u)$$

where

$$\Delta E_\Omega(t,u) = f_{\Omega_1}(t_1)\sum_{t_2=0}^{t-1} f_{\Omega_2}(t_2)\cdots\sum_{t=0}^{t_{\Omega-1}-1} f_{\Omega\Omega}(t_\Omega)u(t_\Omega).$$

From the assumptions on the functions $\Delta m(t)$, hence $m(t)$ is non-decreasing on $\mathbb{N}_0$. Hence we have

$$\begin{aligned}\Delta m(t) &\le \sum_{\Omega=1}^{n} \Delta E_\Omega(t, p+qm)\\ &\le \sum_{\Omega=1}^{n} \Delta E_\Omega(t,p) + \sum_{\Omega=1}^{n} \Delta E_\Omega(t,qm)\\ &\le \sum_{\Omega=1}^{n} \Delta E_\Omega(t,p) + m(t)\sum_{\Omega=1}^{n} \Delta E_\Omega(t,q)\end{aligned}$$

whence

$$m(t+1) - [1 + \sum_{\Omega=1}^{n} \Delta E_\Omega(t,q)]m(t) \le \sum_{\Omega=1}^{n} \Delta E_\Omega(t,p).$$

Multiplying the above inequality by $\prod_{s=0}^{t}[1 + \sum_{\Omega=1}^{n} \Delta E_\Omega(s,q)]^{-1}$ and summing over from 0 to $t-1$, we get

$$m(t)\prod_{s=0}^{t-1}[1 + \sum_{\Omega=1}^{n} \Delta E_\Omega(s,q)]^{-1} \le \sum_{s=0}^{t-1}(\sum_{\Omega=1}^{n} \Delta E_\Omega(s,p)) \prod_{\tau=s+1}^{t-1}[1 + \sum_{\Omega=1}^{n} \Delta E_\Omega(\tau,q)]^{-1},$$

which gives us

$$m(t) \le \sum_{s=0}^{t-1}(\sum_{\Omega=1}^{n} \Delta E_\Omega(s,p)) \prod_{\tau=s+1}^{t-1}[1 + \sum_{\Omega=1}^{n} \Delta E_\Omega(\tau,q)].$$

Substituting this estimate in (2.1.126), we can obtain (2.1.128). □

Remark 2.1.3 Let $n = 1$, $q(t) = 1$ in inequality (2.1.126), then the estimate (2.1.128) is the same as that in Sugiyama [611].

Remark 2.1.4 Let $n = 1$ in equality (2.1.126), then the estimate (2.1.128) is the same as that in Pachpatte [449].

Remark 2.1.5 Several particular cases of Theorem 2.1.26 have been considered by Pachpatte [462, 465, 469, 475], but results are not comparable.

Theorem 2.1.27 (Agarwal-Thandapani [17]) *Assume the following inequality holds for all $t \in \mathbb{N}_0$,*

$$u(t) \leq u_0(t) + \sum_{s=0}^{t-1} f(s)\Big(u(s) + \sum_{\tau=0}^{s-1} g(\tau)u(\tau)\Big). \tag{2.1.129}$$

Then for all $t \in \mathbb{N}_0$,

$$u(t) \leq u_0\Big[1 + \sum_{s=0}^{t-1} f(s)(1 - \phi(s)) \prod_{\tau=0}^{s-1}(1 + f(\tau) + g(\tau))\Big] \tag{2.1.130}$$

where

$$\phi(t) = \sum_{s=0}^{t-1} g(s)\Big(\prod_{\tau=0}^{s-1}(1 + f(\tau) + g(\tau))\Big)^{-1} \sum_{\tau=0}^{s-1} g(\tau).$$

Proof As in the proof of Theorem 2.1.26, we define $m(t)$ as the right-hand side of (2.1.129). Then

$$\begin{aligned} \Delta m(t) &= f(t)[u(t) + \sum_{\tau=0}^{t-1} g(\tau)u(\tau)] \\ &\leq f(t)[m(t) + \sum_{\tau=0}^{t-1} g(\tau)m(\tau)]. \end{aligned} \tag{2.1.131}$$

If we define

$$n(t) = m(t) + \sum_{\tau=0}^{t-1} g(\tau)m(\tau),$$

then we have

$$\Delta n(t) = m(t) + \Delta g(t)m(t) \tag{2.1.132}$$

and

$$m(t) \leq n(t) - u_0 \sum_{\tau=0}^{t-1} g(\tau). \tag{2.1.133}$$

Using (2.1.131) and (2.1.133) in (2.1.132), we conclude

$$\Delta n(t) \leq f(t)n(t) + g(t)n(t) - u_0 g(t) \sum_{\tau=0}^{t-1} g(\tau)$$

or

$$n(t+1) - [1 + f(t) + g(t)]n(t) \leq -u_0 g(t) \sum_{\tau=0}^{t-1} g(\tau).$$

Multiplying the above inequality by $\prod_{s=0}^{t}(1 + f(s) + g(s))^{-1}$ and summing over from 0 to $t-1$, we get

$$n(t) \leq u_0\Big(1 - \phi(t)\Big) \prod_{s=0}^{t-1} \Big(1 + f(s) + g(s)\Big).$$

By substituting the above estimate in (2.1.131) and summing over from 0 to $t-1$, we obtain (2.1.133). □

Remark 2.1.6 For $\phi(t) = 0$ in (2.1.129), the estimate is the same as that in [447]. In fact, almost all the results obtained in [462, 465, 469, 475] can be improved uniformly using the same arguments as in the proof of Theorem 2.1.27.

The next result is the discrete analogue of Willett's inequality [647].

Theorem 2.1.28 (Agarwal-Thandapani [17]) *Assume the following inequality holds for all $t \in \mathbb{N}_0$,*

$$u(t) \leq p_0(t) + \sum_{i=1}^{n} p_i(t)\Big(\sum_{s=0}^{t-1} v_i(s)u(s)\Big). \tag{2.1.134}$$

Then for all $t \in \mathbb{N}_0$,

$$u(t) \leq E_n p_0(t) \tag{2.1.135}$$

where

$$\begin{cases} E_i = D_i D_{i-1} \cdots D_0 \\ D_0 = \omega \\ D_j\omega = \omega + (E_{j-1}p_j)(\sum_{s=0}^{t-1} v_j\omega \prod_{\tau=s+1}^{t-1} (1 + v_j E_{j-1} p_j)), \quad j = 1, 2, \ldots, n. \end{cases}$$

Proof For $n = 1$, it follows from (2.1.134) that

$$\begin{aligned} u(t) &\le p_0(t) + p_1(t) \sum_{s=0}^{t-1} v_1(s) p_0(s) \prod_{\tau=s+1}^{t-1} \Big(1 + v_1(\tau) + p_1(\tau)\Big) \\ &= E_1 p_0(t). \end{aligned}$$

Now, assume that the assertion is true for some k such that $1 \le k \le n-1$, then for $k+1$, we have

$$u(t) \le p_0(t) + \sum_{i=1}^{k} p_i(t) \sum_{s=0}^{t-1} v_1(s) u(s) + p_{k+1}(t) \sum_{s=0}^{t-1} v_{k+1}(s) u(s)$$

whence

$$u(t) \le E_k p * (t)$$

where

$$p * (t) = p_0(t) + p_{k+1}(t) \sum_{s=0}^{t-1} v_{k+1}(s) u(s).$$

From the definition of E_k, we have

$$u(t) \le E_k p_0(t) + E_k p_{k+1} \Big(\sum_{s=0}^{t-1} v_{k+1}(s) u(s) \Big)$$

by using the fact that $\sum_{s=0}^{t-1} v_{k+1}(s) u(s)$ is non-decreasing for all $t \in \mathbb{N}_0$.

Again by using Theorem 2.1.27, we obtain

$$\begin{aligned} u(t) \quad &\le \quad E_k p_0(t) + E_k p_{k+1} \sum_{s=0}^{t-1} v_{k+1}(s) E_k p_0(s) \\ &\quad + \prod_{\tau=s+1}^{t-1} [1 + v_{k+1}(\tau) E_k p_{k+1}(\tau)] \\ &= \quad D_{k+1}(E_k p_0(t)) = D_{k+1} p_0(t), \end{aligned}$$

which implies the assertion by a finite induction. □

Corollary 2.1.13 (Agarwal-Thandapani [17]) *Let, in inequality (2.1.134), $p_i(t) \geq 1$ for all $i = 1, 2, .., n$ and all $t \in \mathbb{N}_0$. Then for all $t \in \mathbb{N}_0$,*

$$u(t) \leq \prod_{i=1}^{n} p_i(t)\Big[p_0(t) + \sum_{s=0}^{t-1} p_0(s)\Big(\sum_{i=1}^{n} h_i(s)\prod_{j=1}^{n} p_j(s)\Big)$$

$$\times \prod_{\tau=s+1}^{t-1}\Big(1 + \sum_{\Omega=1}^{t-1} h_\Omega(\tau)\prod_{i=1}^{n} p_i(t)\Big)\Big].$$

Corollary 2.1.14 (Agarwal-Thandapani [17]) *Let the inequality (2.1.134) be satisfied for all $t \in \mathbb{N}_0$, where (i) $p_0(t)$ is positive and non-decreasing; (ii) $p_i(t) \geq 1$ for all $i = 1, \ldots, n$ and non-decreasing for all $n \geq i \geq 2$. Then for all $t \in \mathbb{N}_0$,*

$$u(t) \leq E_n p_0(t)$$

where

$$\begin{cases} E_0\omega = \omega, \\ E_k\omega = \omega(E_{k-1}p_k)\prod_{s=0}^{t-1}[1 + h_k(s)E_{k-1}(p_k(s))], \ k = 1, 2, \ldots, n. \end{cases}$$

The proof of Corollaries 2.1.13 and 2.1.14 are similar to those of Theorems 2.1.26 and 2.1.28 respectively.

Now we introduce some new discrete inequalities of the Bellman-Bihari type of finite difference equations which are due to Yang [656], and are in a new form in the sense that their right-hand sides are dependent on, not independent of (as is usual), the values of the involved known coefficient functions at the point n.

For any real-valued function $f(n)$ defined on the set $\mathbb{N}_{n_0} = \{n_0, n_0 + 1, n_0 + 2, \cdots\}$, we define

$$\sum_{s=n_0}^{n-1} f(s) = \sum_{p=0}^{n-1-n_0} f(n_0 + p), \quad \prod_{s=n_0}^{n-1} f(s) = \prod_{p=0}^{k-1} f(n_0 + p), \ n = n_0 + k, \tag{2.1.136}$$

and for the convenience of the statement, we shall convent that all empty sums and empty products are equal to zero and one respectively such as

$$\sum_{s=n_0}^{n_0-1} f(s) \equiv 0, \quad \prod_{s=n_0}^{n_0-1} f(s) \equiv 1. \tag{2.1.137}$$

Theorem 2.1.29 (Yang [656]) *Let $x(n)$ and $p(x)$ be real-valued non-negative functions defined on $\mathbb{N}_{n_0}$; and let $f(n,s), g(n,s)$ and $h(n,s)$ be real-valued non-negative functions defined on $\mathbb{N}_{n_0} \times \mathbb{N}_{n_0}$, which are non-decreasing in n when $s \in \mathbb{N}_{n_0}$ fixed. Suppose that the following inequality holds for all $n \in \mathbb{N}_{n_0}$,*

$$x(n) \le p(n) + \sum_{s=n_0}^{n-1} f(n,s)x(s) + \sum_{s=n_0}^{n-1} g(n,s)\left(\sum_{k=n_0}^{n-1} h(s,k)x(k)\right). \quad (2.1.138)$$

Then for all $n \in \mathbb{N}_{n_0}$,

$$x(n) \le p(n) + \sum_{s=n_0}^{n-1} M(n,s)\Big\{\xi(n,s) + p(n_0)M(n,n_0)\prod_{t=n_0+1}^{s-1}\psi(n,t)$$
$$+ \sum_{t=n_0+1}^{s-1} M(n,t)\xi(n,t)\Big(\prod_{k=t+1}^{s-1}\psi(n,k)\Big)\Big\}, \quad (2.1.139)$$

where

$$\begin{cases} M(n,s) = \max[f(n,s), g(n,s)], \quad \text{for every fixed} \quad n \in \mathbb{N}_{n_0}, \\ \xi(n,s) = p(s) + \displaystyle\sum_{k=n_0}^{s-1} h(n,k)p(k), \\ \psi(n,s) = 1 + M(n,s) + h(n,s). \end{cases} \quad (2.1.140)$$

Proof Letting $n = n_0$ in (2.1.140), we get $x(n_0) \le p(n_0)$, which follows from the given inequality (2.1.138). Now fixing an arbitrary integer $r > n_0$ in $\mathbb{N}$, then we obtain from (2.1.138) that for all $n \in [n_0, r]$,

$$x(n) \le p(n) + L(n), \quad (2.1.141)$$

where

$$L(n) = \sum_{s=n_0}^{n-1} f(r,s)x(s) + \sum_{s=n_0}^{n-1} g(r,s)\left(\sum_{k=n_0}^{s-1} h(r,k)x(k)\right), \quad L(n_0) = 0 \quad (2.1.142)$$

whence, from (2.1.141) it follows that for all $n \in [n_0, r]$,

$$\triangle L(n) = f(r,n)x(n) + g(r,n)\sum_{k=n_0}^{n-1} h(r,k)x(k)$$
$$\le M(r,n)\left[L(n) + \sum_{k=n_0}^{n-1} h(r,k)L(k) + \xi(r,n)\right], \quad (2.1.143)$$

where $\xi(r, n)$ is defined by (2.1.140). Now we define a function

$$v(n) = L(n) + \sum_{k=n_0}^{n-1} h(r, k)L(k), \quad v(n_0) = 0, \tag{2.1.144}$$

which implies

$$\triangle v(n) = \triangle L(n) + h(r, n)L(n).$$

Using (2.1.143) and $v(n) \geq L(n)$ from (2.1.144), it follows that

$$\triangle v(n) \leq M(r, n)[v(n) + \xi(r, n)] + h(r, n)v(n),$$

i.e., for all $n \in [n_0, r]$,

$$v(n + 1) \leq \psi(r, n)v(n) + M(r, n)\xi(r, n), \tag{2.1.145}$$

where $\psi(r, n)$ and $\xi(r, n)$ are as defined in (2.1.140). Substituting $n = n_0, n_0 + 1, \ldots, r - 1$, successively in (2.1.145), then we obtain that the following bound on $v(n)$ holds for $n \in [n_0, r]$,

$$v(n) \leq p(n_0)M(r, n_0) \prod_{s=n_0+1}^{n-1} \psi(r, s) + \sum_{s=n_0+1}^{n-1} M(r, s)\xi(r, s) \prod_{k=s+1}^{n-1} \psi(r, k). \tag{2.1.146}$$

In fact, we can easily prove (2.1.146) by induction. Since $v(n_0) = 0$, we derive from (2.1.145) that

$$v(n + 1) \leq M(r, n_0)\xi(r, n_0)$$

which implies (2.1.146) holds when $n = n_0, n_0+1$. Suppose that (2.1.146) is proved for some integer n: $n_0 < n < r$. Then by (2.1.145) and in view of $p(n_0) = \xi(r, n_0)$, we obtain for all $n \in [n_0, r]$,

$$\begin{aligned} v(n + 1) &\leq \psi(r, n)v(n) + M(r, n)\xi(r, n) \\ &= p(n_0)M(r, n_0) \prod_{s=n_0+1}^{n} \psi(r, s) \\ &\quad + \sum_{s=n_0+1}^{n-1} M(r, s)\xi(r, s) \left(\prod_{k=s+1}^{n} \psi(r, k) \right) + M(r, n)\xi(r, n) \end{aligned}$$

$$= p(n_0)M(r, n_0) \prod_{s=n_0+1}^{n} \psi(r, s)$$

$$+ \sum_{s=n_0+1}^{n} M(r, s)\xi(r, s) \left(\prod_{k=s+1}^{n} \psi(r, k) \right),$$

since the empty product introduced is equal to number one (i. e., (2.1.137)), the above inequality proves (2.1.146) holds.

Now, substituting this estimate for $v(n)$ in (2.1.143) and then substituting successively $n = n_0, n_0 + 1, \ldots, r - 1$, and using $L(n_0)$, we can obtain for all $n \in [n_0, r]$,

$$L(n) \leq \sum_{s=n_0}^{n-1} M(r, s) \Big\{ \xi(r, s) + p(n_0)M(r, n_0) \prod_{t=n_0+1}^{s-1} \psi(r, t)$$

$$+ \sum_{t=n_0+1}^{s-1} M(r, t)\xi(r, t) \Big(\prod_{k=t+1}^{s-1} \psi(r, k) \Big) \Big\}.$$

Substituting this bound for $L(n)$ in (2.1.141) and then letting $n = r$ in the obtained inequality, in view of the arbitrariness of the choice of r in $\mathbb{N}_{n_0}$, then the proof is thus complete. □

Note that, if the function $p(n)$ is non-decreasing on $\mathbb{N}_{n_0}$ and does not vanish when $n > n_0$, then a much more simpler estimate for $x(n)$ than (2.1.139) can be obtained.

Theorem 2.1.30 (Yang [656]) *Let the functions $x(n), p(n), f(n, s), g(n, s)$ and $h(n, s)$ be the same as defined in above Theorem 2.1.29; and let $p(n)$ be non-decreasing on $\mathbb{N}_{n_0}$ and $p(n) > 0$ holds for all $n > n_0$. Suppose that the inequality (2.1.138) holds for all $n \in \mathbb{N}_{n_0}$, then for all $n \in \mathbb{N}_{n_0}$,*

$$x(n) \leq p(n) \left[1 + \sum_{s=n_0}^{n-1} M(n, s) \prod_{k=n_0}^{s-1} \psi(n, k) \right], \tag{2.1.147}$$

where $M(n, s)$ and $\psi(n, k)$ are as defined in (2.1.140).

Proof Obviously, the estimate for $x(n)$ in (2.1.147) holds when $n = n_0$. Now we fix an arbitrary integer $r > n_0$ in $\mathbb{N}$, then $p(r) > 0$ and we derive from (2.1.138) that for all $n \in [n_0, r]$,

$$\frac{x(n)}{p(r)} \leq 1 + \sum_{s=n_0}^{n-1} f(r, s) \frac{x(s)}{p(r)} + \sum_{s=n_0}^{n-1} g(r, s) \sum_{k=n_0}^{s-1} h(r, k) \frac{x(k)}{p(r)}. \tag{2.1.148}$$

Define the function $L(n)$ by the right-hand side of (2.1.148), and follow the same argument as in the proof of Theorem 2.1.29, then we can prove the estimate for $x(n)$ in (2.1.147) is valid. Because the argument used here is much more simpler than in Theorem 2.1.29, we leave the details to the reader. □

Remark 2.1.7 In Theorem 2.1.30, if $f(n,s) = g(n,s) = f(s), h(n,s) = g(s)$, then we derive Theorem 2.1 of Pachpatte [458] which, in turn, is a discrete generalization of an integral inequality given in Bellman and Cooke [75], p. 58. We note that the inequality (2.1.138) is more general than those inequalities discussed in Singare and Pachpatte [596], Theorems 1–3 and Pachpatte [468], Theorem 1.

The next result derives a new discrete generalization of the Gronwall-Bellman integral inequality. These generalization should have wide application in the study of finite difference equations and numerical analysis. Theorem 2.1.31 concerns a very general form of linear Bellman type discrete inequalities in one independent variable. It is a discrete analogue of an integral inequality obtained by Yang in [661] and it has extended many discrete inequalities of Agarwal and Thandapani, Pachpatte, and Sugiyama.

A very useful technique in the study of many problems concerning the behavior of solutions of discrete time system is to use recurrent inequalities involving sequences of real numbers, which may be considered as a discrete analogue of the Gronwall-Bellman integral inequality [75] or its generalizations. During the last few years, the area of applications of discrete inequalities has greatly expanded, and now encompasses not only many problems in the theory of finite difference equations and numerical analysis but also some questions of physics, technology, economics, and biological sciences. The discovery of new discrete inequalities and their new applications has attracted much interest from many authors (see, e.g., [17, 119, 656, 659, 684]).

Theorem 2.1.31 (Yang [661]) *Let $x(n), p(n)$ be real-valued non-negative functions defined on $\mathbb{N}_{n_0}$ with p non-decreasing on $\mathbb{N}_{n_0}$, and for $j = 1, 2, \cdots, m$, let $f_j(n,s)$ be real-valued non-negative functions defined on $\mathbb{N}_{n_0} \times \mathbb{N}_{n_0}$, which are non-decreasing in n for every fixed $s \in \mathbb{N}_{n_0}$. Suppose that the discrete inequality holds for all $n \in \mathbb{N}_{n_0}$,*

$$x(n) \leq p(n) + \sum_{s_1=n_0}^{n-1} f_1(n,s_1) \sum_{s_2=n_0}^{s_1-1} f_2(s_1,s_2) \cdots \sum_{s_m=n_0}^{s_{m-1}-1} f_m(s_{m-1},s_m) x(s_m). \tag{2.1.149}$$

Then we have for all $n \in \mathbb{N}_{n_0}$,

$$x(n) \leq p(n) W_m(n), \tag{2.1.150}$$

where $W_m(n) = V_m(n,n)$ *and* $V_m(r,q)$ *are defined by*

$$\begin{cases} V_1(r,q) = \prod_{s=n_0}^{q-1} \left\{ 1 + \sum_{j=1}^{m} f_j(r,s) \right\}, \\ V_k(r,q) = \prod_{s=n_0}^{q-1} g_{m-k+1}(r,s) + \sum_{s=n_0}^{q-1} f_{m-k+1}(r,s) V_{k-1}(r,s) \prod_{t=s+1}^{q-1} g_{m-k+1}(r,t), \\ \qquad k = 2,3,\cdots,m, \end{cases} \tag{2.1.151}$$

where

$$g_h(r,q) = \begin{cases} 1 + \sum_{j=1}^{h-1} f_j(r,q) - f_h(r,q), & \textit{if this expression } \geq 0 \textit{ on } \mathbb{N}_{n_0} \times \mathbb{N}_{n_0}, \\ 1 + \sum_{j=1}^{h-1} f_j(r,q), & \textit{otherwise for } h = 1,2,\cdots,m-1. \end{cases} \tag{2.1.152}$$

Proof For every $c \in \mathbb{N}_{n_0}$ and any real-valued non-negative function $v(n)$ on $\mathbb{N}_{n_0}$, we define

$$\begin{cases} I_h(c,n;v) = f_h(c,n) \sum_{s_{h+1}=n_0}^{n-1} f_{h+1}(c,s_{h+1}) \sum_{s_{h+2}=n_0}^{s_{h+1}-1} f_{h+2}(c,s_{h+2}) \\ \qquad \times \sum_{s_m=n_0}^{s_{m-1}-1} f_m(c,s_m) v(s_m), \qquad h = 1,2,\cdots,m-1, \\ I_m(c,n;v) = f_m(c,n) v(n). \end{cases} \tag{2.1.153}$$

Thus it follows from (2.1.153) that

$$I_{k-1}(c,n;v) = f_{k-1}(c,n) \sum_{s_k=n_0}^{n-1} I_k(c,s_k;v), \qquad k = 2,3,\cdots,m$$

and all $I_j(c,n;v)$ are non-decreasing in v (that is, if $0 \leq x(n) \leq y(n)$ for all $n \in \mathbb{N}_{n_0}$, then $I_j(c,n;x) \leq I_j(c,n;y)$ for all $n \in \mathbb{N}_{n_0}, j = 1,2,\cdots,m$).

Clearly, the estimate for $x(n)$ in (2.1.150) holds when $n = n_0$, since it reduces to the known relation $x(n_0) \leq p(n_0)$ of inequality (2.1.149). Now, fixing an arbitrary integer $n_1 (> n_0)$ from $\mathbb{N}_{n_0}$, then we derive from (2.1.149) for all $n \in \{n_0; n_1\}$,

$$x(n) \leq p(n_1) + \sum_{s_1=n_0}^{n-1} I_1(n_1,s_1;x), \tag{2.1.154}$$

where $\{n_0; n_1\}$ denotes the finite set consisting of the integers $n_0, n_0 + 1, \cdots, n_1$. To derive the upper bound on $x(n)$ from (2.1.154), we define

$$\begin{cases} K_1(n) = p(n_1) + \displaystyle\sum_{s_1=n_0}^{n-1} I_1(n_1, s_1; x), \\ K_k(n) = K_{k-1}(n) + \displaystyle\sum_{s_k=n_0}^{n-1} I_k(n_1, s_k; K_{k-1}), \ for \ \ k = 2, 3, \cdots, m, \ and \ n \in \{n_0; n_1\}. \end{cases} \tag{2.1.155}$$

Obviously, we have

$$\begin{cases} 0 \le p(n_1) = K_j(n_0), \ j = 1, 2, \cdots, m, \\ 0 \le x(n) \le K_1(n) \le K_2(n) \le \cdots \le K_m(n), \ \ n \in \{n_0; n_1\}. \end{cases} \tag{2.1.156}$$

We note that the following discrete inequalities for $K_j(n)$ can be established by induction:

$$\triangle K_h(n) + f_h(n_1, n)K_h(n) \le \sum_{j=1}^{h-1} f_j(n_1, n)K_h(n) + f_h(n_1, n)K_{h+1}(n),$$

$$for \ h = 1, 2, \cdots, m-1; \ \ n \in \{n_0; n_1 - 1\}. \tag{2.1.157}$$

In fact, noting that the $I_j(n_1, n; v)$ are non-decreasing in v, we can use (2.1.156) to derive from the first equality of (2.1.155) that for all $n \in \{n_0; n_1\}$,

$$\triangle K_1(n) = I_1(n_1, n; x) \le I_1(n_1, n; K_1).$$

Adding $f_1(n_1, n)K_1(n)$ to both sides of the above inequality, we obtain for all $n \in \{n_0; n - 1\}$,

$$\begin{aligned} \triangle K_1(n) + f_1(n_1, n)K_1(n) &\le f_1(n_1, n)K_1(n) + I_1(n_1, n; K_1) \\ &= f_1(n_1, n)\left\{K_1(n) + \sum_{s_2=n_0}^{n-1} I_2(n_1, s_2; K_1)\right\} \\ &= f_1(n_1, n)K_2(n), \end{aligned}$$

which establishes (2.1.157) for $h = 1$. We now suppose that (2.1.157) holds for $h = i$, where $1 \le i \le m - 2$. Then it follows from (2.1.155) that for all $n \in \{n_0; n_1 - 1\}$,

$$\begin{aligned} \triangle K_{i+1}(n) &= \triangle K_i(n) + I_{i+1}(n_1, n; K_1) \\ &\le \sum_{j=1}^{i-1} f_j(n_1, n)K_i(n) + f_i(n_1, n)K_{i+1}(n) + I_{i+1}(n_1, n; K_i). \end{aligned}$$

Adding $f_{i+1}(n_1,n)K_{i+1}(n)$ to both sides of the above inequality and using (2.1.156) and the monotonicity of $I_j(n_1,n;v)$ in v, we get for all $n \in \{n_0; n_1-1\}$,

$$\begin{aligned}
&\triangle K_{i+1}(n) + f_{i+1}(n_1,n)K_{i+1}(n)\\
&\leq \sum_{j=1}^{i} f_j(n_1,n)K_{i+1}(n) + f_{i+1}(n_1,n)K_{i+1}(n) + I_{i+1}(n_1,n;K_i)\\
&\leq \sum_{j=1}^{i} f_j(n_1,n)K_{i+1}(n) + f_{i+1}(n,n)\left\{K_{i+1}(n_1,n) + \sum_{s_{i+2}=n_0}^{n-1} I_{i+2}(n_1,s_{i+2};K_{i+1})\right\},
\end{aligned}$$

which, together with (2.1.155) yields (2.1.156).

Next, we shall derive the upper bound on $x(n)$ from the relations (2.1.155), (2.1.156) and (2.1.157). We derive from (2.1.155)–(2.1.156) and the definition of $I_m(n_1,n;v)$ that for all $n \in \{n_0; n_1-1\}$,

$$\begin{aligned}
\triangle K_m(n) &= \triangle K_{m-1}(n) + I_m(n_1,n;K_{m-1})\\
&\leq \sum_{j=1}^{m-2} f_j(n_1,n)K_{m-1}(n) + f_{m-1}(n_1,n)K_m(n) + f_m(n_1,n)K_{m-1}(n)\\
&\leq \sum_{j=1}^{m} f_j(n_1,n)K_m(n).
\end{aligned}$$

Substituting $n = n_0, n_0+1, \cdots, n_1-1$ in the last inequality, then we have for all $n \in \{n_0; n_1-1\}$,

$$\begin{aligned}
K_m(n) &\leq K_m(n_0)\prod_{s=n_0}^{n-1}\left\{1+\sum_{j=1}^{m} f_j(n_1,s)\right\}\\
&\equiv p(n_1)V_1(n_1,n),
\end{aligned}$$

where $V_1(n_1,n)$ is given by (2.1.151). Now substituting this bound for $K_m(n)$ in (2.1.157) with $h = m-1$, we get

$$\begin{aligned}
&\triangle K_{m-1}(n) + f_{m-1}(n_1,n)K_{m-1}(n)\\
&\leq \sum_{j=1}^{m-2} f_j(n_1,n)K_{m-1}(n) + f_{m-1}(n_1,n)p(n_1)V_1(n_1,n),
\end{aligned}$$

i.e., for all $n \in \{n_0; n-1\}$,

$$K_{m-1}(n+1) \leq g_{m-1}(n_1,n)K_{m-1}(n) + f_{m-1}(n_1,n)p(n_1)V_1(n_1,n), \tag{2.1.158}$$

where $g_{m-1}(n_1,n)$ is given by (2.1.152) with $h = m-1$. Substituting in (2.1.151) the numbers $n = n_0, n_0+1, \cdots, n_1-1$, or, more precisely, by using an easy inductive argument, we can obtain that for all $n \in \{n_0+1; n_1\}$,

$$K_{m-1}(n) \leq p(n_1)V_2(n_1,n), \tag{2.1.159}$$

where $V_2(n_1,n)$ is given by (2.1.151). Using this bound for $K_{m-1}(n)$ in (2.1.157) with $h = m-2$, we get

$$K_{m-2}(n+1) \leq g_{m-2}(n_1,n)K_{m-2}(n) + f_{m-2}(n_1,n)p(n_1)V_2(n_1,n), \tag{2.1.160}$$

for all $n \in \{n_0; n-1\}$, where $g_{m-2}(n_1,n)$ is given by (2.1.152). By repeating the same argument as used above from (2.1.158) to (2.1.159), we have for all $n \in \{n_0+1; n_1\}$,

$$K_{m-2}(n) \leq p(n_1)V_3(n_1,n). \tag{2.1.161}$$

Continuing in this way, after $m-1$ applications of the same argument, we derive for all $n \in \{n_0+1; n_1\}$,

$$K_1(n) \leq p(n_1)V_m(n_1,n), \tag{2.1.162}$$

where $V_m(n_1,n)$ is defined by (2.1.152). Now taking $n = n_1$ in (2.1.162), we may prove Theorem 2.1.31. □

We note that if all hypotheses of Theorem 2.1.31 are satisfied except the monotonicity of $p(n)$, then we may replace $p(n)$ by the monotonic function $\tilde{p}(n) = max\,\{p(n_0), p(n_0+1), \cdots, p(n)\}$, and then apply Theorem 2.1.31.

Remark 2.1.8 Theorem 2.1.31 has extended a known discrete inequality due to Sugiyama [614]. A similar result to (2.1.149) (when $m = 3$ and all functions $f_j(n,s)$ are independent of n) can be found in Pachpatte [466], Theorem 6. We also note that the additional assumptions $1 - f_1(n) \geq 0$ and $1 + f_1(n) - f_2(n) \geq 0$ for all $n \in \mathbb{N}$ were also required in [466].

In the next result, we shall introduce a more general case which is due to Yang [661]. To this end, define

$$J_m(n;v) = \sum_{s_1=n_0}^{n-1} f_{i1}(n,s_1) \sum_{s_2=n_0}^{s_1-1} f_{i2}(s_1,s_2) \cdots \sum_{s_m=n_0}^{s_{m-1}-1} f_{im}(s_{m-1},s_m)v(s_m).$$

Theorem 2.1.32 (Yang [661]) *Let $x(n), p(n)$ be the same as in Theorem 2.1.31 and let $f_{ik}(n,s)$ be real-valued non-negative functions on $\mathbb{N}_{n_0} \times \mathbb{N}_{n_0}$, non-decreasing in n for every fixed $s \in \mathbb{N}_{n_0}$ (here $i = 1, 2, \cdots, r; k = 1, 2, \cdots, m$). Suppose that the*

discrete inequality holds for all $n \in \mathbb{N}_{n_0}$,

$$x(n) \le p(n) + \sum_{i=1}^{r} J_{im}(n; x). \tag{2.1.163}$$

Then we have for all $n \in \mathbb{N}_{n_0}$,

$$x(n) \le p(n) \prod_{i=1}^{r} U^{(i)}(n), \tag{2.1.164}$$

where $U^{(i)}(n) = G_m^{(i)}(n, n)$, *and here* $G_m^{(i)}(r, n)$ *are given (in the increasing order of the index i) by*

$$\begin{cases} G_1^{(i)}(r, n) = \prod_{s=n_0}^{n-1} \left\{ 1 + \sum_{k=1}^{m} f_{ik}(r, s) \right\}, \\ G_j^{(i)}(r, n) = \prod_{s=n_0}^{n-1} \theta_{i,m-j+1}(r, s) + \sum_{s=n_0}^{n-1} F_{i,m-j+1}(r, s) G_{j-1}^{(i)}(r, s) \prod_{t=s+1}^{n-1} \theta_{i,m-j+1}(r, t) \end{cases} \tag{2.1.165}$$

for $i = 1, 2, \cdots, r, j = 2, 3, \cdots, m$, *where*

$$\theta_{ih}(r, n) = \begin{cases} \psi_{ih}(r, n) - F_{ih}(r, n), \text{ if this expression } \ge 0 \text{ on } \mathbb{N}_{n_0} \times \mathbb{N}_{n_0}, \\ \psi_{ih}(r, n), \qquad \text{otherwise}, \end{cases} \tag{2.1.166}$$

$$\psi_{ih}(r, n) = 1 + \sum_{j=1}^{h-1} F_{ij}(r, n), \quad h = 1, 2, \cdots, m-1, \tag{2.1.167}$$

and

$$F_{i1}(n, s) = f_{i1}(n, s) \prod_{q=1}^{i-1} U^{(q)}(n), F_{ij}(n, s) = f_{ij}(n, s),$$
$$\text{for } i = 1, 2, \cdots, r, \; j = 2, 3, \cdots, m. \tag{2.1.168}$$

Proof First we may rewrite inequality (2.1.163) as for all $n \in \mathbb{N}_{n_0}$,

$$x(n) \le A_1(n) + J_{1m}(n; x), \tag{2.1.169}$$

with

$$A_1(n) = p(n) + \sum_{i=2}^{r} J_{im}(n;x).$$

Obviously, $A_1(n)$ is non-negative and non-decreasing on $\mathbb{N}_{n_0}$, so by Theorem 2.1.31, we derive from (2.1.169) that for all $n \in \mathbb{N}_{n_0}$,

$$x(n) \leq A_1(n)U^{(1)}(n), \tag{2.1.170}$$

where $U^{(1)}(n) = G_m^{(1)}(n,n)$ and $G_m^{(1)}(r,n)$ is given by (2.1.165)–(2.1.168) with $i = 1$. Obviously, (2.1.169) can be rewritten as for all $n \in \mathbb{N}_{n_0}$,

$$x(n) \leq A_2(n) + J_{2m}^*(n;x), \tag{2.1.171}$$

with

$$A_2(n) = U^{(1)}(n)\left\{p(n) + \sum_{i=3}^{r} J_{im}(n;x)\right\},$$

where $J_{2m}^*(n;x)$ is obtained from $J_{2m}(n;x)$ by changing the $f_{21}(n,s)$ to the function $U^{(1)}(n)f_{21}(n,s)$.

Now applying Theorem 2.1.31 to (2.1.171) yields that for all $n \in \mathbb{N}_{n_0}$,

$$x(n) \leq \prod_{q=1}^{2} U^{(q)}(n)\left\{p(n) + \sum_{i=3}^{r} J_{im}(n;x)\right\}, \tag{2.1.172}$$

where $U^{(2)}(n) = G_m^{(2)}(n,n)$ and $G_m^{(2)}(r,n)$ is given by (2.1.165)–(2.1.168) with $i = 2$. If $r \geq 4$, we may rewrite (2.1.172) as for all $n \in \mathbb{N}_{n_0}$,

$$x(n) \leq A_3(n) + J_{3m}^*(n;x), \tag{2.1.173}$$

where

$$A_3(n) = \prod_{q=1}^{2} U^{(q)}(n)\left\{p(n) + \sum_{i=4}^{r} J_{im}(n;x)\right\},$$

where $J_{3m}^*(n;x)$ is obtained from $J_{3m}(n;x)$ by replacing $f_{31}(n,s)$ by the function $f_{31}(n,s)\prod_{q=1}^{2} U^{(q)}(n)$.

Applying Theorem 2.1.31 once again to (2.1.173), we can get for all $n \in \mathbb{N}_{n_0}$,

$$x(n) \le \prod_{q=1}^{3} U^{(q)}(n) \left\{ p(n) + \sum_{i=4}^{r} J_{im}(n;x) \right\} .$$

Proceeding in this way, we then obtain the desired bound on $x(n)$ in (2.1.164). □

Now we define

$$J_i^{(j)}(n;v) = \sum_{s_1=n_0}^{n-1} f_{i1}^{(j)}(n,s_1) \times \sum_{s_2=n_0}^{s_1-1} f_{i2}^{(j)}(s_1,s_2) \cdots \sum_{s_j=n_0}^{s_{j-1}-1} f_{ij}^{(j)}(s_{j-1},s_j) v(s_j).$$

The next result deals with a very general form of linear discrete inequalities of the Gronwall type in one independent variable. This is an analogue of an integral inequality established by Yang [657, Theorem 4].

Theorem 2.1.33 (Yang [661]) *Let $x(n), p(n)$ be the same as in Theorem 2.1.32; let $f_{ik}^{(j)}(n,s)$ be real-valued non-negative functions defined on $\mathbb{N}_{n_0} \times \mathbb{N}_{n_0}$, which are non-decreasing in n for every fixed $s \in \mathbb{N}_{n_0}$. Suppose that for all $n \in \mathbb{N}_{n_0}$, the following discrete inequality holds,*

$$x(n) \le p(n) + \sum_{j=1}^{q} \sum_{i=1}^{r_j} J_i^{(j)}(n;x), \tag{2.1.174}$$

where r_j are known positive integers. Then we have for all $n \in \mathbb{N}_{n_0}$,

$$x(n) \le p(n) \sum_{j=1}^{q} \left(\sum_{i=1}^{r_j} B_j^{(i)}(n) \right), \tag{2.1.175}$$

where $B_j^{(i)}(n) = H_{jj}^{(i)}(n,n)$, and $H_{jj}^{(i)}(n,n)$ are defined inductively on the index j by

$$\begin{cases} H_{1j}^{(i)}(r,n) = \prod_{s=n_0}^{n-1} \left\{ 1 + \sum_{k=1}^{j} F_{ik}^{(j)}(r,s) \right\} , \\ H_{kj}^{(i)}(r,n) = \prod_{s=n_0}^{n-1} a_{i,j-k+1}^{(j)}(r,s) + \sum_{s=n_0}^{n-1} F_{i,j-k+1}^{(j)}(r,s) H_{k-1,j}^{(i)}(r,s) \\ \qquad \times \prod_{t=s+1}^{n-1} a_{i,j-k+1}^{(j)}(r,t) \end{cases} \tag{2.1.176}$$

for $j = 1, 2, \cdots, q;\ i = 1, 2, \cdots, r_j;\ k = 2, 3, \cdots, j$, *where*

$$a_{ih}^{(j)}(r,n) = \begin{cases} c_{ih}^{(j)}(r,n) - F_{ih}^{(j)}(r,n), & \text{if this expression } \geq 0 \text{ on } \mathbb{N}_{n_0} \times \mathbb{N}_{n_0}, \\ c_{ih}^{(j)}(r,n), & \text{otherwise}, \end{cases} \tag{2.1.177}$$

$$c_{ih}^{(j)}(r,n) = 1 + \sum_{i=1}^{h-1} F_{ik}^{(j)}(r,n), \quad h = 1, 2, \cdots, j-1 \tag{2.1.178}$$

and

$$F_{i1}^{(j)}(n,s) = f_{i1}^{(j)}(n,s) \prod_{k=1}^{i-1} \left(\prod_{m=1}^{r_h} B_h^{(m)}(n) \right), \quad F_{ik}^{(j)}(n,s) = f_{ik}^{(j)}(n,s), \quad k = 2, 3, \cdots, i. \tag{2.1.179}$$

Proof The proof can be done by using Theorem 2.1.32 and an inductive argument. To finish the argument, we only give a few steps here. First we may rewrite (2.1.174) as for all $n \in \mathbb{N}_{n_0}$,

$$x(n) \leq E_1(n) + \sum_{i=1}^{r_1} J_i^{(1)}(n;x), \tag{2.1.180}$$

where

$$E_1(n) = p(n) + \sum_{j=2}^{q} \sum_{i=1}^{r_j} J_i^{(j)}(n;x).$$

Applying Theorem 2.1.32 to (2.1.180) yields that for all $n \in \mathbb{N}_{n_0}$,

$$x(n) \leq E_1(n) \prod_{i=1}^{r_1} B_1^{(i)}(n), \tag{2.1.181}$$

where $B_1^{(i)}(n)$ are given by (2.1.176) and (2.1.178) with $j = 1$. Second, we also rewrite the last inequality as for all $n \in \mathbb{N}_{n_0}$,

$$x(n) \leq E_2(n) + \sum_{i=1}^{r_2} \left(\prod_{k=1}^{r_1} B_1^{(k)}(n) \right) J_i^{(2)}(n;x), \tag{2.1.182}$$

with

$$E_2(n) = \left\{p(n) + \sum_{j=3}^{q}\sum_{i=1}^{r_j} J_i^{(j)}(n;x)\right\} \prod_{k=1}^{r_1} B_{k=1}^{(r_1)}(n). \tag{2.1.183}$$

Now a suitable application of Theorem 2.1.32 to the above inequality gives us for all $n \in \mathbb{N}_{n_0}$,

$$\begin{aligned} x(n) &\leq E_2(n) \prod_{k=1}^{r_2} B_2^{(k)}(n) \\ &\leq \prod_{j=1}^{2}\left(\prod_{k=1}^{r_j} B_j^{(k)}(n)\right)\left\{p(n) + \sum_{j=3}^{q}\sum_{i=1}^{r_j} J_i^{(j)}(n;x)\right\}. \end{aligned}$$

Continuing in this way, we then obtain the desired bound for $x(n)$ in (2.1.175). □

Remark 2.1.9 Many particular cases of (2.1.174) when $r_j = 1$ hold for $j = 1, 2, \cdots, q$ have been studied by Pachpatte [469, Theorems 1–3] and Agarwal and Thandapani [17, Theorems 1–3]. However, present consequences for these special cases are not comparable with those known results. Further, the special case of (2.1.174) when $r_j = 1, j = 1, 2$, was discussed by Yang [656], under the additional condition such that $p(n) > 0$ for all $n > n_0$.

The next corollary is Theorem 3 proved by Yang in [661].

Corollary 2.1.15 (Yang [661]) *Let $v(n), c(n) : \mathbb{N}_0 \to \mathbb{R}_+, \mathbb{N}_0 = \{0, 1, 2, \ldots.\}$, with $c(n)$ non-decreasing. Let $g_i(n,s) : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}_+$ $(i = 1, 2, 3)$ be non-decreasing in n for every fixed $s \in \mathbb{N}_0$. Then the discrete inequality for all $n \in \mathbb{N}_{n_0}$,*

$$v(n) \leq c(n) + \sum_{s=0}^{n-1} g_1(n,s)v(s) + \sum_{s=0}^{n-1} g_2(n,s) \sum_{t=0}^{n-1} g_3(s,t)v(t),$$

implies for all $n \in \mathbb{N}_{n_0}$,

$$v(n) \leq c(n)\gamma(n) \prod_{s=0}^{n-1}\Big(1 + \gamma(n)g_2(n,s) + g_3(n,s)\Big),$$

where $\gamma(n) := \prod_{s=0}^{n-1}\Big(1 + g_1(n,s)\Big)$. Here the empty sum and product are taken to be zero and one, respectively.

We know that one of the most fundamental contributions to the mathematical theory of systems has been the formulation and characterization of the property of controllability, which was first done by Kalman [309] for linear, time-invariant,

discrete-time systems. Many studies on this subject have done for discrete time control systems in several directions, we may refer to, Freeman [218], Kalman and Bertram [309], Lin and Varaiya [362], Michel and Wu [401], Pachpatte [444] and Weiss [642], etc. Although the "Lyapunov technique" has been employed in one form or another in the development of conditions for stability and asymptotic stability of the discrete time control systems, in the late 1950s, Lyapunov's method seemed to have been exhausted, and at about that time Popov introduced the frequency method, which is currently being used in differential control systems. Pachpatte [447] developed a theory which deals with the stability problems of a class of discrete time control systems of the more general type, by first establishing two fundamental finite difference inequalities which provide us a powerful technique to study the desired discrete time control systems.

Now we introduce such two fundamental finite difference inequalities which can be used as a powerful tool to study of discrete time control systems of the more general type, these results are due to Pachpatte [475].

Theorem 2.1.34 (Pachpatte [475]) *Let $x(n), p(n), h(n)$, and $q(n)$ be real-valued non-negative functions defined on $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, satisfying the inequality for all $n \in \mathbb{N}_0$,*

$$x(n) \le x_0 + \sum_{s=n_0}^{n-1} p(s)[x(s) + h(s)] + \sum_{s=n_0}^{n-1} p(s) \left(\sum_{\tau=n_0}^{s-1} q(\tau) x(\tau) \right) \tag{2.1.184}$$

where x_0 is a non-negative constant. Then for all $n \in \mathbb{N}_0$,

$$\begin{aligned} x(n) \le x_0 + \sum_{s=n_0}^{n-1} p(s) \Big[h(s) + x_0 \prod_{t=n_0}^{s-1} \Big(1 + p(t) + q(t)\Big) \\ + \sum_{t=n_0}^{s-1} p(t) h(t) \prod_{\tau=t+1}^{s-1} (1 + p(\tau) + q(\tau)) \Big]. \end{aligned} \tag{2.1.185}$$

Proof Define a function $m(n)$ by the right-hand side of (2.1.184). Then we have

$$\triangle m(n) = p(n)[x(n) + h(n)] + p(n) \sum_{\tau=n_0}^{n-1} q(\tau) x(\tau), \ m(n_0) = x_0,$$

which, along with (2.1.184), implies

$$\triangle m(n) \le p(n) h(n) + p(n) \Big[m(n) + \sum_{\tau=n_0}^{n-1} q(\tau) m(\tau) \Big]. \tag{2.1.186}$$

If we put

$$v(n) = m(n) + \sum_{\tau=n_0}^{n-1} q(\tau)m(\tau), \quad v(n_0) = m(n_0) = x_0, \tag{2.1.187}$$

then it follows from (2.1.186), (2.1.187) and the fact that $m(n) \leq v(n)$, that the following inequality holds for all $n \in \mathbb{N}_0$,

$$v(n+1) - [1 + p(n) + q(n)]v(n) \leq p(n)h(n). \tag{2.1.188}$$

Multiplying by $\prod_{s=n_0}^{n} \Big(1 + p(s) + q(s)\Big)^{-1}$ to both sides of (2.1.188) and summing up from n_0 to $n-1$, we obtain

$$v(n) \leq x_0 \prod_{s=n_0}^{n-1} (1 + p(s) + q(s)) + \sum_{s=n_0}^{n-1} p(s)h(s) \prod_{\tau=s+1}^{n-1} \Big(1 + p(\tau) + q(\tau)\Big).$$

Substituting this value of $v(n)$ into (2.1.186), we have

$$\begin{aligned} \triangle m(n) \leq p(n)\Big[h(n) &+ x_0 \prod_{s=n_0}^{n-1} \Big(1 + p(s) + q(s)\Big) \\ &+ \sum_{s=n_0}^{n-1} p(s)h(s) \prod_{\tau=s+1}^{n-1} (1 + p(\tau) + q(\tau))\Big] \end{aligned}$$

which implies

$$\begin{aligned} m(n) \leq x_0 + \sum_{s=n_0}^{n-1} p(s)\Big[h(s) &+ x_0 \prod_{t=n_0}^{s-1} (1 + p(t) + q(t)) \\ &+ \Big(\sum_{t=n_0}^{s-1} p(t)h(t) \prod_{\tau=t+1}^{s-1} \Big(1 + p(\tau) + q(\tau)\Big)\Big)\Big]. \end{aligned}$$

Now substituting this value of $m(n)$ into (2.1.184), we can obtain (2.1.185). □

We note that a special form of Theorem 2.1.34 when $h(n) = 0$ was already established and employed in the analysis of finite difference equations in Pachpatte [447].

In what follows, unless otherwise stated, all the functions which appear in the inequalities are assumed to be defined and non-negative in their domains of definition.

Next, we shall give some linear discrete Gronwall-Bellman inequalities and linear difference Gronwall-Bellman inequalities.

Let $\alpha, \beta \in \mathbb{Z}, \mathbb{N}_\alpha = \{n \in \mathbb{Z} : n \geq \alpha\}, \mathbb{N}_{\alpha,\beta} = \{n \in \mathbb{Z} : \alpha \leq n \leq \beta\}$, and let $x_n = x(n)$ be a sequence of real numbers, defined for all $n \in \mathbb{N}_\alpha$.

Let $\sum_{j=\alpha}^{\beta} x_j$ and $\Pi_{j=\alpha}^{\beta} x_j$ be the sum and product respectively of the numbers $x_j, j \in \mathbb{N}_{\alpha,\beta}$, and assume that $\sum_{j=\alpha}^{\alpha-1} x_j = 0, \Pi_{j=\alpha}^{\alpha-1} x_j = 1$. The first difference of the sequence x_j is the sequence $\triangle x_n = x_{n+1} - x_n, n \in \mathbb{N}_\alpha$, and the k-th difference (difference of order k) $\triangle^k x_n$ is inductively defined by for all $n \in \mathbb{N}_\alpha$,

$$\triangle^k x_n = \triangle(\triangle^{k-1} x_n) = \triangle^{k-1} x_{n+1} - \triangle^{k-1} x_n.$$

All sequences considered here consist of real numbers.

For two sequences $f_i, g_i \in \mathbb{N}_\alpha$, the following summation by parts formula holds for all $n \in \mathbb{N}_\alpha$,

$$\sum_{i=\alpha}^{n-1} f_i \triangle g_i = f_n g_n - f_\alpha g_\alpha - \sum_{i=\alpha}^{n-1} g_{i+1} \triangle f_i. \tag{2.1.189}$$

Moreover, for all $n \in \mathbb{N}_\alpha$,

$$\sum_{i=\alpha}^{n-1} \triangle g_i = g_n - g_\alpha, \tag{2.1.190}$$

and for all $n \in \mathbb{N}_\alpha$,

$$1 + \sum_{i=\alpha}^{n-1} f_i \prod_{j=i+1}^{n-1} (1 + f_j) = \prod_{j=\alpha}^{n-1} (1 + f_j), \tag{2.1.191}$$

provided that, $1 + f_i \neq 0, j \in \mathbb{N}_\alpha$.

Theorem 2.1.35 (Pachpatte [475]) *Assume that the following inequality holds for all $k \in \mathbb{N}_a$,*

$$u(k) \leq p(k) + q(k) \sum_{l=a}^{k-1} f(l) u(l). \tag{2.1.192}$$

Then for all $k \in \mathbb{N}_a$,

$$u(k) \leq p(k) + q(k) \sum_{l=a}^{k-1} f(l) p(l) \prod_{\tau=l+1}^{k+1} (1 + q(\tau) f(\tau)). \tag{2.1.193}$$

Proof Define a function $v(k)$ on $\mathbb{N}_a$ as follows

$$v(k) = \sum_{l=a}^{k-1} f(l)u(l).$$

Thus, we have

$$\triangle v(k) = f(k)u(k), \quad v(a) = 0. \tag{2.1.194}$$

Since $u(k) \le p(k) + q(k)v(k)$, and $f(k) \ge 0$, from (2.1.194) it follows

$$v(k+1) - (1 + q(k)f(k))v(k) \le p(k)f(k). \tag{2.1.195}$$

Because $1 + q(k)f(k) > 0$ for all $k \in \mathbb{N}_a$, we can multiply (2.1.195) by $\prod_{l=a}^{k}(1 + q(l)f(l))^{-1}$ to obtain

$$\triangle\Big(\prod_{l=a}^{k-1}(1 + q(l)f(l))^{-1}v(k)\Big) \le p(k)f(k)\prod_{l=a}^{k}(1 + q(l)f(l))^{-1}.$$

Summing the above inequality from a to $k-1$, and using $v(a) = 0$, we may get

$$\prod_{l=a}^{k-1}\Big(1 + q(l)f(l)\Big)^{-1} v(k) \le \sum_{l=a}^{k-1} p(l)f(l)\prod_{l=a}^{l}\Big(1 + q(\tau)f(\tau)\Big)^{-1}$$

which yields

$$v(k) \le \sum_{l=a}^{k-1} p(l)f(l) \prod_{\tau=l+1}^{k-1} (1 + q(\tau)f(\tau)). \tag{2.1.196}$$

Therefore, the assertion (2.1.193) follows from (2.1.196) and the inequality $u(k) \le p(k) + q(k)v(k)$. $\square$

Remark 2.1.10 The above proof obviously holds if $p(k)$ and $u(k)$ in Theorem 2.1.35 change sign on $\mathbb{N}_a$. Further, the inequality (2.1.193) is the best possible in the sense that equality in (2.1.192) implies an equality in (2.1.193).

The next result is another generalization of Corollary 2.1.4 which may be convenient in some applications.

Theorem 2.1.36 (Pachpatte [469]) *Let $x(n)$, $f(n)$, $g(n)$, $h(n)$ and $k(n)$ be real-valued non-negative functions defined on $\mathbb{N}$ satisfying the following inequality for*

all $n \in \mathbb{N}$,

$$x(n) \le x_0 + \sum_{s=n_0}^{n-1} f(s)x(s) + \sum_{s=n_0}^{n-1} g(s)\Big(\sum_{t=n_0}^{s-1} h(t)x(t) \times \Big(\sum_{\tau=n_0}^{t-1} k(\tau)x(\tau)\Big)\Big), \tag{2.1.197}$$

where x_0 *is a positive constant. If* $1 + f(n) - g(n) \ge 0$ *for all* $n \in \mathbb{N}$, *then for all* $n \in \mathbb{N}$,

$$x(n) \le x_0 \prod_{s=n_0}^{n-1} [1 + f(s) - g(s)] + \sum_{s=n_0}^{n-1} g(s)R(s) \times \prod_{t=s+1}^{n-1} [1 + f(\tau) - g(\tau)], \tag{2.1.198}$$

where for all $n \in \mathbb{N}$,

$$R(n) = \frac{x_0 \prod_{s=n_0}^{n-1} [1 + f(s) + g(s) + h(s)Q(s)]}{1 + x_0 \sum_{s=n_0}^{n-1} h(s) \prod_{t=n_0}^{s} [1 + f(t) + g(t) + h(t)Q(t)]}, \tag{2.1.199}$$

in which for all $n \in \mathbb{N}$,

$$Q(n) = \frac{x_0 \prod_{s=n_0}^{n-1} [1 + f(s) + g(s) + k(s)]}{1 - x_0 \sum_{s=n_0}^{n-1} h(s) \prod_{t=n_0}^{s} [1 + f(t) + g(t) + k(t)]}, \tag{2.1.200}$$

and

$$\sum_{s=n_0}^{n-1} h(s) \prod_{t=n_0}^{s} [1 + f(t) - g(t) + k(t)] < x_0^{-1}.$$

Proof Define a function $m(n)$ by the right-hand side of (2.1.197). Then

$$\Delta m(n) = f(n)x(n) + g(n)\Big(\sum_{t=n_0}^{n-1} h(t)x(t)\Big(\sum_{\tau=n_0}^{t-1} k(\tau)x(\tau)\Big)\Big),$$

$$m(n_0) = x_0,$$

which, in view of (2.1.197), implies

$$\Delta m(n) \leq f(n)m(n) + g(n)\Big(\sum_{t=n_0}^{n-1} h(t)m(t)\Big(\sum_{\tau=n_0}^{t-1} k(\tau)m(\tau)\Big)\Big).$$

Adding $g(n)m(n)$ to both sides of the above inequality, we have

$$\Delta m(n) + g(n)m(n) \leq f(n)m(n) + g(n)\Big[m(n) + \sum_{t=n_0}^{n-1} h(t)m(t)\Big(\sum_{\tau=n_0}^{t-1} k(\tau)m(\tau)\Big)\Big]. \tag{2.1.201}$$

If we put

$$\begin{cases} v(n) = m(n) + \sum_{t=n_0}^{n-1} h(t)m(t)\Big(\sum_{\tau=n_0}^{t-1} k(\tau)m(\tau)\Big), \\ v(n_0) = m(n_0) = x_0, \end{cases} \tag{2.1.202}$$

then it follows that

$$\Delta v(n) = \Delta m(n) + h(n)m(n)\Big(\sum_{\tau=n_0}^{t-1} k(\tau)m(\tau)\Big). \tag{2.1.203}$$

Using the facts $\Delta m(n) \leq f(n)m(n) + g(n)v(n)$ and $m(n) \leq v(n)$ from (2.1.201) and (2.1.202) in (2.1.203), we see that there holds that

$$\Delta v(n) \leq [f(n) + g(n)]v(n) + h(n)v(n)\Big(\sum_{\tau=n_0}^{t-1} k(\tau)m(\tau)\Big).$$

Adding $h(n)v^2(n)$ to both sides of the above inequality, we have

$$\Delta v(n) + b(n)v^2(n) \leq [f(n) + g(n)]v(n) + h(n)v(n)\Big(v(n) + \sum_{\tau=n_0}^{t-1} k(\tau)m(\tau)\Big). \tag{2.1.204}$$

Put

$$\begin{cases} u(n) = v(n) + \sum_{\tau=n_0}^{t-1} k(\tau)m(\tau), \\ u(n_0) = v(n_0) = x_0, \end{cases} \tag{2.1.205}$$

then

$$\Delta u(n) = \Delta v(n) + k(n)v(n). \tag{2.1.206}$$

Using the facts

$$\Delta v(n) \leq [f(n) + g(n)]v(n) + h(n)v(n)u(n)$$

and

$$v(n) \leq u(n)$$

from (2.1.204) and (2.1.205) in (2.1.206), we see that the following inequality holds

$$u(n+1) - [1 + f(n) + g(n) + k(n)]u(n) \leq h(n)u^2(n). \tag{2.1.207}$$

Define

$$\begin{cases} e_1(n) = \prod_{s=n_0}^{n-1}[1 + f(s) + g(s) + k(s)]^{-1}, \\ e_1(n_0) = 1, \end{cases} \tag{2.1.208}$$

then

$$e_1(n+1) - e_1(n) = -[f(n) + g(n) + k(n)]e_1(n+1). \tag{2.1.209}$$

Multiplying by $e_1(n+1)$ to both sides of (2.1.207) and using (2.1.209), we obtain

$$u(n+1)e_1(n+1) - u(n)e_1(n) \leq h(n)e_1^{-1}(n+1) \times [u(n)e_1(n+1)]^2. \tag{2.1.210}$$

Because $u(n)$ is monotone increasing, $e_1(n)$ is monotone decreasing, we know

$$[u(n)e_1(n+1)]^{-2} \geq z^{-2},$$

for all values of z between $u(n)e_1(n)$ and $u(n+1)e_1(n+1)$. So if we apply the mean value theorem to the function $F(z) = z^{-1}/-1$, we conclude that

$$\begin{aligned} \frac{[u(n+1)e_1(n+1)_{-1} - [u(n)e_1(n)]_{-1}}{-1} \leq {} & [u(n)e_1(n+1)]_{-2} \\ & \times[u(n+1)e_1(n+1) - u(n)e_1(n)]. \end{aligned} \tag{2.1.211}$$

Therefore from (2.1.210) and (2.1.211) it follows

$$[u(n+1)e_1(n+1)]_{-1} - [u(n)e_1(n)]_{-1} \geq -h(n)e^{-1}(n+1). \qquad (2.1.212)$$

Summing up both sides of (2.1.212) from n_0 to $n-1$ and substituting the values of $e_1(n)$ and $e_1(n+1)$ from (2.1.208), we obtain

$$u(n) \leq \frac{x_0 \prod_{s=n_0}^{n-1} [1 + f(s) + g(s) + k(s)]}{1 - x_0 \sum_{s=n_0}^{n-1} h(s) \prod_{t=n_0}^{s} [1 + f(t) + g(t) + k(t)]} = Q(n). \qquad (2.1.213)$$

Substituting this value of $u(n)$ in (2.1.204), we obtain

$$v(n+1) - [1 + f(n) + g(n) + h(n)Q(n)]v(n) \leq -h(n)v^2(n).$$

Now, defining

$$\begin{cases} e_2(n) = \prod_{s=n_0}^{n-1} [1 + f(s) + g(s) + h(s)Q(s)]^{-1}, \\ e_2(n_0) = 1, \end{cases}$$

and following the above argument with suitable modifications, we can obtain

$$v(n) \leq \frac{x_0 \prod_{s=n_0}^{n-1} [1 + f(s) + g(s) + h(s)Q(s)]}{1 + x_0 \sum_{s=n_0}^{n-1} h(s) \prod_{t=n_0}^{s} [1 + f(t) + g(t) + h(t)Q(t)]} = R(n).$$

Substituting this value of $v(n)$ in (2.1.201), we have

$$m(n+1) - [1 + f(n) - g(n)]m(n) \leq g(n)R(n).$$

Multiplying by $\sum_{s=n_0}^{n} [1 + f(n) - g(n)]^{-1}$ to both sides of the above inequality and summing up from n_0 to $n-1$, we conclude

$$m(n) \leq x_0 \prod_{s=n_0}^{n-1} [1 + f(s) - g(s)] + \sum_{s=n_0}^{n-1} g(s)R(s) \times \prod_{t=s+1}^{n-1} [1 + f(t) - g(t)],$$

which, substituted in (2.1.197), implies (2.1.198). $\square$

Theorem 2.1.37 (Agarwal [10]) *Let $u(k)$ be defined on $\mathbb{N}_a$, then for all $k \in \mathbb{N}_a$,*

$$1 + \sum_{l=a}^{k-1} u(l) \prod_{\tau=l+1}^{k-1} (1 + u(\tau)) = \prod_{l=a}^{k-1} (1 + u(l)). \tag{2.1.214}$$

Corollary 2.1.16 (Agarwal [10]) *Let in Theorem 2.1.35, $p(k) = p$ and $q(k) = q$ for all $k \in \mathbb{N}_a$. Then for all $k \in \mathbb{N}_a$,*

$$u(k) \le p \prod_{l=a}^{k-1} (1 + qf(l)). \tag{2.1.215}$$

Proof Estimate (2.1.163) follows from (2.1.163) and Theorem 2.1.35. □

Corollary 2.1.17 (Agarwal [10]) *Let in Theorem 2.1.35, $p(k)$ be non-decreasing and $q(k) \ge 1$ for all $k \in \mathbb{N}_a$. Then, for all $k \in \mathbb{N}_a$,*

$$u(k) \le p(k)q(k) \prod_{l=a}^{k-1} \Big(1 + q(l)f(l)\Big).$$

Proof In fact, for such $p(k)$ and $q(k)$, the inequality (2.1.193) implies

$$u(k) \le p(k)q(k) \Big(1 + \sum_{l=a}^{k-1} f(l)p(l) \prod_{\tau=l+1}^{k+1} (1 + q(\tau)f(\tau))\Big).$$

Therefore the result follows from Theorem 2.1.35 immediately. □

Theorem 2.1.38 (Agarwal [10]) *Assume the following inequality holds for all $k \in \mathbb{N}_a$,*

$$u(k) \le p(k) + q(k) \sum_{i=1}^{r} E_i(k, u), \tag{2.1.216}$$

where

$$E_i(k, u) = \sum_{l_1=a}^{k-1} f_{i1}(l_1) \sum_{l_2=a}^{l_1-1} f_{i2}(l_2) \cdots \sum_{l_i=a}^{l_{i-1}-1} f_{ii}(l_i)u(l_i). \tag{2.1.217}$$

Then for all $k \in \mathbb{N}_a$,

$$u(k) \le p(k) + q(k) \sum_{l=a}^{k-1} \Big(\sum_{i=1}^{r} \triangle E_i(l, p)\Big) \prod_{\tau=l+1}^{k-1} \Big(1 + \sum_{i=1}^{r} \triangle E_i(\tau, q)\Big). \tag{2.1.218}$$

Proof Define a function $v(k)$ on $\mathbb{N}_a$ as

$$v(k) = \sum_{i=1}^{r} \triangle E_i(k, u),$$

which yields

$$\triangle v(k) = \sum_{i=1}^{r} \triangle E_i(k, u), \quad v(a) = 0. \tag{2.1.219}$$

Since $u(k) \le p(k) + q(k)v(k)$, and $v(k)$ is non-decreasing in k, from (2.1.219) it follows

$$\begin{aligned}
\triangle v(k) &\le \sum_{i=1}^{r} \triangle E_i(k, p + qv) \\
&= \sum_{i=1}^{r} \triangle E_i(k, p) + \sum_{i=1}^{r} \triangle E_i(k, qv) \\
&\le \sum_{i=1}^{r} \triangle E_i(k, p) + v(k)\sum_{i=1}^{r} \triangle E_i(k, q).
\end{aligned}$$

The rest of the proof is similar to that of Theorem 2.1.35. □

Definition 2.1.1 (Condition (c)) We say that condition (c) is satisfied if for all $k \in \mathbb{N}_a$, the inequality (2.1.218) holds, where

$$f_{ii}(k) = f_i(k),\ 1 \le i \le r; \quad f_{i+1,i}(k) = f_{i+2,i}(k) = \cdots = f_{r,i}(k) = g_i(k),\ 1 \le i \le r-1.$$

In the next result, for all $k \in \mathbb{N}_a$, we shall denote

$$\begin{aligned}
\phi_j(k) = \max\Big\{0, \sum_{i=1}^{r-j+1} q(k)f_i(k) - g_{r-j+1}(k); \\
g_i(k) - g_{r-j+1}(k),\ 1 \le i \le r-j\Big\},\quad 1 \le j \le r
\end{aligned}$$

where $g_r(k) = 0$ for all $k \in \mathbb{N}_a$.

Theorem 2.1.39 (Agarwal [10]) *Assume the condition* **(c)** *holds. Then for all* $k \in \mathbb{N}_a$,

$$u(k) \le p(k) + q(k)\psi_j(k), \quad 1 \le j \le r, \tag{2.1.220}$$

where

$$\psi_j(k) = \sum_{l=a}^{k-1}\Big(p(l)\sum_{i=1}^{r-j+1} f_i(l) + g_{r-j+1}(l)\psi_{j-1}(l)\Big)\prod_{\tau=l+1}^{k-1}(1+\phi_j(r)),\ 1 \le j \le r.$$

Proof Indeed, if the condition **(c)** holds, then the inequality (2.1.216) is equivalent to the system

$$\begin{cases} u_1(k) & \le\ p(k) + q(k)\sum_{l=a}^{k-1}(f_1(l)u_1(l) + g_1(l)u_2(l)), & (2.1.221)\\ u_{j-1}(k) & =\ \sum_{l=a}^{k-1}(f_{j-1}(l)u_1(l) + g_{j-1}(l)u_j(l)),\ 3 \le j \le r, & (2.1.222)\\ u_r(k) & =\ \sum_{l=a}^{k-1} f_m(l)u_1(l). & (2.1.223)\end{cases}$$

Define

$$v_1(k) = \sum_{l=a}^{k-1}\Big(f_1(l)u_1(l) + g_1(l)u_2(l)\Big)$$

and $v_j(k) = u_j(k),\ 2 \le j \le r$, then from (2.1.221)–(2.1.223) it follows

$$\begin{cases} \triangle v_1(k) & \le\ f_1(k)\Big(p(k) + q(k)v_1(k)\Big) + g_1(k)v_2(k), & & (2.1.224)\\ \triangle v_{j-1}(k) & \le\ f_{j-1}(k)(p(k) + q(k)v_1(k)) + g_{j-1}(k)v_j(k), & 3 \le j \le r, & (2.1.225)\\ \triangle v_r(k) & \le\ f_r(k)(p(k) + q(k)v_1(k)). & & (2.1.226)\end{cases}$$

Thus adding up (2.1.224)–(2.1.226), we obtain

$$\begin{aligned}\triangle\Big(\sum_{i=1}^{r} v_i(k)\Big) &\le p(k)\sum_{i=1}^{r} f_i(k) + q(k)\sum_{i=1}^{r} f_i(k)v_i(k) + \sum_{i=1}^{r-1} g_i(k)v_{i+1}(k)\\ &\le p(k)\sum_{i=1}^{r} f_i(k) + \phi_1(k)\Big(\sum_{i=1}^{r} v_i(k)\Big).\end{aligned}$$

Now as in Theorem 2.1.35, we have

$$\sum_{i=1}^{r} v_i(k) \le \psi_1(k). \tag{2.1.227}$$

Adding up (2.1.224) and (2.1.225), $3 \le j \le r$ and using (2.1.227), we obtain

$$\begin{aligned}\triangle\Big(\sum_{i=1}^{r-1} v_i(k)\Big) &\le p(k)\sum_{i=1}^{r-1} f_i(k) + q(k)\sum_{i=1}^{r-1} f_i(k)v_i(k) \\ &\quad + \sum_{i=1}^{r-2} g_i(k)v_{i+1}(k) + g_{r-1}(k)\Big(\psi_1(k) - \sum_{i=1}^{r-1} v_i(k)\Big) \\ &\le \Big(p(k)\sum_{i=1}^{r-1} f_i(k) + g_{r-1}(k)\psi_1(k)\Big) + \phi_2(k)\Big(\sum_{i=1}^{r-1} v_i(k)\Big).\end{aligned}$$

Now similarly as in Theorem 2.1.35, we can get

$$\sum_{i=1}^{r-1} v_i(k) \le \psi_2(k). \tag{2.1.228}$$

Continuing in this manner, we finally get

$$\sum_{i=1}^{r-j+1} v_i(k) = \psi_j(k), \quad 3 \le j \le r. \tag{2.1.229}$$

Since $u(k) = u_1(k) \le p(k) + q(k)v_1(k)$, the result (2.1.220), $1 \le j \le r$, follows from (2.1.227)–(2.1.229), $3 \le j \le r$. □

Theorem 2.1.40 (Agarwal [10]) *Assume that the following inequality holds for all* $k \in \mathbb{N}_a$,

$$u(k) \le p_0(k) + \sum_{i=1}^{r} p_i(k)\sum_{l=a}^{k-1} q_i(l)u(l). \tag{2.1.230}$$

Then for all $k \in \mathbb{N}_a$,

$$u(k) \le F_r(p_0(k)), \tag{2.1.231}$$

where

$$\begin{cases} F_i = D_i D_{i-1} \cdots D_0, \\ D_0(w) = w, \\ D_j(w) = w + (F_{j-1}(p_j))\Big(\displaystyle\sum_{l=a}^{k-1} q_j(l)w(l)\prod_{\tau=l+1}^{k-1}(1 + q_j(\tau)F_{j-1}(p_j(\tau)))\Big), \quad 1 \le j \le r. \end{cases}$$

Proof The proof is by induction. For $r = 1$, inequality (2.1.230) reduces to (2.1.192) with $p(k) = p_0(k), q(k) = p_1(k)$ and $f(k) = q_1(k)$. Thus from Theorem 2.1.35, it follows that $u(k) \le D_1(p_0(k)) = F_1(p_0(k))$, i.e., (2.1.231) is true. Assume that the result is true for some j, where $1 \le j \le r-1$. Then, to prove for $j+1$, we have

$$u(k) \le \Big(p_0(k) + p_{j+1}(k)\sum_{l=a}^{k-1} q_{j+1}(l)u(l)\Big) + \sum_{i=1}^{j} p_i(k)\sum_{l=a}^{k-1} q_i(l)u(l) \tag{2.1.232}$$

and hence from (2.1.231) it follows that

$$u(k) \le F_j\Big(p_0(k) + p_{j+1}(k)\sum_{l=a}^{k-1} q_{j+1}(l)u(l)\Big).$$

where we have used the definition of F_j and the fact that $\sum_{l=a}^{k-1} q_{j+1}(l)u(l)$ is nondecreasing for all $k \in \mathbb{N}_a$, and further

$$\begin{aligned} u(k) &\le F_j(p_0(k)) + F_j\Big(p_{j+1}(k)\sum_{l=a}^{k-1} q_{j+1}(l)u(l)\Big) \\ &\le F_j(p_0(k)) + F_j(p_{j+1}(k))\sum_{l=a}^{k-1} q_{j+1}(l)u(l). \end{aligned}$$

Therefore applying Theorem 2.1.35 to the above inequality, we conclude

$$\begin{aligned} u(k) &\le F_j(p_0(k)) + F_j(p_{j+1}(k))\sum_{l=a}^{k-1} q_{j+1}(l)F_j(p_0(l)) \cdot \prod_{\tau=l+1}^{k-1}\Big(1 + q_{j+1}(\tau)F_j(p_{j+1}(\tau))\Big) \\ &= F_{j+1}(p_0(k)). \end{aligned}$$

Corollary 2.1.18 (Agarwal [10]) *Assume, in addition to hypotheses of Theorem 2.1.40, $p_i(k) \ge 1$ for all $k \in \mathbb{N}_a, 1 \le i \le r$. Then for all $k \in \mathbb{N}_a$, there holds that*

$$u(k) \le \prod_{j=1}^{r} p_j(k)\Bigg(p_0(k) + \sum_{l=a}^{k-1}\Big(\sum_{i=1}^{r} q_i(l)\prod_{j=0}^{r} p_j(l)\Big) \cdot \prod_{\tau=l+1}^{k-1}\Big(1 + \sum_{i=1}^{r} q_i(\tau)\sum_{j=1}^{r} p_j(\tau)p_j(\tau)\Big)\Bigg). \tag{2.1.233}$$

Proof For such $p_i(k), 1 \le i \le r$, inequality (2.1.230) can be written as (2.1.192) with $p(k) = \prod_{i=0}^{r} p_i(k)$, $q(k) = \prod_{i=1}^{r} p_i(k)$ and $f(k) = \sum_{i=1}^{r} q_i(k)$. □

Lemma 2.1.1 (Agarwal [10]) *Assume in Theorem 2.1.35, $p(k) = q(k)$, for all $k \in \mathbb{N}_a$, then for all $k \in \mathbb{N}_a$, we have*

$$u(k) \leq p(k) \prod_{l=a}^{k-1} (1 + p(l)f(l)). \tag{2.1.234}$$

Proof The proof is left to the reader as an exercise. □

Corollary 2.1.19 (Agarwal [10]) *Assume, in addition to hypotheses of Theorem 2.1.40, $p_0(k) > 0$ and non-decreasing; $p_i(k) \geq 1$, $1 \leq i \leq r$ and non-decreasing when $2 \leq i \leq r$ for all $k \in \mathbb{N}_a$. Then for all $k \in \mathbb{N}_a$,*

$$u(k) \leq G_r(p_0(k)), \tag{2.1.235}$$

where

$$\begin{cases} G_0(w) = w, \\ G_j(w) = wG_{j-1}(p_j) \displaystyle\prod_{l=a}^{k-1} \Big(1 + q_j G_{j-1}(p_j)\Big), \quad 1 \leq j \leq r. \end{cases} \tag{2.1.236}$$

Proof The proof can be done by induction. For $r = 1$, Corollary 2.1.17 implies that $u(k) \leq G_1(p_0(k))$. Let the result be true for some j, where $1 < j \leq r - 1$, then to prove for $j + 1$, we have (2.1.232). Since in (2.1.232), the part in brackets is positive and non-decreasing, we get

$$u(k) \leq G_j\Big(p_0(k) + p_{j+1}(k) \sum_{l=a}^{k-1} q_{j+1}(l)u(l)\Big).$$

In the above inequality, using the definition of G_j, we obtain

$$u(k) \leq G_j(p_0(k)) + p_{j+1}(k)G_j(p_0(k)) \sum_{l=a}^{k-1} q_{j+1}(l) \frac{u(l)}{p_0(l)},$$

which also yields

$$\begin{aligned} \frac{u(k)}{p_0(k)} &\leq \frac{p_{j+1}(k)G_j(p_0(k))}{p_0(k)} \Big(1 + \sum_{l=a}^{k-1} q_{j+1}(l) \frac{u(l)}{p_0(l)}\Big) \\ &= G_j(p_{j+1}(k)) \Big(1 + \sum_{l=a}^{k-1} q_{j+1}(l) \frac{u(l)}{p_0(l)}\Big). \end{aligned}$$

Now applying Lemma 2.1.1 to the above inequality, we conclude that $u(k) \leq G_{j+1}(p_0(k))$. □

Remark 2.1.11 In Corollary 2.1.19, the requirement $p_0(k) > 0$ is not essential. In fact, if $p_0(k) = 0$ for some k, then we can replace $p_0(k)$ by $p_0(k) + \epsilon$ for any $\epsilon > 0$. The conclusion then follows by letting $\epsilon \to 0$ in the resulting inequalities.

Theorem 2.1.41 (Agarwal [10]) *Assume that for all $k, r \in \mathbb{N}_a$ such that $k \leq r$, the following inequality holds*

$$u(r) \geq u(k) - q(r) \sum_{l=k+1}^{r} f(l)u(l), \tag{2.1.237}$$

where $u(k)$ is not necessarily non-negative. Then for all $k, r \in \mathbb{N}_a, k \leq r$,

$$u(r) \geq u(k) \prod_{l=k+1}^{r} \Big(1 + q(r)f(l)\Big)^{-1}, \tag{2.1.238}$$

and (2.1.238) is the best possible.

Proof Obviously, inequality (2.1.237) can be rewritten as

$$u(k) \leq u(r) + q(r) \sum_{l=k+1}^{r} f(l)u(l). \tag{2.1.239}$$

Let $v(k)$ be the right-hand side of (2.1.239), then for all $k, r \in \mathbb{N}_a, k \leq r$, it follows that $u(k) \leq v(k)$ and

$$\triangle v(k) = -q(r)f(k+1)u(k+1), \quad v(r) = u(r).$$

Since $q(r)f(k+1) \geq 0$ and $u(k+1) \leq v(k+1)$, we get

$$v(k) \leq \Big(1 + q(r)f(k+1)\Big)v(k+1), \quad v(r) = u(r)$$

which readily yields

$$v(k) \leq \prod_{l=k+1}^{r} \Big(1 + q(r)f(l)\Big)u(r). \tag{2.1.240}$$

Thus (2.1.238) readily follows from $u(k) \leq v(k)$ and (2.1.240). □

Theorem 2.1.42 (Agarwal [10]) *Assume that the following inequality holds for all $k \in \mathbb{N}_0$,*

$$u(k) \leq c_2 + h^{1/2} c_1 \sum_{l=0}^{k-1} (k-l)^{-1/2} u(l), \tag{2.1.241}$$

where $c_1 > 0, c_2 > 0$ *and* $h > 0$. *Then for all* $k \in \mathbb{N}_0$,

$$u(k) \le c_2\Big(1 + hc_1^{1/2} + 2c_1(kh)^{1/2}\Big)(1 + hc_1^2\pi)^k. \tag{2.1.242}$$

Proof Clearly, from (2.1.241) it follows

$$\begin{aligned}
u(k) &\le c_2 + h^{1/2}c_1\sum_{l=0}^{k-1}(k-l)^{-1/2}\Big(c_2 + h^{1/2}c_1\sum_{\tau=0}^{l-1}(l-\tau)^{-1/2}u(\tau)\Big)\\
&= c_2 + h^{1/2}c_1c_2\sum_{l=0}^{k-1}(k-l)^{-1/2} + hc_1^2\sum_{l=0}^{k-1}\sum_{\tau=0}^{l-1}(k-l)^{-1/2}(l-\tau)^{-1/2}u(\tau)\\
&= c_2 + h^{1/2}c_1c_2k^{-1/2} + h^{1/2}c_1c_2\sum_{l=0}^{k-1}(k-l)^{-1/2}\\
&\quad + hc_1^2\sum_{\tau=0}^{k-2}\Big(\sum_{l=\tau+1}^{k-1}(k-l)^{-1/2}(l-\tau)^{-1/2}\Big)u(\tau)\\
&\le c_2 + h^{1/2}c_1c_2 + h^{1/2}c_1c_2\sum_{l=0}^{k-1}(k-l)^{-1/2}\\
&\quad + hc_1^2\sum_{\tau=0}^{k-2}\Big(\sum_{l=1}^{k-\tau-1}(k-\tau-l)^{-1/2}l^{-1/2}\Big)u(\tau).
\end{aligned} \tag{2.1.243}$$

Now consider the function $\phi(t) = (k-\tau-t)^{-1/2}t^{-1/2}, 0 < t < k-\tau$ (≥ 2), which is strictly convex on the given interval and attains its minimum at $t = \frac{k-\tau}{2}$. Thus,

$$\sum_{l=1}^{k-\tau-1}(k-\tau-l)^{-1/2}l^{-1/2} = \sum_{l=1}^{k-\tau-1}\phi(l) \le \int_0^{k-\tau}\phi(t)dt,$$

which is an immediate consequence of interpreting the given sum as a lower Riemann sum, with the rectangle for the subinterval $[\frac{k-\tau}{2}, \frac{k-\tau}{2}+1]$ (if $k-\tau$ is even), or $[\frac{k-\tau-1}{2}, \frac{k-\tau+1}{2}]$ (if $k-\tau$ is odd) missing. But

$$\int_0^{k-\tau}\phi(t)dt = \int_0^1(1-t_1)^{-1/2}t_1^{-1/2}dt_1 = B(\frac{1}{2}, \frac{1}{2}) = \pi. \tag{2.1.244}$$

In the same manner, we have

$$\sum_{l=1}^{k-1}(k-l)^{-1/2} \le \int_0^k(k-t)^{-1/2}dt = 2k^{1/2}. \tag{2.1.245}$$

Inserting (2.1.244) and (2.1.245) into (2.1.243), we can obtain

$$u(k) \leq c_2\Big(1 + c_1 h^{1/2} + 2c_1(kh)^{1/2}\Big) + \sum_{\tau=0}^{k-1}(hc_1^2\pi)u(\tau).$$

Therefore the result (2.1.242) follows by applying Corollary 2.1.18 to the above inequality. □

Theorem 2.1.43 (Agarwal [10]) *Assume that in Theorem 2.1.38,* $r = 2, p(k) = u_0, q(k) = 1$ *and* $f_{11}(k) = f_{21}(k)$ *for all* $k \in \mathbb{N}_a$. *Then for all* $k \in \mathbb{N}_a$,

$$u(k) \leq u_0\Big(1 + \sum_{l=a}^{k-1} f_{11}(l)(1 - v(l))\prod_{\tau=a}^{l-1}(1 + f_{11}(\tau) + f_{22}(\tau))\Big), \quad (2.1.246)$$

where

$$v(k) = \sum_{l=a}^{k-1} f_{22}(l)\Big(\prod_{\tau=a}^{l}(1 + f_{11}(\tau) + f_{22}(\tau))\Big)^{-1}\sum_{\tau=a}^{l-1} f_{22}(\tau). \quad (2.1.247)$$

Proof The proof is left to the reader as an exercise. □

Now if we define $N(0, K) = \{0, 1, \cdots, K\}$, then we have the following result.

Theorem 2.1.44 (Agarwal [10]) *Assume that the following inequality holds for all* $k \in \mathbb{N}_a$,

$$u(k) \leq p(k) + \sum_{l=a}^{k-1} q(k, l)u(l). \quad (2.1.248)$$

Then for all $k \in \mathbb{N}_a$,

$$u(k) \leq P(k)\prod_{l=a}^{k-1}(1 + Q(k, l)), \quad (2.1.249)$$

where $P(k) = \max\{p(\tau) : \tau \in N(a, k)\}$ *and* $Q(k, l) = \max\{q(\tau, l) : \tau \in N(a, k)\}$.

Proof The proof is left to the reader as an exercise. □

Theorem 2.1.45 (Agarwal [10]) *Assume that the following inequality holds for all* $k \in \mathbb{N}_a$

$$u(k + 1) \leq p + qu(k), \quad (2.1.250)$$

where p and $u(k)$ are not necessarily non-negative. Then for all $k \in \mathbb{N}_a$,

$$u(k) \le q^{k-a}u(a) + \begin{cases} \frac{q^{k-a}-1}{q-1}p, & \text{if} \quad q \ne 1, \\ \\ (k-a)p, & \text{if} \quad q = 1. \end{cases} \tag{2.1.251}$$

Proof The proof is left to the reader as an exercise. □

Theorem 2.1.46 (Agarwal [10]) *Assume that for all $k, r-1 \in \mathbb{N}_a$ with $k \le r-1$, the following inequality holds*

$$u(r) \ge u(k) - q(r)\sum_{l=k}^{r-1} f(l)u(l), \tag{2.1.252}$$

where $u(k)$ is not necessarily non-negative. Then for all $k, r-1 \in \mathbb{N}_a, k \le r-1$,

$$u(r) \ge u(k)\prod_{l=k}^{r-1}\Big(1 - q(r)f(l)\Big) \tag{2.1.253}$$

as long as $1 - q(r)f(l) > 0$. Furthermore, the inequality (2.1.253) is the best possible.

Proof The proof is left to the reader as an exercise. □

Theorem 2.1.47 (Agarwal [10]) *Assume that the following inequality holds for all $k \in N(0, K)$*

$$u(k) \le c_2 + h^{1-\alpha}c_1\sum_{l=0}^{k-1}(k-l)^{-\alpha}u(l), \tag{2.1.254}$$

where $0 < \alpha < 1, c_1 > 0, c_2 > 0$ and $h > 0$. Furthermore, let ν be the smallest positive integer satisfying $\nu(1-a) \ge 1$. Then for all $k \in N(0, K)$,

$$u(k) \le c_2\Big(c_2' + hc_1'(Kh)^{\nu(1-\alpha)-1}\Big)\exp\Big(c_1'(Kh)^{\nu(1-\alpha)}\Big), \tag{2.1.255}$$

where

$$c_1' = \Big(c_1\Gamma(1-\alpha)\Big)^{\nu}\Gamma(\nu(1-\alpha)), \quad c_2' = \Big(1 + c_1h^{1-\alpha}\Big)\sum_{j=0}^{\nu-2}\lambda^j + \lambda^{\nu-1},$$
$$\lambda = c_1\frac{(Kh)^{1-\alpha}}{1-\alpha}. \tag{2.1.256}$$

Proof The proof is left to the reader as an exercise. □

Now we begin to introduce the discrete inequalities involving difference.

Theorem 2.1.48 (Agarwal [10]) *Assume that the following inequality holds for all* $k \in \mathbb{N}_a$,

$$\triangle^n u(k) \leq p(k) + q(k) \sum_{i=0}^{n} \sum_{l=a}^{k-1} q_i(l) \triangle^i u(l). \tag{2.1.257}$$

Then for all $k \in \mathbb{N}_a$,

$$\triangle^n u(k) \leq p(k) + q(k) \sum_{l=a}^{k-1} \phi_1(l) \prod_{\tau=l+1}^{k-1} (1 + \phi_2(\tau)), \tag{2.1.258}$$

where

$$\begin{aligned} \phi_1(k) = p(k) q_n(k) &+ \sum_{i=0}^{n} \sum_{j=0}^{i} \triangle^i u(a) q_i(k) \frac{(k-a)^{(i-j)}}{(i-j)!} \\ &+ \sum_{i=0}^{n-1} q_{n-i-1}(k) \sum_{l=a}^{k-i-1} \frac{(k-l-1)^{(i)}}{i!} p(l) \end{aligned} \tag{2.1.259}$$

and

$$\phi_2(k) = q(k) q_n(k) + \sum_{i=0}^{n-1} q_{n-i-1}(k) \sum_{l=a}^{k-i-1} \frac{(k-l-1)^{(i)}}{i!} q(l). \tag{2.1.260}$$

Proof Define a function $v(k)$ on $\mathbb{N}_a$ as

$$v(k) = \sum_{i=0}^{n} \sum_{l=a}^{k-1} q_i(l) \triangle^i u(l).$$

Then (2.1.257) can be rewritten as

$$\triangle^n u(k) \leq p(k) + q(k) v(k). \tag{2.1.261}$$

From the definition of $v(k)$, it follows readily

$$\triangle v(k) = \sum_{i=0}^{n} q_i(k) \triangle^i u(k).$$

Thus from the discrete Taylor's formula and (2.1.261), we can derive

$$\triangle v(k) \le q_n(k)(p(k) + q(k)v(k)) + \sum_{i=0}^{n-1} q_i(k)\Big(\sum_{j=i}^{n-1} \frac{(k-a)^{(j-i)}}{(j-i)!} \triangle^j u(a)$$

$$+ \frac{1}{(n-i-1)!} \sum_{l=a}^{k-n+i} (k-l-1)^{(n-i-1)} \triangle^n u(l)\Big)$$

$$\le p(k)q_n(k) + \sum_{i=0}^{n-1}\sum_{j=0}^{i} \triangle^i u(a) q_i(k) \frac{(k-a)^{(i-j)}}{(i-j)!} + q(k)q_n(k)v(k)$$

$$+ \sum_{i=0}^{n-1} q_{n-i-1}(k) \sum_{l=a}^{k-i-1} \frac{(k-l-1)^{(i)}}{i!}(p(l) + q(l)v(l)),$$

which yields by using the non-decreasing nature of $v(k)$,

$$\triangle v(k) \le \phi_1(k) + \phi_2(k)v(k).$$

The rest of the proof is similar to that of Theorem 2.1.35. □

Corollary 2.1.20 *Assume that in Theorem 2.1.48, $\triangle^i u(a) = 0, 0 \le i \le n-1, p(k)$ be non-decreasing and $q(k) = 1$ for all $k \in \mathbb{N}_a$. Then for all $k \in \mathbb{N}_a$,*

$$\triangle^n u(k) \le p(k) \sum_{l=a}^{k-1} (1 + \phi_3(l)),$$

where

$$\phi_3(k) = \sum_{i=0}^{n} \frac{(k-a)^{(i)}}{i!} q_{n-i}(k).$$

Proof The proof is similar to that of Corollary 2.1.17 and by using the equality

$$q_n(k) + \sum_{i=0}^{n-1} q_{n-i-1}(k) \sum_{l=a}^{k-i-1} \frac{(k-l-1)^{(i)}}{i!} = \sum_{i=0}^{n} \frac{(k-a)^{(i)}}{i!} q_{n-i}(k).$$

□

Theorem 2.1.49 (Agarwal [10]) *Assume that in addition to hypotheses of Theorem 2.1.48, $q_i(k) = q^*(k), 0 \le i \le n$, and $q(k) \ge 1$ for all $k \in \mathbb{N}_a$. Then for all $k \in \mathbb{N}_a$,*

$$\triangle^n u(k) \le p(k) + q(k)B_i(k), \quad 1 \le i \le n+1 \tag{2.1.262}$$

where

$$\begin{cases} B_1(k) = \sum_{l=a}^{k-1} q^*(l)\phi_4(l) \prod_{\tau=l+1}^{k-1} \Big(q^*(\tau)q(\tau) + q^*(\tau) + nq(\tau) + n\Big), \\ B_i(k) = \sum_{l=a}^{k-1} \Big(q^*(l)\phi_4(l) + B_{i-1}(l)\Big) \prod_{\tau=l+1}^{k-1} \Big(q^*(\tau)q(\tau) + q^*(\tau) \\ \qquad +(n-i-1)q(\tau) + n - i\Big),\ 2 \le i \le n, \end{cases}$$

and $\phi_4(k)$ is the same as $\phi_1(k)$ with $q_i(k) = 1, 0 \le i \le n$.

Proof Define

$$v_1(k) = \sum_{i=0}^{n} \sum_{l=a}^{k-1} q^*(l) \triangle^i u(l).$$

Then similarly as in Theorem 2.1.48, we may derive

$$\triangle v_1(k) + q^*(k)v_1(k) \le q^*(k)\phi_4(k) + q^*(k)q(k)v_1(k) + q^*(k)v_2(k), \quad (2.1.263)$$

where

$$v_2(k) = v_1(k) + \sum_{l=a}^{k-i-1} \sum_{i=0}^{n-1} \frac{(k-l-1)^{(i)}}{i!} q(l)v_1(l).$$

Similarly as again in Theorem 2.1.48 on using $v_1(k) \le v_2(k)$, we also get

$$\triangle v_2(k) + v_2(k) \le q^*(k)\phi_4(k) + (q^*(k)q(k) + q^*(k) + q(k))v_2(k) + v_3(k),$$

where

$$v_3(k) = v_2(k) + \sum_{l=a}^{k-i-1} \sum_{i=0}^{n-2} \frac{(k-l-1)^{(i)}}{i!} q(l)v_2(l).$$

Using $v_2(k) \le v_3(k)$, we can derive

$$\triangle v_3(k) + v_3(k) \le q^*(k)\phi_4(k) + (q^*(k)q(k) + q^*(k) + 2q(k) + 1)v_3(k) + v_4(k),$$

where

$$v_4(k) = v_3(k) + \sum_{l=a}^{k-i-1} \sum_{i=0}^{n-3} \frac{(k-l-1)^{(i)}}{i!} q(l)v_3(l).$$

Continuing this way, we may get

$$\triangle v_n(k) + v_n(k) \leq q^*(k)\phi_4(k) + (q^*(k)q(k) + q^*(k) \\ +(n-1)q(k) + (n-2))v_n(k) + v_{n+1}(k), \quad (2.1.264)$$

where

$$v_{n+1}(k) = v_n(k) + \sum_{l=a}^{k-1} q(l)v_n(l)$$

so that from $v_n(k) \leq v_{n+1}(k)$, it follows that

$$\triangle v_{n+1}(k) \leq q^*(k)\phi_4(k) + (q^*(k)q(k) + q^*(k) + nq(k) + (n-1))v_{n+1}(k). \quad (2.1.265)$$

Obviously, from the above definitions $v_1(k) \leq v_2(k) \leq \cdots \leq v_{n+1}(k)$ and $v_i(a) = 0, 1 \leq i \leq n+1$. Thus, as in Theorem 2.1.35, (2.1.265) implies $v_{n+1}(k) \leq B_1(k)$ and (2.1.262) follows from $\triangle^n u(k) \leq p(k) + q(k)v_{n+1}(k)$. Next, using $v_{n+1}(k) \leq B_1(k)$ in (2.1.264), we get

$$\triangle v_n(k) + v_n(k) \leq (q^*(k)\phi_4(k) + B_1(k)) + (q^*(k)q(k) + q^*(k) \\ +(n-1)q(k) + (n-2))v_n(k),$$

which yields $v_n(k) \leq B_2(k)$. Continuing this way, we easily find $v_i(k) \leq B_{n-i+2}(k); i = n+1, n, \ldots, 2$. Finally, we can use $v_2(k) \leq B_n(k)$ in (2.1.263) to obtain

$$\triangle v_1(k) \leq q^*(k)(\phi_4(k) + B_n(k)) + q^*(k)(q(k) - 1)v_1(k),$$

which implies $v_1(k) \leq B_{n+1}(k)$. □

Remark 2.1.12 In Theorem 2.1.49, we need $q(k) \geq 1$ only to prove the conclusion (2.1.262). Therefore, instead of $q(k) \geq 1$, it is enough to assume that $1 + q^*(k)(q(k) - 1) \geq 0$ for all $k \in \mathbb{N}_a$. Further, if there is no condition on $q(k)$, then an immediate upper estimate can be obtained from the inequality

$$\triangle v_1(k) \leq q^*(k)(\phi_4(k) + B_n(k)) + q^*(k)q(k)v_1(k).$$

Theorem 2.1.50 (Agarwal [10]) *Assume that the following inequality holds for all $k \in \mathbb{N}_a$,*

$$\triangle^n u(k) \leq p(k) + \sum_{i=0}^{n}\sum_{l=a}^{k-1} q_i(l)\triangle^n u(l), \quad (2.1.266)$$

where $p(k)$ is positive and non-decreasing. Then for all $k \in \mathbb{N}_a$,

$$\triangle^n u(k) \leq \frac{p(k)e^{-1}(k)}{1 - \sum_{l=a}^{k-1} p(l)\phi_3(l)e^{-1}(l+1)}, \tag{2.1.267}$$

where

$$e(k) = \prod_{l=a}^{k-1}(1 + \phi_5(l))^{-1}$$

and $\phi_5(k)$ is the same as $\phi_1(k)$ with $p(k) = 0$, as long as $1 - \sum_{l=a}^{k-1} p(l)\phi_3(l)e^{-1}(l+1) > 0$.

Proof Since $p(k)$ is positive and non-decreasing, inequality (2.1.266) implies

$$\frac{\triangle^n u(k)}{p(k)} \leq 1 + \sum_{i=0}^{n}\sum_{l=a}^{k-1} q_i(l)\triangle^i u(l)\frac{\triangle^n u(l)}{p(l)}. \tag{2.1.268}$$

Let $v(k)$ be the right-hand side of (2.1.268), then

$$\begin{aligned}
\triangle v(k) &= \sum_{i=0}^{n} q_i(l)\triangle^i u(k)\frac{\triangle^n u(k)}{p(k)} \\
&\leq q_n(k)p(k)v^2(k) + \sum_{i=0}^{n} q_i(k)v(k)\Big(\sum_{i=0}^{n}\frac{(k-a)^{(j-i)}}{(j-i)!}\triangle^j u(a) \\
&\quad + \frac{1}{(n-i-1)!}\sum_{l=a}^{k-n+i}(k-l-1)^{(n-i-1)}\triangle^n u(l)\Big) \\
&\leq p(k)\phi_3(k)v^2(k) + \phi_5(k)v(k),
\end{aligned}$$

which implies

$$\triangle(e(k)v(k)) \leq p(k)\phi_3(k)e^{-1}(k+1)\Big(e(k+1)v(k)\Big)^2. \tag{2.1.269}$$

Now since $v(k)$ is non-decreasing and $e(k)$ is non-increasing, we have

$$-\triangle(e(k)v(k))^{-1} = \int_k^{k+1}\frac{d(e(t)v(t))}{(e(t)v(t))^2} \leq \frac{\triangle(e(t)v(t))}{(e(t)v(t))^2}.$$

Thus from (2.1.269) it follows

$$-\triangle(e(k)v(k))^{-1} \leq p(k)\phi_3(k)e^{-1}(k+1),$$

which, by using $v(a) = 1$, yields

$$v(k) \leq \frac{e^{-1}(k)}{1 - \sum_{l=a}^{k-1} p(l)\phi_3(l)e^{-1}(l+1)}.$$

Therefore, substituting the above inequality into (2.1.268) readily implies (2.1.267). □

In the following two results, we consider discrete inequalities involving higher-order differences.

Theorem 2.1.51 (Agarwal-Thandapani [14]) *Let $u(t) \geq 0, p(t) \geq 0, q(t) \geq 0$, $\triangle^j u(t) \geq 0, h_j(t) \geq 0, j = 0, 1, \cdots, k$, be sequences defined for all $t \in \mathbb{N}_0$ such that for all $t \in \mathbb{N}_\alpha$,*

$$\triangle^k u(t) \leq p(t) + q(t) \sum_{j=0}^{k} \sum_{s=0}^{t-1} h_j(s) \triangle^j u(s). \tag{2.1.270}$$

Then

$$\triangle^k u(t) \leq p(t) + q(t) \sum_{s=0}^{t-1} \phi_1(s) \prod_{\tau=s+1}^{t-1} \Big(1 + \phi_2(\tau)\Big), \tag{2.1.271}$$

where

$$\begin{cases} \phi_1(t) = p(t)h_k(t) + \displaystyle\sum_{j=0}^{k-1} \sum_{i=0}^{j-1} \triangle^j u(0) h_i(t) \binom{t}{j-i} \\ \qquad + \displaystyle\sum_{j=0}^{k-1} h_{k-j-1}(t) \sum_{s=0}^{t-j-s} \binom{t-s-1}{j} p(s), \\ \phi_2(t) = q(t)h_k(t) + \displaystyle\sum_{j=0}^{k-1} h_{k-j-1}(t) \sum_{s=0}^{t-j-1} \binom{t-s-1}{j} q(s). \end{cases}$$

Proof Let $0 \leq j \leq k-1$. Then it is easy to verify

$$\triangle^j u(t) = \sum_{i=j}^{k-1} \binom{t}{i-j} \triangle u^i(0) + \sum_{s=0}^{t-k+j} \binom{t-s-1}{k-j-1} \triangle^k u(s). \tag{2.1.272}$$

Set $R(t) = \sum_{j=0}^{k} \sum_{s=0}^{k-1} h_j(s) \triangle^j u(s)$, then (2.1.270) can be rewritten as

$$\triangle^k u(t) \leq p(t) + q(t)R(t) \tag{2.1.273}$$

with $R(0) = 0$. From the definition of $R(t)$, we obtain by using (2.1.272) and (2.1.273)

$$\begin{aligned}\triangle R(t) &= \sum_{j=0}^{k} h_j(t)\triangle u(t)\\ &= h_k(t)\triangle^k u(t) + \sum_{j=0}^{k-1} h_j(t)\left[\sum_{i=j}^{k-1}\binom{t}{i-j}\triangle^i u(0) + \sum_{s=0}^{t-k+j}\binom{t-s-1}{k-j-1}\triangle^k u(s)\right]\\ &\leq \sum_{j=0}^{k-1}\sum_{i=0}^{j}\triangle^j u(0)h_i(t)\binom{t}{j-i} + h_k(t)[p(t)+q(t)R(t)]\\ &\quad + \sum_{j=0}^{k-1} h_{k-j-1}(t)\sum_{s=0}^{t-j-1}\binom{t-s-1}{j}[p(s)+q(s)R(s)].\end{aligned}$$

Since $R(t)$ is non-decreasing, we conclude

$$\triangle R(t) \leq \phi_1(t) + \phi_2(t)R(t),$$

or

$$R(t+1) - [1+\phi_2(t)]R(t) \leq \phi_1(t).$$

Multiplying the above inequalities by $\prod_{s=0}^{t}[1+\phi_2(s)]^{-1}$ and summing over t from 0 to $t_1 - 1 \in \mathbb{N}$, we get

$$R(t_1)\prod_{s=0}^{t_1-1}[1+\phi_2(t)]^{-1} \leq \sum_{s=0}^{t_1-1}\phi_1(s)\prod_{\tau=0}^{s}[1+\phi_2(\tau)]^{-1},$$

or

$$R(t_1) \leq \sum_{s=0}^{t_1-1}\phi_1(s)\prod_{\tau=s+1}^{t_1-1}[1+\phi_2(\tau)]^{-1},$$

which implies (2.1.271) by using (2.1.273). □

Corollary 2.1.21 (Agarwal-Thandapani [14]) *If, in Theorem 2.1.50,* $\triangle^j u(0) = 0, j = 0, \cdots, k-1$, *and* $p(t)$ *is non-decreasing in* $\mathbb{N}_0$, *then (2.1.270) implies that*

$$\triangle^k u(t) \leq p(t)\left[1 + q(t)\sum_{s=0}^{t-1}\phi_3(s)\sum_{\tau=s+1}^{t-1}[1+\phi_4(\tau)]\right], \tag{2.1.274}$$

where $\phi_3(t) = \sum_{j=0}^{k} h_{k-j}(t)\binom{t}{j}$, $\phi_4(t) = \phi_2(t)$.

Remark 2.1.13 If we take $k = 1$, $h_0(t) = h_1(t) = h(t)$ in the inequality (2.1.270), then we have

$$\Delta u(t) \le p(t) + q(t) \sum_{s=0}^{t-1} \phi_5(s) \prod_{\tau=s+1}^{t-1} [1 + \phi_6(\tau)],$$

where

$$\phi_5(t) = h(t)[u(0) + p(t) + \sum_{s=0}^{t-1} p(s)],\ \phi_6(t) = h(t)[q(t) + \sum_{s=0}^{t-1} q(s)],$$

which is not comparable with the result obtained in [444] whose generalization for any k can be stated in the next theorem.

Theorem 2.1.52 (Agarwal-Thandapani [14]) *In the inequality (2.1.270), let $h_j(t) = h(t)$ $(0 \le j \le k)$, $q(t) \ge 1$. Then*

$$\Delta^k u(t) \le p(t) + q(t) \sum_{s=0}^{t-1} h(s)\Big(A(s) + B_k(s)\Big) \prod_{\tau=s+1}^{t-1} \Big(1 + h(\tau)[q(\tau) - 1]\Big), \tag{2.1.275}$$

where

$$\begin{cases} A(t) = p(t) + \displaystyle\sum_{j=0}^{k-1} \sum_{i=0}^{j} \Delta^j u(0) \frac{(t)^{(i)}}{i!} + \sum_{j=0}^{k-1} \frac{1}{j!} \sum_{s=0}^{t-j-1} (t-s-1)^{(j)} p(s), \\ B_1(t) = \displaystyle\sum_{s=0}^{t-1} h(s) A(s) \prod_{\tau=s+1}^{t-1} \Big\{1 + [h(\tau)q(\tau) + h(\tau) + kq(\tau) + k - 1]\Big\}, \\ B_i(t) = \displaystyle\sum_{s=0}^{t-1} [h(s)A(s) + B_{i-1}(s)] \\ \qquad \times \displaystyle\prod_{\tau=s+1}^{t-1} \{1 + [h(\tau)q(\tau) + h(\tau)(k-i+1)q(\tau) + (k-i-1)]\}, \\ \qquad i = 2, 3, \cdots, k. \end{cases}$$

Proof Let

$$R_1(t) = \sum_{s=0}^{t-1} h(s) \sum_{j=0}^{k} \Delta^j u(s).$$

Then, as in Theorem 2.1.51, we have

$$\Delta R_1(t) + h(t)R_1(t) \leq h(t)A(t) + h(t)q(t)R_1(t) + h(t)R_2(t),$$

where

$$R_2(t) = R_1(t) + \sum_{j=0}^{k-1} \frac{1}{j!} \sum_{s=0}^{t-j-1} (t-s-1)^{(j)} q(s) R_1(s).$$

Similarly to Theorem 2.1.51, we know by using $R_1(t) \leq R_2(t)$,

$$\Delta R_2(t) + R_2(t) \leq h(t)A(t) + [h(t)q(t) + h(t) + q(t)]R_2(t) + R_3(t),$$

where

$$R_3(t) = R_2(t) + \sum_{j=0}^{k-2} \frac{1}{j!} \sum_{s=0}^{t-j-1} (t-s-1)^{(j)} q(s) R_2(s).$$

Noting that $R_2(t) \leq R_3(t)$, we get

$$\Delta R_3(t) + R_3(t) \leq h(t)A(t) + \Big(h(t)q(t) + h(t) + 2q(t) + 1\Big)R_3(t) + R_4(t),$$

where

$$R_4(t) = R_3(t) + \sum_{j=0}^{k-3} \frac{1}{j!} \sum_{s=0}^{t-j-1} (t-s-1)^{(j)} q(s) R_3(s).$$

Continuing in this way, we finally derive

$$\Delta R_{k+1}(t) \leq h(t)A(t) + [h(t)q(t) + h(t) + kq(t) + (k-1)]R_{k+1}(t),$$

where

$$R_i(0) = 0, \qquad i = 1, 2, \ldots, k+1.$$

Now, as in Theorem 2.1.51, multiplying the last inequality by $\prod_{s=0}^{t}[1 + h(s)q(s) + h(s) + kq(s) + (k-1)]^{-1}$ and summing over t from 0 to $t_1 - 1$, we get $R_{k+1}(t_1) \leq B_1(t_1)$ for all $t_1 - 1 \in \mathbb{N}$, and hence

$$\Delta R_k(t) + R_k(t) \leq h(t)A(t) + [h(t)q(t) + h(t) + (k-1)q(t) + (k-2)]R_k(t) + B_1(t),$$

which implies $B_1(t) \le B_2(t)$. Continuing in this way, we obtain

$$\Delta R_1(t) \le h(t)[A(t) + B_k(t)] + h(t)[q(t) - 1]R_1(t),$$

which yields the desired result. □

Remark 2.1.14 For the case when $q(t) \le 1$, we may obtain an estimate from the inequality

$$\Delta R_1(t) \le h(t)[A(t) + B_k(t)] + h(t)[q(t)]R_1(t).$$

We note that the estimate in Theorem 2.1.52 can be improved uniformly, and to justify this, we may consider the case of Remark 2.1.13.

Theorem 2.1.53 (Agarwal-Thandapani [14]) *Let* $k = 1$, $h_0(t) = h_1(t) = h(t)$, $q(t) \ge 1$ *in the inequality (2.1.270). Then*

$$\Delta u(t) \le p(t) + q(t)\sum_{s=0}^{t-1}[\phi_6(t) + h(s)\phi_7(s)]\prod_{\tau=s+1}^{t-1}[1 + q(\tau)h(\tau) - h(\tau)], \quad (2.1.276)$$

where

$$\phi_7(t) = \sum_{s=0}^{t-1}[\phi_6(t) - \psi(s)]\prod_{\tau=s+1}^{t-1}[1 + q(\tau) + h(\tau) + q(\tau)h(\tau)],$$

with

$$\psi(t) = u(0)q(t)[1 + h(t)]\sum_{s=0}^{t-1} q(s)\Big(\sum_{\tau=0}^{s+1} h(\tau)\Big).$$

Proof Similarly to the proof of Theorem 2.1.52, we know

$$\Delta R_1(t) + h(t)R_1(t) \le \phi_6(t) + h(t)q(t)R_1(t) + h(t)R_2(t), \quad (2.1.277)$$

where

$$R_2(t) = R_1(t) + \sum_{s=0}^{t-1} q(s)R_1(s).$$

Thus we get

$$\Delta R_2(t) \le \phi_6(t) + [h(t)q(t) + q(t)]R_1(t) + h(t)R_2(t).$$

Now noting that

$$R_1(t) \leq R_2(t) - u(0) \sum_{s=0}^{t-1} q(s) (\sum_{\tau=0}^{s-1} h(\tau)),$$

we obtain

$$\Delta R_2(t) \leq [\phi_6(t) - \psi(t)] + [h(t)q(t) + h(t) + q(t)] R_2(t),$$

which easily implies $R_2(t) \leq \phi_7(t)$.

Substituting $R_2(t) \leq \phi_7(t)$ in the inequality (2.1.277) and using the same method as in the previous theorems, we can obtain the desired result. □

The next result concerns the following inequality

$$\Delta^k u(t) \leq p(t) + q(t) \Big[\sum_{r=1}^{n-1} E_r(t, \sum_{i=0}^{k} \Delta^i u) + E_n(t, \Delta^k u) \Big], \tag{2.1.278}$$

where

$$E_r(t, *) = \sum_{t_1=0}^{t-1} f_{r1}(t_1) \sum_{t_2=0}^{t_1-1} f_{r2}(t_2) \cdots \sum_{t_r=0}^{t_{r-1}-1} f_{rr}(t_r) * (t_r).$$

Theorem 2.1.54 (Agarwal-Thandapani [14]) *Assume that the inequality (2.1.278) holds. Then*

$$\Delta^k u(t) \leq p(t) + q(t) \sum_{s=0}^{t-1} \left[\sum_{r=1}^{n-1} \Delta E_r(s, \phi_8) + \Delta E_n(s, p) \right] \times \prod_{\tau=s+1}^{t-1} \left[1 + (\sum_{r=1}^{n-1} \Delta E_r(\tau, \phi_9) + \Delta E_n(\tau, q)) \right],$$

where $\phi_8(t)$ and $\phi_9(t)$ are same as $\phi_1(t)$ and $\phi_2(t)$ with $h_i(t) = 1$, $i = 0, 1, \ldots, k$.

Proof Let $R(t)$ be the part of right-hand side appearing in the bracket of (2.1.278). Then similar to Theorem 2.1.51 we have,

$$\begin{aligned} \Delta R(t) &= \sum_{r=1}^{n-1} \Delta E_r(t, \sum_{i=0}^{k} \Delta^i u) + \Delta E_n(t, \Delta^k u)] \\ &= \sum_{r=1}^{n-1} \Delta E_r(t, \sum_{j=0}^{k-1} \Delta^j u) + \sum_{r=1}^{n} \Delta E_r(t, \Delta^k u) \end{aligned}$$

$$\leq \sum_{r=1}^{n-1} \Delta E_r(t, \phi_8 + \phi_9 R - p - qR) + \sum_{r=1}^{n} \Delta E_r(t, p + qR)$$

$$\leq \sum_{r=1}^{n-1} \Delta E_r(t, \phi_8 + \phi_9 R) + \Delta E_n(t, p + qR)$$

$$\leq \left[\sum_{r=1}^{n-1} \Delta E_r(t, \phi_8) + \Delta E_n(t, p)\right] + \left[\sum_{r=1}^{n-1} \Delta E_r(t, \phi_9) + \Delta E_n(t, q)\right] R(t),$$

or

$$R(t+1) - [1 + (\sum_{r=1}^{n-1} \Delta E_r(t, \phi_9) + \Delta E_n(t, q))]R(t)$$

$$\leq [\sum_{r=1}^{n-1} \Delta E_r(t, \phi_8) + \Delta E_n(t, p)].$$

The rest of the proof is the same as that for the previous theorems. □

Several particular cases of Theorem 2.1.54 have been discussed by Pachpatte [443–446]; however, the present result here, due to Agarwal [10] cannot be compared with his results. Nevertheless, as in Theorem 2.1.54 above, all his results can be improved uniformly, and in the next theorem, we shall give the improved version of Theorem 4 in [444].

Theorem 2.1.55 (Agarwal-Thandapani [14]) *In the inequality (2.1.278), let* $n = 2$, $k = 1$, $p(t) = u(0)$, $q(t) = 1$, $f_{11}(t) = f_{21}(t) = a(t)$, *and* $f_{22}(t) = b(t)$. *Then*

$$\Delta u(t) \leq u(0)\Big\{1 + \sum_{s=0}^{t-1}(2 - \phi_{10}(s))a(s) \prod_{\tau=0}^{s-1}[2 + a(\tau) + b(\tau)]\Big\},$$

where

$$\phi_{10}(t) = \sum_{s=0}^{t-1} \Big(1 + b(s)\Big)(1 + s + \sum_{\tau=0}^{s-1} b(\tau)) \prod_{\tau=0}^{s} \Big(2 + a(\tau) + b(\tau)\Big)^{-1}.$$

Proof For this particular case, let $R_1(t)$ be the right-hand side of the inequality (2.1.278). Then $R_1(0) = u(0)$ and

$$\Delta R_1(t) = a(t)[u(t) + \Delta u(t) + \sum_{s=0}^{t-1} b(s)\Delta u(s)]. \tag{2.1.279}$$

Since $\Delta u(t) \leq R_1(t)$ and $u(t) \leq u(0) + \sum_{s=0}^{t-1} R_1(s)$, we can derive

$$\Delta R_1(t) \leq a(t)R_2(t), \tag{2.1.280}$$

where $R_2(0) = 2u(0)$ and

$$R_2(t) = u(0) + R_1(t) + \sum_{s=0}^{t-1} R_1(s) + \sum_{s=0}^{t-1} b(s)R_1(s). \tag{2.1.281}$$

Therefore from (2.1.281) it follows

$$R_1(t) \leq R_2 - u(0)[1 + t + \sum_{s=0}^{t-1} b(s)].$$

Similarly,

$$\Delta R_2(t) \leq a(t)R_2(t) + [1 + b(t)]\Big(R_2(t) - u(0)(1 + t + \sum_{s=0}^{t-1} b(s))\Big),$$

or

$$R_2(t+1) - [2 + a(t) + b(t)]R_2(t) \leq -u(0)[1 + b(t)](1 + t + \sum_{s=0}^{t-1} b(s)),$$

which yields

$$R_2(t) \leq u(0)[2 - \phi_{10}(t)] \prod_{s=0}^{t-1}[2 + a(s) + b(s)].$$

Substituting the above estimate in (2.1.281) and summing over from 0 to $t_1 - 1 \in \mathbb{N}$, we can get the desired result. □

The discrete analogue of Theorem 1.2.60 is given in the following theorem (see, e.g., [500]).

Theorem 2.1.56 (Pachpatte [500]) *Let $u(n)$ be a real-valued non-negative function defined on $\mathbb{Z}_{\alpha,\beta} = \{n \in \mathbb{Z}, \alpha \leq n \leq \beta, \alpha, \beta \in \mathbb{Z}\}$, $E = \{(n,s) \in \mathbb{Z}^2 : \alpha \leq s \leq n \leq \beta\}$. Let $a(n,s)$, $b(n,s)$ be real-valued non-negative functions defined on E and non-decreasing in n for each $s \in \mathbb{Z}_{\alpha,\beta}$ and suppose that for all $n \in \mathbb{Z}_{\alpha,\beta}$,*

$$u(n) \leq c + \sum_{\sigma=\alpha}^{n-1} a(n,\sigma)u(\sigma) + \sum_{\sigma=\alpha}^{\beta} b(n,\sigma)u(\sigma), \tag{2.1.282}$$

where $c \geq 0$ is a constant. If

$$r(n) = \sum_{s=\alpha}^{\beta} b(n,s) \prod_{\sigma=\alpha}^{s-1} \Big(1 + a(s,\sigma)\Big) < 1, \tag{2.1.283}$$

then for all $n \in \mathbb{Z}_{\alpha,\beta}$,

$$u(n) \leq \frac{c}{1 - r(n)} \prod_{\sigma=\alpha}^{n-1} \Big(1 + a(n,\sigma)\Big). \tag{2.1.284}$$

Proof Fix any $m \in \mathbb{Z}_{\alpha,\beta}$, $\alpha \leq m \leq \beta$, then for $\alpha \leq n \leq m$, we have

$$u(n) \leq c + \sum_{s=\alpha}^{n-1} a(m,s)u(s) + \sum_{s=\alpha}^{\beta} b(m,s)u(s). \tag{2.1.285}$$

Define a function $z(n)$, $\alpha \leq n \leq m$ by the right-hand side of (2.1.285). Then $u(n) \leq z(n)$, $\alpha \leq n \leq m$,

$$z(\alpha) = c + \sum_{s=\alpha}^{\beta} b(m,s)u(s), \tag{2.1.286}$$

and

$$\begin{aligned} z(n+1) - z(n) &= a(m,n)u(n) \\ &\leq a(m,n)z(n), \end{aligned}$$

i.e., for $\alpha \leq n \leq m$,

$$z(n+1) \leq \Big(1 + a(m,n)\Big)z(n), \tag{2.1.287}$$

By setting $n = \sigma$ in (2.1.287) and $\sigma = \alpha, \alpha + 1, \cdots, m - 1$ successively, we obtain

$$z(m) \leq z(\alpha) \prod_{\sigma=\alpha}^{m-1} \Big(1 + a(m,\sigma)\Big). \tag{2.1.288}$$

Since m is arbitrary, from (2.1.288) and (2.1.286) with m replaced by n and using $u(n) \leq z(n)$, it follows

$$u(n) \leq z(\alpha) \prod_{\sigma=\alpha}^{n-1} \Big(1 + a(n,\sigma)\Big), \tag{2.1.289}$$

where

$$z(\alpha) = c + \sum_{s=\alpha}^{\beta} b(n,s)u(s). \tag{2.1.290}$$

Using (2.1.289) on the right-hand side of (2.1.290) and (2.1.283), we may obtain

$$z(\alpha) \le \frac{c}{1-\alpha}. \tag{2.1.291}$$

Therefore, inserting (2.1.291) into (2.1.289), we can get (2.1.283) and thus the proof is complete. □

In what follows, we shall also assume that all the sums and products involved in our discussion exist on the respective domains of their definitions.

The next result deals with a useful finite difference inequality similar to that of given in Theorem 2.1.13.

Theorem 2.1.57 (Pachpatte [497]) *Let $u(n)$, $a(n)$, $b(n)$ be real-valued non-negative functions defined for all $n \in \mathbb{N}_0$ and suppose that $a(n)$ is non-increasing for all $n \in \mathbb{N}_0$. If for all $n \in \mathbb{N}_0$,*

$$u(n) \le a(n) + \sum_{s=n+1}^{+\infty} b(s)u(s), \tag{2.1.292}$$

then for all $n \in \mathbb{N}_0$,

$$u(n) \le a(n) \prod_{s=n+1}^{+\infty} [1 + b(s)]. \tag{2.1.293}$$

Proof First we assume that $a(n) > 0$ for all $n \in \mathbb{N}_0$. Therefore from (2.1.292) it follows

$$\frac{u(n)}{a(n)} \le 1 + \sum_{s=n+1}^{+\infty} b(s)\frac{u(s)}{a(s)}. \tag{2.1.294}$$

Define a function $z(n)$ by the right-hand side of (2.1.294), then $\frac{u(n)}{a(n)} \le z(n)$ and

$$\begin{aligned} z(n) - z(n+1) &= b(n+1)\frac{u(n+1)}{a(n+1)} \\ &\le b(n+1)z(n+1), \end{aligned} \tag{2.1.295}$$

which implies

$$z(n) \le [1 + b(n+1)]z(n+1). \tag{2.1.296}$$

By setting $n = s$ in (2.1.296) and then substituting $s = n, n+1, \ldots, m-1$ $(m \ge n+1$ is arbitrary in $\mathbb{N}_0)$ successively, we obtain

$$z(n) \le z(m) \prod_{s=n+1}^{m} [1 + b(s)]. \tag{2.1.297}$$

Noting that $\lim_{m\to+\infty} z(m) = 1$ and by letting $m \to +\infty$ in (2.1.297), we get

$$z(n) \le \prod_{s=n+1}^{m} [1 + b(s)]. \tag{2.1.298}$$

Using (2.1.298) in $\frac{u(n)}{a(n)} \le z(n)$, we get the desired inequality in (2.1.293). If $a(n)$ is non-negative, we carry out the above procedure with $a(n) + \epsilon$ instead of $a(n)$, where $\epsilon > 0$ is an arbitrarily small constant, and subsequently pass to the limit as $\epsilon \to 0$ to obtain (2.1.293). The proof is hence complete. □

A sub-additive function is a function $f : A \to B$, having a domain A and an ordered codomain B that are both closed under addition, with the following property:

$$\forall x, y \in A, f(x+y) \le f(x) + f(y).$$

A sub-multiplicative function is a function $f : A \to B$, which satisfies

$$\forall x, y \in A, f(xy) \le f(x)f(y).$$

Another interesting and useful finite difference inequality is given in the following theorem.

Theorem 2.1.58 (Pachpatte [497]) *Let $u(n), a(n), b(n)$ be real-valued non-negative functions defined for all $n \in \mathbb{N}_0$, let $W(r)$ be a real-valued continuous, positive, non-decreasing, sub-additive and sub-multiplicative function on $\mathbb{R}_+$ and $H(r)$ be a real-valued, continuous, positive and non-decreasing function on $\mathbb{R}_+$. If for all $n \in \mathbb{N}_0$,*

$$u(n) \le f(n) + g(n)H\left(\sum_{s=n+1}^{+\infty} h(s)W(u(s))\right), \tag{2.1.299}$$

then for all $0 \le n \le n_1,\ n,\ n_1 \in \mathbb{N}_0$,

$$u(n) \le f(n) + g(n)H\left(G^{-1}[G(\sum_{s=n+1}^{\infty} W(f(s)) + \sum_{s=n+1}^{\infty} h(s)W(g(s)))]\right), \quad (2.1.300)$$

where

$$G(r) = \int_{r_0}^{r} \frac{ds}{W(H(s))}, \qquad r > 0, \quad (2.1.301)$$

and $r_0 > 0$ *is arbitrary,* G^{-1} *is the inverse function of* G *and for all* $0 \le n \le n_1,\ n,\ n_1 \in \mathbb{N}_0$,

$$G\left(\sum_{s=n+1}^{+\infty} h(s)W(f(s)) + \sum_{s=n+1}^{+\infty} h(s)W(g(s)) \in Dom(G^{-1})\right).$$

Proof Define a function $z(n)$ by

$$z(n) = \sum_{s=n+1}^{+\infty} h(s)W(u(s)). \quad (2.1.302)$$

Then from (2.1.299) it follows

$$u(n) \le f(n) + g(n)H(z(s)). \quad (2.1.303)$$

Now from (2.1.302) and (2.1.303), we derive that

$$\begin{aligned} z(n) &\le \sum_{s=n+1}^{+\infty} h(s)W(f(s) + g(s)H(z(s))) \\ &\le \sum_{s=n+1}^{+\infty} h(s)[W(f(s)) + W(g(s))W(H(z(s)))] \\ &\le \sum_{s=n+1}^{+\infty} h(s)W(f(s)) + \sum_{s=n+1}^{+\infty} W(g(s))W(H(z(s))). \end{aligned} \quad (2.1.304)$$

Define a function $v(n) = \epsilon + y(n)$, where $y(n)$ is defined by the right-hand side of (2.1.304) and $\epsilon > 0$ is an arbitrarily small constant. Then $z(n) \le v(n)$ and

$$\begin{aligned} v(n) - v(n+1) &= h(n+1)W(g(n+1))W(H(z(n+1))) \\ &\le h(n+1)W(g(n+1))W(H(v(n+1))). \end{aligned} \quad (2.1.305)$$

Thus from (2.1.301) and (2.1.305), we derive

$$G(v(n)) - G(v(n+1)) = \int_{v(n+1)}^{v(n)} \frac{ds}{W(H(s))} \leq \frac{[v(n) - v(n+1)]}{W(H(v(n+1)))} \leq h(n+1)W(g(n+1)). \tag{2.1.306}$$

Substituting $n = s$ and taking the sum over s from n to $p-1$ $(p \geq n+1$ is arbitrary in $\mathbb{N}_0)$, we obtain

$$G(v(n)) - G(v(n+1)) \leq \sum_{s=n+1}^{p} h(s)W(g(s)). \tag{2.1.307}$$

Noting that $\lim_{p\to+\infty} v(p) = \sum_{s=1}^{+\infty} h(s)W(f(s)) + \epsilon$ and by taking $p \to +\infty$ in (2.1.307), we get

$$v(n) \leq G^{-1}[G(\sum_{s=1}^{+\infty} h(s)W(f(s)) + \epsilon) + \sum_{s=n+1}^{+\infty} h(s)W(g(s))]. \tag{2.1.308}$$

Hence the desired inequality (2.1.300) follows from the fact that $z(n) \leq v(n)$, $\epsilon \to 0$ in (2.1.308) and (2.1.303). The sub-domain $0 \leq n \leq n_1$ is obvious. □

Theorem 2.1.59 (Pachpatte [497]) *Let $u(n), a(n), b(n)$ be real-valued non-negative functions defined for all $n \in \mathbb{N}_0$ and $L : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ be a function which satisfies the condition for all $u \geq v \geq 0$,*

$$0 \leq L(n,u) - L(n,v) \leq M(n,v)(u-v),$$

where $M(u,v)$ is a real-valued non-negative function defined for all $n \in \mathbb{N}_0$, $v \in \mathbb{R}_+$. If for all $n \in \mathbb{N}_0$,

$$u(n) \leq a(n) + b(n) \sum_{s=n+1}^{+\infty} L(s,u(s)), \tag{2.1.309}$$

then for all $n \in \mathbb{N}_0$,

$$u(n) \leq a(n) + b(n)e(n) \prod_{s=n+1}^{+\infty} [1 + M(s,a(s)b(s))], \tag{2.1.310}$$

where for all $n \in \mathbb{N}_0$,

$$e(n) = \sum_{s=n+1}^{+\infty} L(s, a(s)). \tag{2.1.311}$$

Proof Define a function $z(n)$ by

$$z(n) = \sum_{s=n+1}^{+\infty} L(s, u(s)), \tag{2.1.312}$$

then from (2.1.309) it follows

$$u(n) \leq a(n) + b(n)z(s). \tag{2.1.313}$$

On the other hand, from (2.1.312) and (2.1.313) and the hypotheses on L, we can see that

$$\begin{aligned} z(n) &\leq \sum_{s=n+1}^{+\infty} [L(s, a(s) + b(s)z(s)) - L(s, a(s)) + L(s, a(s))] \\ &\leq e(n) + \sum_{s=n+1}^{+\infty} M(s, a(s))b(s)z(s), \end{aligned} \tag{2.1.314}$$

when $e(s)$ is defined by (2.1.311). Clearly, $e(n)$ is real-valued non-negative and non-increasing in $n \in \mathbb{N}_0$. Now applying Theorem 2.1.57 to (2.1.314) yields

$$z(n) \leq e(n) \prod_{s=n+1}^{+\infty} [1 + M(s, a(s))b(s)] \leq h(n+1)W(g(n+1))W(H(v(n+1))). \tag{2.1.315}$$

Therefore the desired inequality (2.1.310) follows from (2.1.313) and (2.1.315). □

Theorem 2.1.60 (Pachpatte [497]) *Let* $u(n)$, $a(n)$, $b(n)$ *be real-valued non-negative functions defined for all* $n \in \mathbb{N}_0$ *and* $L : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$ *be a function which satisfies the condition for* $u \geq v \geq 0$,

$$0 \leq L(n, u) - L(n, v) \leq M(n, v)\phi^{-1}(u - v),$$

where $M(u, v)$ *is defined as in Theorem 2.1.59,* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be continuous and strictly increasing function with* $\phi(0) = 0$, ϕ^{-1} *is the inverse function of* ϕ *and for*

all $u,\ v \in \mathbb{R}_+$,

$$\phi^{-1}(uv) \leq \phi^{-1}(u)\phi^{-1}(v).$$

If for all $n \in \mathbb{N}_0$,

$$u(n) \leq a(n) + b(n)\phi\left(\sum_{s=n+1}^{+\infty} L(s, u(s))\right), \tag{2.1.316}$$

then for all $n \in \mathbb{N}_0$,

$$u(n) \leq a(n) + b(n)\phi\left(e(n)\prod_{s=n+1}^{+\infty} [1 + M(s, a(s)b(s))]\right) \tag{2.1.317}$$

where $e(n)$ *is defined by (2.1.311).*

Proof Define a function $z(n)$ by (2.1.312), then from (2.1.316) it follows

$$u(n) \leq a(n) + b(n)z(s). \tag{2.1.318}$$

Thus from (2.1.312) and (2.1.318) and the hypotheses on L and ϕ, we derive

$$\begin{aligned} z(n) &\leq \sum_{s=n+1}^{+\infty} [L(s, a(s) + b(s)\phi(z(s))) - L(s, a(s)) + L(s, a(s))] \\ &\leq e(n) + \sum_{s=n+1}^{+\infty} M(s, a(s))\phi^{-1}(b(s)\phi(z(s))) \\ &\leq e(n) + \sum_{s=n+1}^{+\infty} M(s, a(s))\phi^{-1}(b(s))z(s), \end{aligned}$$

when $e(s)$ is defined by (2.1.311). Now following the same argument as in the proof of Theorem 2.1.59, we can get (2.1.317). □

2.2 Systems of Linear One-Dimensional Difference Inequalities

In this section, we shall introduce some results on systems of linear difference inequalities.

First we begin with some systems of two discrete inequalities of Gronwall type which are due to Salem [564].

It is well-known that there have been a number of works written on the discrete analogue of the Gronwall inequality and its nonlinear version due to Bhiari [465, 473, 656, 661].

The expression $\sum_{s=0}^{t-1} b(s)$ represents the solution of the linear difference equation $\triangle x(t) = b(t)$ for all $t \in \mathbb{N}_0$ under the initial condition $x(0) = 0$, where the operator $\triangle$ is defined by $\triangle x(t) = x(t+1) - x(t)$. Also the expression $\prod_{s=0}^{t-1} C(s)$ represents the solution of the linear difference equation $x(t+1) = C(t)x(t)$ for all $t \in \mathbb{N}_0$ under the initial condition $x(0) = 1$. It is assumed that $\sum_{s=0}^{-1} b(s) = 0$ and $\prod_{s=0}^{-1} C(s) = 1$.

Theorem 2.2.1 (Salem [564]) *Assume the following system of two linear inequalities holds for all $t \in \mathbb{N}_0$,*

$$\begin{cases} u_1(t) \le a_1(t) + p_1(t) \displaystyle\sum_{s=0}^{t-1} E(s, u_1(s)) + q_1(t) \sum_{s=0}^{t-1} E(s, u_2(s)), & (2.2.1) \\ u_2(t) \le a_2(t) + p_2(t) \displaystyle\sum_{s=0}^{t-1} E(s, u_1(s)) + q_2(t) \sum_{s=0}^{t-1} E(s, u_2(s)), & (2.2.2) \end{cases}$$

where $a_1(t), a_2(t), p_1(t), p_2(t), q_1(t)$, and $q_2(t)$ are positive and non-decreasing functions, for all $t \in \mathbb{N}_0$, and

$$E(t, u) = \sum_{t_1=0}^{t-1} f_1(t_1) \sum_{t_2=0}^{t_1-1} f_2(t_2) \cdots \sum_{t_n=0}^{t_{n-1}-1} f_n(t_n) u(t_n), \tag{2.2.3}$$

$f_i(t_i)$ are real-valued and non-negative functions, $1 \le i \le n$, $t_i \in \mathbb{N}_0$. Then for all $t \in \mathbb{N}_0$,

$$\begin{cases} u_1(t) \le a_1(t) + p_1(t)\phi_1(t) + q_1(t)\psi_1(t), \\ u_2(t) \le a_2(t) + p_2(t)\phi_1(t) + q_2(t)\psi_1(t) \end{cases} \tag{2.2.4}$$

where

$$\begin{cases} \phi_1(t) = \displaystyle\sum_{s=0}^{t-1} \left(E(s, a_1(s)) + E(s, q_1(s))\psi(s)\right) \prod_{\tau=s+1}^{t-1} \left(1 + E(\tau, p_1(\tau))\right), \\ \psi_1(t) = \displaystyle\sum_{s=0}^{t-1} \left(E(s, a_2(s)) + E(s, p_2(s))\psi(s)\right) \prod_{\tau=s+1}^{t-1} \left(1 + E(\tau, q_2(\tau))\right), \end{cases} \tag{2.2.5}$$

and

$$\psi(t) = \sum_{s=0}^{t-1} (E(s, a_1(s) + a_2(t)) \prod_{\tau=s+1}^{t-1} (1 + E(\tau, p_1(\tau) + p_2(\tau) + q_1(\tau) + q_2(\tau))) . \tag{2.2.6}$$

Proof In fact, inequalities (2.2.1) and (2.2.2) can be rewritten as

$$\begin{cases} u_1(t) \le a_1(t) + p_1(t)R_1(t) + q_1(t)R_2(t), & (2.2.7) \\ u_2(t) \le a_2(t) + p_2(t)R_1(t) + q_2(t)R_2(t) & (2.2.8) \end{cases}$$

where

$$R_1(t) = \sum_{s=0}^{t-1} E(s, u_1(s)), \quad R_2(t) = \sum_{s=0}^{t-1} E(s, u_2(s)). \tag{2.2.9}$$

Since $R_1(t)$ and $R_2(t)$ are non-decreasing for all $t \in \mathbb{N}_0$, from (2.2.7)–(2.2.9) it follows

$$\begin{cases} \triangle R_1(t) \le E(t, a_1(t)) + E(t, p_1(t))R_1(t) + E(t, q_1(t))R_2(t), & (2.2.10) \\ \triangle R_2(t) \le E(t, a_2(t)) + E(t, p_2(t))R_1(t) + E(t, q_2(t))R_2(t). & (2.2.11) \end{cases}$$

We derive from (2.2.10) and (2.2.11) that

$$\begin{aligned} &R_1(t+1) + R_2(t+1) - [1 + E(t, p_1(t) + p_2(t) + q_1(t) + q_2(t))](R_1(t) + R_2(t)) \\ &\le E(t, a_1(t) + a_2(t))). \end{aligned} \tag{2.2.12}$$

Multiplying both sides of the above inequality by $\prod_{s=0}^{t-1} \Big(1 + E(t, p_1(t) + p_2(t) + q_1(t) + q_2(t))\Big)^{-1}$ and summing up from 0 to $t-1$, we can get

$$R_1(t) + R_2(t) \le \Phi(t). \tag{2.2.13}$$

Thus from (2.2.3) and (2.2.4), it follows

$$R_1(t+1) - [1 + E(t, p_1(t))]R_1(t) \le E(t, a_1(t)) + E(t, q_1(t))\Psi(t). \tag{2.2.14}$$

Multiplying both sides of the above inequality by $\prod_{s=0}^{t-1} \Big(1 + E(t, p_1(t))\Big)^{-1}$ and summing up from 0 to $t-1$, we can get

$$R_1(t) \le \phi_1(t). \tag{2.2.15}$$

Similarly, from (2.2.12) and (2.2.13) it follows

$$R_2(t) \le \psi_1(t). \tag{2.2.16}$$

Hence the desired result (2.2.4) follows from (2.2.7), (2.2.8), (2.2.15), and (2.2.16). □

Theorem 2.2.2 (Salem [564]) *Assume the following system of two linear inequalities holds for all $t \in \mathbb{N}_0$,*

$$\begin{cases} u_1(t) \le a_1(t) + p_1(t)\displaystyle\sum_{r=1}^{n} E_1^r(t, u_1(t)) + q_1(t)\sum_{r=1}^{n} E_1^r(t, u_2(t)), & (2.2.17)\\ u_2(t) \le a_2(t) + p_2(t)\displaystyle\sum_{r=1}^{n} E_1^r(t, u_1(t)) + q_2(t)\sum_{r=1}^{n} E_1^r(t, u_2(t)), & (2.2.18)\end{cases}$$

where

(i) $a_1(t), a_2(t), p_1(t), p_2(t), q_1(t)$, *and* $q_2(t)$ *are real-valued, positive and non-decreasing for all* $t \in \mathbb{N}_0$,
(ii) $E_1^r(t,u) = \sum_{t_1=0}^{t-1} f_{r1}(t_1) \sum_{t_2=0}^{t_1-1} f_{r2}(t_2) \cdots \sum_{t_2=0}^{t_{r-1}-1} f_{rr}(t_r) u(t_r)$,
(iii) $f_{ri}(t_j)$ *are real-valued and non-negative functions,* $t_j \in \mathbb{N}_0$. *Then for all* $t \in \mathbb{N}_0$,

$$\begin{cases} u_1(t) \le a_1(t) + p_1(t)\phi_2(t) + q_1(t)\psi_2(t),\\ u_2(t) \le a_2(t) + p_2(t)\phi_2(t) + q_2(t)\psi_2(t)\end{cases} \tag{2.2.19}$$

where

$$\begin{cases} \phi_2(t) = \displaystyle\sum_{s=0}^{t-1}\Big(A(a_1(s)) + A(q_1(s))\psi(s)\Big)\prod_{\tau=s+1}^{t-1}\Big(1 + A(p_1(\tau))\Big), & (2.2.20)\\ \psi_1(t) = \displaystyle\sum_{s=0}^{t-1}\Big(A(a_1(s)) + A(p_2(s))\psi(s)\Big)\prod_{\tau=s+1}^{t-1}\Big(1 + A(q_2(\tau))\Big), & (2.2.21)\\ A(a(t)) = \displaystyle\sum_{r=1}^{n} \triangle E^r(t, a(t)), & (2.2.22)\\ \psi(t) = \displaystyle\sum_{s=0}^{t-1} A(a_1(s) + a_2(t))\prod_{\tau=s+1}^{t-1}\Big(1 + A(p_1(\tau) + p_2(\tau) + q_1(\tau) + q_2(\tau))\Big). & (2.2.23)\end{cases}$$

Proof The proof is similar to that of Theorem 2.2.1. □

Theorem 2.2.3 (Salem [564]) *Assume the following system of two linear inequalities holds for all $t \in \mathbb{N}_0$,*

$$\begin{cases} u_1(t) \le a_1(t) + p_1(t) \sum_{r=1}^{n} E_1^r(t, u_1(t)) + q_1(t) \sum_{r=1}^{n} E_2^r(t, u_2(t)), & (2.2.24) \\ u_2(t) \le a_2(t) + p_2(t) \sum_{r=1}^{n} E_1^r(t, u_1(t)) + q_2(t) \sum_{r=1}^{n} E_2^r(t, u_2(t)) & (2.2.25) \end{cases}$$

where $a_1(t), a_2(t), p_1(t), p_2(t), q_1(t), q_2(t)$, and $E_1^r(t, u_1(t))$ are defined in Theorem 2.2.2, and

$$E_2^r(t, u) = \sum_{t_1=0}^{t-1} g_{r1}(t_1) \sum_{t_2=0}^{t_1-1} g_{r2}(t_2) \cdots \sum_{t_r=0}^{t_{r-1}-1} g_{rr}(t) u(t_r), \tag{2.2.26}$$

$g_{rj}(t_j)$ are real-valued and non-negative functions, $t \in \mathbb{N}_0$, $1 \le j \le r$.

Then for all $t \in \mathbb{N}_0$,

$$\begin{cases} u_1(t) \le a_1(t) + p_1(t)\phi_3(t) + q_1(t)\psi_3(t), \\ u_2(t) \le a_2(t) + p_2(t)\phi_3(t) + q_2(t)\psi_3(t) \end{cases} \tag{2.2.27}$$

where

$$\begin{cases} \phi_3(t) = \sum_{s=0}^{t-1} (B_1(a_1(s)) + B_1(q_1(s))\psi(s)) \prod_{\tau=s+1}^{t-1} (1 + B_1(p_1(\tau))), \\ \psi_3(t) = \sum_{s=0}^{t-1} (B_2(a_2(s)) + B_2(p_2(s))\psi(s)) \prod_{\tau=s+1}^{t-1} (1 + B_2(\tau, q_2(\tau))), \\ B_1(a(t)) = \sum_{r=1}^{n} \triangle E_1^r(t, a(t)), \quad B_2(b(t)) = \sum_{r=1}^{n} \triangle E_2^r(t, b(t)), \\ \psi(t) = \sum_{s=0}^{t-1} A(a_1(s), a_2(s)) \prod_{\tau=s+1}^{t-1} (1 + A(p_1(\tau) + q_1(\tau), p_2(\tau) + q_2(\tau))), \\ A(a(t), b(t)) = B_1(a(t)) + B_2(b(t)). \end{cases} \tag{2.2.28}$$

Proof The proof is similar to those of Theorems 2.2.1–2.2.2 □

Theorem 2.2.4 (Salem [564]) *Let the following system of two linear inequalities be satisfied,*

$$\begin{cases} u_1(t) \leq a_1(t) + p_1(t) \sum_{s=0}^{t-1} H(u_1(s)) + q_1(t) \sum_{s=0}^{t-1} H(u_2(s)), & (2.2.29) \\ u_2(t) \leq a_2(s) + p_2(t) \sum_{s=0}^{t-1} H(u_1(s)) + q_2(t) \sum_{s=0}^{t-1} H(u_2(s)) & (2.2.30) \end{cases}$$

where $a_1(t), a_2(t), p_1(t), p_2(t), q_1(t)$, and $q_2(t)$ are defined in Theorem 2.2.1, H is positive, continuous, sub-additive, and sub-multiplicative. Then

$$\begin{cases} u_1(t) \leq a_1(t) + p_2(t)\phi_4(t) + q_1(t)\psi_4(t) & (2.2.31) \\ u_2(t) \leq a_2(t) + p_2(t)\phi_4(t) + q_2(t)\psi_4(t), & (2.2.32) \end{cases}$$

where

$$\begin{cases} \phi_4(t) = \sum_{s=0}^{t-1} \{H(a_1(s) - p_1(s) - q_1(s)) + (H(p_1(s)) + H(q_1(s)))H(\psi(s))\}, \\ \psi_4(t) = \sum_{s=0}^{t-1} \{H(a_2(s) - p_2(s) - q_2(s)) + (H(p_2(s)) + H(q_2(s)))H(\psi(s))\}, \\ \psi(t) = G^{-1} \left\{ G(2) + \sum_{s=0}^{t-1} (A(s) + B(s)) \right\}, \end{cases}$$

and

$$\begin{cases} A(t) = H(a_1(s) - p_1(s) - q_1(s)) + H(a_2(s) - p_2(s) - q_2(s)), \\ B(t) = H(p_1(t)) + H(q_1(t)) + H(p_2(t)) + H(q_2(t)), \end{cases}$$

and

$$G(r) = \int_{r_0}^{r} \frac{ds}{s + H(s)}, \; 0 < r_0 \leq r,$$

as long as

$$G(2) + \sum_{s=0}^{t-1} (A(s) + B(s)) \in Dom\ (G^{-1}).$$

Proof Indeed, inequalities (2.2.29) and (2.2.30) can be rewritten as

$$u_1(t) \leq (a_1(t) - p_1(t) - q_1(t)) + p_1(t)R_1(t) + q_1(t)R_2(t) \tag{2.2.33}$$

and

$$u_2(t) \leq (a_2(t) - p_2(t) - q_2(t)) + p_2(t)R_1(t) + q_2(t)R_2(t), \tag{2.2.34}$$

where

$$R_1(t) = \sum_{s=0}^{t-1} H(u_1(s)) + 1$$

and

$$R_2(t) = \sum_{s=0}^{t-1} H(u_2(s)) + 1.$$

Then

$$\triangle R_1(t) \leq H(a_1(t) - p_1(t) - q_1(t)) + H(p_1(t))H(R_1(t)) + H(q_1(t))H(R_2(t)) \tag{2.2.35}$$

and

$$\triangle R_2(t) \leq H(a_2(t) - p_2(t) - q_2(t)) + H(p_2(t))H(R_1(t)) + H(q_2(t))H(R_2(t)). \tag{2.2.36}$$

Since $R_1(t)$ and $R_2(t)$ are non-decreasing and $R_1(0) = 1, R_2(0) = 1$, then from the definition of G, it follows

$$\begin{aligned} G(R_1(t-1) + R_2(t+1)) - G(R_1(t) + R_2(t)) &= \int_{R_1(t)+R_2(t)}^{R_1(t+1)+R_2(t+1)} \frac{ds}{s + H(s)} \\ &\leq \frac{\triangle(R_1(t) + R_2(t))}{R_1(t) + R_2(t) + H(R_1(t) + R_2(t))} \\ &\leq A(t) + B(t). \end{aligned}$$

Summing up from 0 to $t - 1$, we get

$$R_1(t) + R_2(t) \leq \psi(t). \tag{2.2.37}$$

From (2.2.35) and (2.2.37), we have

$$\triangle R_1(t) \leq H(a_1(t) - p_1(t) - q_1(t)) + (H(p_1(t)) + H(q_1(t)))H(\psi(t)),$$

and summing up from 0 to $t-1$, we can get

$$R_1(t) \leq \Phi_4(t). \tag{2.2.38}$$

Also from (2.2.36) and (2.2.37), we can derive

$$R_2(t) \leq \psi_4(t). \tag{2.2.39}$$

Hence (2.2.31)–(2.2.32) follow from (2.2.33), (2.2.34), (2.2.38), and (2.2.39). □

Theorem 2.2.5 (Salem [564]) *Let the following system of two inequalities be satisfied,*

$$\begin{cases} u_1(t) \leq a_1(t) + p_1(t)\sum_{s=0}^{t-1} e_1(s)u_1(s) + p_2(t)\sum_{s=0}^{t-1} e_2(s)u_2(s) \\ \qquad +p_3(t)\sum_{s=0}^{t-1} e_3(s)H(u_1(s)) + p_4(t)\sum_{s=0}^{t-1} e_4(s)H(u_2(s)), & (2.2.40) \\ u_2(t) \leq a_2(t) + q_1(t)\sum_{s=0}^{t-1} h_1(s)u_1(s) + q_2(t)\sum_{s=0}^{t-1} h_2(s)u_2(s) \\ \qquad +q_3(t)\sum_{s=0}^{t-1} h_3(s)H(u_1(s)) + q_4(t)\sum_{s=0}^{t-1} h_4(s)H(u_2(s)), & (2.2.41) \end{cases}$$

where all given functions are real-valued, positive, non-decreasing, and continuous functions, H is defined in Theorem 2.2.4, and for all $t \in \mathbb{R}$, $a_1 \geq p_1 + p_2 + p_3 + p_4$ and $a_2 \geq q_1 + q_2 + q_3 + q_4$. Then

$$\begin{aligned} u_1(t) \leq\ & a_1(t) + p_1(t)\sum_{s=0}^{t-1} e_1(s)a_1(s)\psi_6(s) + p_2(t)\sum_{s=0}^{t-1} e_2(s)a_2(s)\psi_7(s) \\ & +p_3(t)\sum_{s=0}^{t-1} e_3(s)\phi_6(s) + p_4(t)\sum_{s=0}^{t-1} e_4(s)\phi_7(s) \end{aligned} \tag{2.2.42}$$

and

$$u_2(t) \le a_2(t) + q_1(t)\sum_{s=0}^{t-1} h_1(s)a_1(s)\psi_6(s) + q_2(t)\sum_{s=0}^{t-1} h_2(s)a_2(s)\psi_7(s)$$
$$+q_3(t)\sum_{s=0}^{t-1} h_3(s)\phi_6(s) + q_4(t)\sum_{s=0}^{t-1} h_4(s)\phi_7(s), \tag{2.2.43}$$

where

$$\begin{cases} \phi_6(t) = H(a_1 - p_1 - p_2 - p_3 - p_4) + (H(p_1) + H(p_2)) + H(p_3) + H(p_4))H(\psi_6), \\ \phi_7(t) = H(a_2 - q_1 - q_2 - q_3 - q_4) + (H(q_1) + H(q_2)) + H(q_3) + H(q_4))H(\psi_7), \end{cases}$$

$$\begin{cases} \psi_6(t) = 4 + \displaystyle\sum_{s=0}^{t-1} \left[A_1(s) + B_1(s)\psi(s) + c_1(s)H(\psi(s))\right], \\ \psi_7(t) = 4 + \displaystyle\sum_{s=0}^{t-1} \left[A_2(s) + B_2(s) + c_2(s)H(\psi(s))\right], \end{cases}$$

$$\psi(t) = G^{-1}\left\{G(8) + \sum_{s=0}^{t-1}(A(s) + B(s) + c(s))\right\},$$

$$\begin{cases} A_1(t) = e_1(a_1 - P) + e_2(a_2 - Q) + e_3H(a_1 - P) + e_4H(a_2 - Q), \\ B_1(t) = e_1P + e_2Q, \\ C_1(t) = e_3(H(p_1) + H(P_2) + H(p_3) + H(p_4)) + e_4(H(q_1) + H(q_2) + H(q_3) + H(q_4)), \end{cases}$$

$$\begin{cases} A_2(t) = h_1(a_1 - P) + h_2(a_2 - Q) + h_3H(a_1 - P) + h_4H(a_2 - Q), \\ B_2(t) = h_1P + h_2Q, \\ C_2(t) = h_3(H(p_1) + H(P_2) + H(p_4)) + h_4(H(q_1) + H(q_2) + H(q_3) + H(q_4)), \end{cases}$$

$$\begin{cases} A(t) = A_1(t) + A_2(t), \\ B(t) = B_1(t) + B_2(t), \\ C(t) = C_1(t) + C_2(t), \\ P = p_1 + p_2 + p_3 + p_4, \quad Q = q_1 + q_2 + q_3 + q_4, \end{cases}$$

and

$$G(r) = \int_{r_0}^{r} \frac{ds}{s + H(s)}, \; 0 < r_0 \le r.$$

as long as

$$G(8) + \sum_{s=0}^{t-1}(A(s) + B(s) + C(s)) \in Dom \ \ (G^{-1}).$$

Proof The proof is similar to that of previous theorems. □

Now we begin to introduce some results on finite systems of inequalities. First, we introduce some concepts. Let the subscript i range over the integers $1, 2, \cdots, n$ and r be some fixed positive integer such that $1 \le r \le n$. The subscripts p and q range over the integers $1, 2, \cdots, r$ and $r+1, r+2, \cdots, n$ respectively.

Definition 2.2.1 (Agarwal [10]) The function $\mathcal{F}(k, \mu)$ is said to possess mixed monotone property if

(i) $f_p(k, \mu)$ is non-decreasing in $u_1, \cdots, u_r$ and non-increasing in $u_{r+1}, \cdots, u_n$ for all fixed $k \in \mathbb{N}_a$, and
(ii) $f_q(k, \mu)$ is non-increasing in $u_1, \cdots, u_r$ and non-decreasing in $u_{r+1}, \cdots, u_n$. In particular, $\mathcal{F}(k, \mu)$ is said to possess non-decreasing property if $f_i(k, \mu)$ is non-decreasing in $u_1, \cdots, u_n$ for all fixed $k \in \mathbb{N}_a$.

Definition 2.2.2 (Agarwal [10]) The function ν defined on $\mathbb{N}_a$ is said to be a r under and $(n-r)$ over function with respect to the system $\mu(k+1) = \mathcal{F}(k, \mu(k))$ if $v_p(k+1) \le f_p(k, \nu(k))$ and $v_q(k+1) \ge f_q(k, \nu(k))$ for all $k \in \mathbb{N}_a$. If $\nu(k)$ satisfies the reverse inequalities, then it is said to be r over and $(n-r)$ under function.

Theorem 2.2.6 (Agarwal [10]) *Let the function $\mathcal{F}(k, \mu)$ possess mixed monotone property. Further, let there exist two functions $\nu(k), \omega(k)$ defined on $\mathbb{N}_a$ such that*

$$\begin{cases} v_p(k+1) \le f_p(k, \nu(k)), \quad v_q(k+1) \le f_q(k, \nu(k)), & (2.2.44) \\ w_p(k+1) \le f_p(k, \omega(k)), \quad w_q(k+1) \le f_q(k, \omega(k)), & (2.2.45) \\ v_p(a) \le w_p(a), \quad v_q(a) \ge w_q(a). & (2.2.46) \end{cases}$$

Then for all $k \in \mathbb{N}_a$,

$$v_p(k) \le w_p(k), \quad v_q(k) \ge w_q(k). \tag{2.2.47}$$

Proof Define a function $z(k)$: $z_p(k) = w_p(k) - v_p(k)$ and $z_q(k) = w_q(k) - v_q(k)$. By induction, we shall show that $z_i(k) \ge 0$ for all $k \in \mathbb{N}_a$. For this, from (2.2.44)–(2.2.46), $z_i(a) \ge 0$. Let $z_i(k) \ge 0$ for some fixed $k \in \mathbb{N}_{a+1}$, then since $\mathcal{F}(k, \mu)$ is mixed monotone, we have

$$v_p(k+1) \le f_p(k, \nu(k)) \le f_p(k, \omega(k)) \le w_p(k+1)$$

and

$$w_q(k+1) \le f_q(k, \omega(k)) \le f_q(k, \nu(k)) \le v_q(k+1),$$

i.e., $z_i(k+1) \ge 0$. □

Corollary 2.2.1 (Agarwal [10]) *Let the function $\mathcal{F}(k, \mu)$ be non-decreasing. Further, let there exist two functions $\nu(k), \omega(k)$ defined on $\mathbb{N}_a$ such that*

$$\nu(k+1) \le \mathcal{F}(k, \nu(k)), \quad \omega(k+1) \ge \mathcal{F}(k, \omega(k)), \quad \nu(a) \le \omega(a).$$

Then for all $k \in \mathbb{N}_a$, $\nu(k) \le \omega(k)$.

Corollary 2.2.2 (Agarwal [10]) *Let the functions $\nu(k)$, $\omega(k)$ be r under and $(n-r)$ over, r over and $(n-r)$ under function with respect to the system $\mu(k+1) = \mathcal{F}(k, \mu(k))$ respectively. Further, let the vector valued function $\mathcal{F}(k, \mu(k))$ possess mixed monotone property. If $\nu(a) = \omega(a) = \mu(a) = \mu^0$, where $\mu(k)$ is the solution of the problem $\mu(k+1) = \mathcal{F}(k, \mu(k))$, then for all $k \in \mathbb{N}_a$, we have*

$$v_p(k) \le u_p(k) \le w_p(k), \quad v_q(k) \le u_q(k) \le w_q(k).$$

Theorem 2.2.7 (Agarwal [10]) *Assume that the following inequality holds for all $k \in \mathbb{N}_a$,*

$$\mu(k) \le \varphi(k) + B(k)\sum_{l=a}^{k-1} C(l)u(l), \tag{2.2.48}$$

where $\mu(k), \varphi(k)$ are not necessarily non-negative. Then for all $k \in \mathbb{N}_a$, we have

$$\mu(k) \le \varphi(k) + B(k) \sum_{l=a+1}^{k} \prod_{\tau=0}^{k-1-l} \Big(1 + C(k-1-\tau)B(k-1-\tau)\Big) C(l-1)\varphi(l-1). \tag{2.2.49}$$

Proof Define a function $\nu(k)$ on $\mathbb{N}_a$ as

$$\nu(k) = \sum_{l=a}^{k-1} C(l)\mu(l).$$

Then, similarly as in Theorem 2.2.6, we have

$$\nu(k+1) \le (1 + C(k)B(k))\nu(k) + C(k)\varphi(k), \quad \nu(a) = 0.$$

Applying Corollary 2.2.2 to the above inequality, we know that $\nu(k) \le \omega(k)$, where $\omega(k)$ is the solution of the problem

$$\nu(k+1) = (1 + C(k)B(k))\omega(k) + C(k)\varphi(k), \quad \omega(a) = 0.$$

Thus it follows from some property of Green's matrix on linear system that

$$\nu(k) \le \omega(k) = \sum_{l=a+1}^{k} \prod_{\tau=0}^{k-1-l} (1 + C(k-1-\tau)B(k-1-\tau))C(l-1)\varphi(l-1).$$

Thus the result (2.2.49) now follows from the inequality $\mu(k) \le \varphi(k) + B(k)$ $\nu(k)$. □

Remark 2.2.1 The inequality (2.2.49) is the best possible, however, at the cost of several matrix multiplications which may not be feasible. Thus from a practical point of view, it is not of much use. In the next two results, we shall provide explicit upper estimates, which are not the best possible.

Theorem 2.2.8 (Agarwal [10]) *Assume that the inequality (2.2.48) holds for all $k \in \mathbb{N}_a$, and $\varphi(k)$ is not necessarily non-negative. Then for all $k \in \mathbb{N}_a$, we have*

$$u_i(k) \le p_i(k) + \max_{1\le j\le n} b_{ij}(k) \sum_{l=a}^{k-1} \alpha(l) \prod_{\tau=l+1}^{k-1} (1 + \beta(\tau)), \quad i = 1, 2, \cdots, n, \tag{2.2.50}$$

where

$$\alpha(k) = \sum_{j,r=1}^{n} c_{jr}(k) p_r(k), \quad \beta(k) = \max_{1\le s\le n} \sum_{j,r=1}^{n} c_{jr}(k) b_{rs}(k).$$

Proof Taking components of (2.2.48), we obtain

$$u_i(k) \le p_i(k) + \sum_{j=1}^{n} b_{ij}(k) v_j(k), \tag{2.2.51}$$

where

$$v_j(k) = \sum_{r=1}^{n} \sum_{l=a}^{k-1} c_{jr}(l) u_r(l). \tag{2.2.52}$$

Define $v(k) = \sum_{j=1}^{n} v_j(k)$, then it follows that

$$\triangle v(k) = \sum_{j=1}^{n} \sum_{r=1}^{n} c_{jr}(k) u_r(k) \le \sum_{j,r=1}^{n} c_{jr}(k) \Big(p_r(k) + \sum_{s=1}^{n} b_{rs}(k) v_s(k) \Big) \le \alpha(k) + \beta(k) v(k)$$

which, as in Theorem 2.2.4, yields

$$v(k) \le \sum_{l=a}^{k-1} \alpha(l) \prod_{\tau=l+1}^{k-1} (1 + \beta(\tau)).$$

Thus the result (2.2.50) now follows from (2.2.51). □

Theorem 2.2.9 (Agarwal [10]) *Assume that the inequality (2.2.48) holds for all $k \in \mathbb{N}_a$, and $\varphi(k)$ is not necessarily non-negative. Then for all $k \in \mathbb{N}_a$,*

$$u^*(k) \le p^*(k) + b^*(k) \sum_{l=a}^{k-1} p^*(l) c^*(l) \prod_{\tau=l+1}^{k-1} \Big(1 + b^*(\tau) c^*(\tau)\Big), \tag{2.2.53}$$

where

$$\begin{cases} u^*(k) = \max\limits_{1 \le i \le n} u_i(k), \quad p^*(k) = \max\limits_{1 \le i \le n} p_i(k), \\ b^*(k) = \sum\limits_{j=1}^{n} \max\limits_{1 \le i \le n} b_{ij}(k), \quad c^*(k) = \max\limits_{1 \le j \le n} \Big(\sum\limits_{r=1}^{n} c_{rj}(k) \Big). \end{cases}$$

Proof First, taking maxima in (2.2.51) over $1 \le i \le n$, we obtain

$$u^*(k) \le p^*(k) + \sum_{j=1}^{n} b_j^*(k) v_j(k), \tag{2.2.54}$$

where $b_j^*(k) = \max_{1 \le i \le n} b_{ij}(k)$. Next, from (2.2.52) it follows

$$v_j(k) \le \sum_{r=1}^{n} \sum_{l=a}^{k-1} c_{jr}(l) u^*(l) = \sum_{l=a}^{k-1} c_j(l) u^*(l), \tag{2.2.55}$$

where $c_j(k) = \sum_{r=1}^{n} c_{jr}(k)$.

Using (2.2.54) in (2.2.53), we get

$$\begin{aligned} u^*(k) &\le p^*(k) + \sum_{j=1}^{n} b_j^*(k) \sum_{l=a}^{k-1} u^*(l) c_j(l) \\ &\le p^*(k) + b^*(k) \sum_{l=a}^{k-1} u^*(l) c^*(l). \end{aligned} \tag{2.2.56}$$

Thus the result (2.2.53) follows from Theorem 2.2.4. □

Remark 2.2.2 An explicit upper estimate for $u^*(k)$ can also be provided by the inequality (2.2.55) which has been considered in Theorem 2.1.39.

2.3 Linear One-Dimensional Discrete Inequalities in Distributive Lattice

In this section, we shall consider discrete inequalities in a partially ordered space. Many inequalities can be investigated along the general framework outlined below. We first introduce some basic concepts in a partially ordered space.

Definition 2.3.1 A binary relation R in a set in X is said to be a partial order relation if R is reflexive, transitive, and antisymmetric, i.e., xRx for all $x \in X$, xRy and yRz imply xRz, and xRy and yRx imply $x = y$. We denote a partial order relation by the symbol "$\leq$", and say that "x is smaller than y" if $x \leq y$. The notation $x < y$ means that $x \leq y$ and $x \neq y$. A set X endowed with a partial order relation is called a partially ordered set.

Remark 2.3.1 If X is a partially ordered set, then it is not always true that one of the relations $x \leq y$ or $y \leq x$ holds.

Definition 2.3.2 If X is partially ordered by $\leq$, then an operator $T : X \to X$ is said to be isotone (monotone, or order-preserving) if $x \leq y$ implies $Tx \leq Ty$.

Definition 2.3.3 A linear space X with zero θ is said to be a linear partially ordered space if X is partially ordered by a relation "$\leq$" and the following conditions hold

$$\theta \leq x \Rightarrow \theta \leq \lambda x \quad for \;\; all \;\; \lambda \geq 0, \tag{2.3.1}$$

$$x_1 \leq y_1, x_2 \leq y_2 \Rightarrow x_1 + x_2 \leq y_1 + y_2, \tag{2.3.2}$$

i.e., the partial order is compatible with the algebraic operations in X.

Definition 2.3.4 In a linear partially ordered space X, the interval $[u; v] = \{x \in X : u \leq x \leq v\}$ is a convex set. A closed subset K of a real Banach space X is said to be a cone if

$$x \in K, \;\; x \neq \theta \Rightarrow \lambda x \in K \;\; for \;\; all \;\; \lambda \geq 0, \;\; and \;\; -x \notin K$$

where θ is the zero element in X.

Example 2.3.1 In each of the space C, L^p, Orlicz spaces, the set of non-negative functions is a cone. The set $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geq 0, i = 1, \cdots, n\}$ is a cone in $\mathbb{R}^n$.

A cone K in a space X induces a partial order in X by setting

$$x \leq y \Longleftrightarrow y - x \in K$$

(also, $x < y \Longleftrightarrow y - x \in \ Int\, K$, the interior of K).

Definition 2.3.5 If A is a partially ordered Banach space with partial order "$\leq$", then $X^+ = \{x \in X : \theta \leq x\}$ is called the positive cone of X. In this case, an operator $S : X \to X$ is said to be positive if $S(X^+) \subset X^+$. If S is a linear operator, then S is positive if and only if S is isotone.

Let A be a non-empty subset of a linear partially ordered space X. An element $w \in X$ is the least upper bound of A if $x \leq w$ for all $x \in A$, and $x \leq y$ for all $x \in A$ implies $w \leq y$. The least upper bound w of A (if it exists) is also denoted by

$$w = \sup A.$$

Similarly, we define the greatest lower bound $v = infA$.

Definition 2.3.6 A linear partially ordered space X is said to be a lattice (or $K-$ lineal) if any two elements $x, y \in X$ have a least upper bound, denoted by

$$x \vee y = \sup\{x, y\}.$$

If X is a lattice, we can prove that any two sequences $x, y \in X$ also have a greatest lower bound,

$$x \wedge y = \inf\{x, y\}.$$

In this case, any finite set $\{x_i\}_{i=1}^n$ of elements of X has a least upper bound $\vee_{i=1}^n x_i$ and a greatest lower bound $\wedge_{i=1}^n x_i$.

The relations $x \vee y$ and $x \wedge y$ define in the lattice X binary operations with the following properties.

Lemma 2.3.1 (Bainov-Simeonov [42]) *For all $x, y, z \in X$, the following relations hold,*

$$x \vee x = x, \tag{2.3.3}$$

$$x \vee y = y \vee x, \tag{2.3.4}$$

$$(x \vee y) \vee z = x \vee (y \vee z), \tag{2.3.5}$$

$$x \wedge (x \vee y) = x, \tag{2.3.6}$$

$$x \wedge x = x, \tag{2.3.7}$$

$$x \wedge y = y \wedge x, \tag{2.3.8}$$

$$(x \wedge y) \wedge z = x \wedge (y \wedge z), \tag{2.3.9}$$

$$x \vee (x \wedge y) = x. \tag{2.3.10}$$

Any element x of the lattice X has a representation $x = x^+ - x^-$, where $x^+ = \theta \vee x, x^- = \theta \vee (-x)$. The element $|x| = x^+ + x^-$ is called the modulus of $x \in X$. A linear normed space X is said to be a $KB-$lineal if it is a lattice and the following condition is satisfied

$$x, y \in X, \quad |x| \leq |y| \Rightarrow \|x\| \leq \|y\| \quad (\textit{monotonicity of the norm}).$$

In a KB-lineal X, we have $\|x\| = \||x|\|$ for all $x \in X$.

An example of a KB-lineal is the space $C([a, b], \mathbb{R}^n)$ with norm $\|x\| = \sup_{t\in[a,b]} |x(t)|$ and positive cone $C^+ = C([a, b], \mathbb{R}^n_+)$.

A set X is said to be a distributive lattice if it is a lattice in which the following distributive law holds

$$(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z), \tag{2.3.11}$$

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z). \tag{2.3.12}$$

Let E be a distributive lattice, and let $\wedge_{j=1}^n x_j$ be the greatest lower bound and $\vee_{j=1}^n x_j$ the least upper bound of a system $\{x_j\}, j \in \mathbb{Z}_{1,n}$, of elements of E. We use the symbol "$\leq$" for the order relation. For any $x \in E$,

$$x \vee \bigvee_{j=1}^{0} = x \wedge \bigwedge_{j=1}^{0} = x.$$

We recall some facts in the following:

i) $x \leq y$ implies $x \wedge y = x$ and $x \vee y = y$;
ii) $x \leq x \vee y$ and $x \wedge y \leq x$ for $x, y \in E$;
iii) $x_i \leq y_i, \ i = 1, 2$, implies

$$x_1 \wedge x_2 \leq x_i \leq y_i \leq y_1 \vee y_2, \ x_1 \wedge x_2 \leq y_1 \wedge y_2, \quad x_1 \vee x_2 \leq y_1 \vee y_2;$$

iv) $\vee_{j=1}^{k+1} x_j = (\vee_{j=1}^{k} x_j) \vee x_{k+1}, \quad \wedge_{j=1}^{k+1} x_j = (\wedge_{j=1}^{k} x_j) \wedge x_{k+1}$ for any element $x_j \in E, j = 1, \cdots, k+1$.

We have the following two results, due to Popenda [528], on discrete inequalities in distributive lattice.

Theorem 2.3.1 (Popenda [528]) *Let* $x, a, b : \mathbb{N}_1 \to E$ *and for all* $n \in \mathbb{N}_1$,

$$x_{n+1} \le a_n \vee \bigvee_{j=1}^{n} (b_j \wedge x_j). \tag{2.3.13}$$

Then for all $n \in \mathbb{N}_1$,

$$x_{n+1} \le a_n \vee (x_1 \wedge b_1) \vee \bigvee_{j=1}^{n-1} (b_{j+1} \wedge a_j). \tag{2.3.14}$$

Proof For all $n \in \mathbb{N}_1$, we set

$$p_n = \bigvee_{j=1}^{n} (b_j \wedge x_j), \quad q_n = (x_1 \wedge b_1) \vee \bigvee_{j=1}^{n-1} (b_{j+1} \wedge a_j).$$

Note that if p_n is comparable with q_n and

$$p_n \le q_n, \tag{2.3.15}$$

then, by iii), $a_n \vee p_n \le a_n \vee q_n$. Hence, by (2.3.13) and the transitivity of the order relation, x_{n+1} is comparable with $a_n \vee q_n$, and (2.3.14) holds for this n, i.e.,

$$x_{n+1} \le a_n \vee q_n.$$

It thus suffices to prove (2.3.15) for all $n \in \mathbb{N}_1$. This part of the proof is by induction. In fact, for $n = 1$, we have $p_1 = b_1 \wedge x_1 = q_1$. Now we assume that (2.3.15) holds for some $n = k$, and we consider p_{k+1}. First, by iv), we get

$$p_{k+1} = p_k \vee (b_{k+1} \wedge x_{k+1}), \quad q_{k+1} = q_k \vee (b_{k+1} \wedge a_k). \tag{2.3.16}$$

Noting the first part of the proof, we have $x_{n+1} \le a_k \vee q_k$, whence, by iii),

$$b_{k+1} \wedge x_{k+1} \le b_{k+1} \wedge (a_k \vee q_k).$$

Applying to this inequality, the finite distributive laws, i) and iii) imply

$$b_{k+1} \wedge x_{k+1} \le (b_{k+1} \wedge a_k) \vee (b_{k+1} \wedge q_k) \le (b_{k+1} \wedge a_k) \vee q_k,$$

which, together with (2.3.16), leads to

$$x_{k+1} \wedge b_{k+1} \le q_{k+1}.$$

By the induction hypothesis, this gives us $p_{k+1} = p_k \vee (b_{k+1} \wedge x_{k+1}) \le q_k \vee q_{k+1}$. However, $q_k \le q_{k+1}$ so that $p_{k+1} \le q_{k+1}$. $\square$

Theorem 2.3.2 (Popenda [528]) *Let $x, a, b : \mathbb{N}_1 \to E$. Then for all $t \in \mathbb{N}_1$,*

(1) $x_{n+1} \leq a_n \wedge \bigwedge_{j=1}^{n}(b_j \wedge x_j)$ *implies* $x_{n+1} \leq a_n \wedge x_1 \wedge b_1$;
(2) $x_{n+1} \leq a_n \wedge \bigwedge_{j=1}^{n}(b_j \wedge x_j)$ *implies* $x_{n+1} \leq x_1 \wedge \bigvee_{j=1}^{n}(a_j \wedge b_j)$;
(3) $x_{n+1} \leq a_n \vee \bigwedge_{j=1}^{n}(b_j \wedge x_j)$ *implies* $x_{n+1} \leq a_n \vee (x_1 \wedge \bigwedge_{j=1}^{n} b_j)$;
(4) $x_{n+1} \leq a_n \wedge \bigvee_{j=1}^{n}(b_j \vee x_j)$ *implies* $x_{n+1} \leq a_n \wedge (x_1 \vee \bigvee_{j=1}^{n} b_j)$;
(5) $x_{n+1} \leq a_n \wedge \bigwedge_{j=1}^{n}(b_j \vee x_j)$ *implies* $x_{n+1} \leq a_n \wedge (x_1 \vee b_1) \wedge \bigwedge_{j=1}^{n}(a_j \vee b_{j+1})$;
(6) $x_{n+1} \leq a_n \vee \bigvee_{j=1}^{n}(b_j \vee x_j)$ *implies* $x_{n+1} \leq x_1 \vee \bigvee_{j=1}^{n}(b_j \vee a_j)$;
(7) $x_{n+1} \leq a_n \vee \bigwedge_{j=1}^{n}(b_j \vee x_j)$ *implies* $x_{n+1} \leq a_n \vee \vee b_1$.

Proof The proof is left to the reader. □

Chapter 3
Linear One-Dimensional Discontinuous Integral Inequalities

3.1 One-Dimensional Discontinuous Gronwall-Bellman Integral Inequalities

3.1.1 Linear One-Dimensional Bellman-Bihari Integral Inequalities for Discontinuous Functions-Linear Impulse Integral Inequalities

It is well known that linear impulse integral inequalities arise in the study of impulse differential equations. To introduce such integral inequalities, we need now to give a brief description of the following class of impulse differential equations

$$\begin{cases} x'(t) = f(t,x) \quad \text{if} \quad \phi(t,x) \neq 0, \\ \triangle x = I(t,x) \quad \text{if} \quad \phi(t,x) = 0, \end{cases} \tag{3.1.1}$$

where $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n, \phi : \mathbb{R}_+ \times \Omega \to \mathbb{R}, I : \mathbb{R}_+ \times \Omega \to \mathbb{R}^n, \Omega$ is a domain in $\mathbb{R}^n, t \in \mathbb{R}_+, x \in \Omega$.

We recall that Eq. (3.1.1) describe processes and phenomena that at certain moments in their development (called instants of impulse effect) are subject to short-time effects and change their state jumpwise. The instants of impulse effect occur when the mapping point $(t, x(t))$ meets the hypersurface σ defined by $\phi(t,x) = 0$, i.e., when $\phi(t, x(t)) = 0$. At the instant of impulse effect $t = t_i$, the mapping point jumps instantaneously from position $(t_i, x(t_i))$ to position $(t_i, x(t_i^+))$, where $x(t_i^+) = x(t_i) + \triangle x(t_i) = x(t_i) + I(t_i, x(t_i))$. We assume that in an interval $(t_i, t_{i+1}]$ between two successive instants of impulse effect the solution $x(t)$ of problem (3.1.1) coincides with the solution of the initial value problem

$$y'(t) = f(t, y(t)), \quad y(t_i) = x(t_i^+), \quad t_i \leq t \leq t_{i+1}, \tag{3.1.2}$$

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and, $x(t)$ is left-continuous at each instant of impulse effect, i.e., $x(t_i^-) = x(t_i)$. The most simple models to study are impulse differential equations for which the impulse occur at fixed times, given by an increasing sequence $\{t_i\}, t_i < t_{i+1}$. For this case, problem (3.1.1) can be written as

$$x'(t) = f(t,x), \quad t \neq t_k; \quad \triangle x = I_k(x), \ t = t_k \tag{3.1.3}$$

or as

$$x'(t) = f(t,x), \quad t \neq t_k, \quad x(t_k^+) = \psi_k(x(t_k)), \tag{3.1.4}$$

where $\psi_k(x) = x + I_k(x), \quad k = 1, 2, \cdots$.

A more general form than problem (3.1.1)–(3.1.2) is as follows

$$x'(t) = f(t,x), t \neq \tau_k(x); \quad \triangle x = I_k(x), \quad t = \tau_k(x) \tag{3.1.5}$$

where $\tau_k : \Omega \to \mathbb{R}_+$ and $0 < \tau_1(x) < \tau_2(x) < \cdots$ for all $x \in \Omega$.

The instants t_i of impulse effect for the solution $x(t)$ of problem (3.1.5) occur when the mapping point $(t, x(t))$ meets some hypersurface σ_k with equation $t = \tau_k(x)$, i.e., when $t_i = \tau_k(x(t_i))$ for some k.

The theory of integral inequalities [10, 24, 66, 82, 338, 342, 388] and its numerous linear, nonlinear generalizations for continuous, discontinuous functions of one and n independent variables have been very important in investigating different qualitative characteristics of solutions, both for ordinary and partial differential equations (functional-differential equations, integrodifferential equations, impulsive differential equations, etc.) such as boundedness, existence, uniqueness, continuous dependence of parameters, stability, attraction, practical stability [66, 82, 388], etc. In the one-dimensional case, all the main results in the theory of integral inequalities for continuous functions are based on the solvability of Chaplygin's problem [36, 543] for the integral inequality

$$u(t) \leq \phi(t) + \int_{t_0}^{t} K(t, s, u(s))ds, \tag{3.1.6}$$

for establishing the estimate $u(t) \leq \sigma_\phi(t)$, where $\sigma_\phi(t)$ is the solution of Volterra's integral equation

$$\sigma(t) = \phi(t) + \int_{t_0}^{t} K(t, s, \sigma(s))ds \tag{3.1.7}$$

as mentioned in [388].

Similarly, for the discrete case like for the continuous case, Chaplygin's problem solvability was investigated for the inequality

$$u(t) \leq \phi(t) + \sum_{i} K(t, t_i, u(t_i)). \tag{3.1.8}$$

The solvability of Chaplygin's problem was studied in the works [568] in which the inequalities of a certain type were investigated

$$u(t) \leq \phi(t) + \int_{t_0}^{t} K(t, s, u(s))ds + \sum_{t_0 < t_k < t} \psi(t, t_k)\beta_k(u(t_k - 0)), \tag{3.1.9}$$

where $u(t)$ is a non-negative piecewise continuous function with first kind discontinuities at the points $\{t_i\}$, $t_0 < t_1 < \cdots$, $\lim_{i \to +\infty} t_i = +\infty$.

As described in [101], the solution of Chaplygin's problem to (3.1.9), finds its numerous generalizations for functional variables (integrosum functional inequalities) in the multi-dimensional case in [568] in which the works [568] and [101] were devoted to the applications of the method of integro-sum inequalities for solving problems of boundedness, Lyapunov stability, Chetaev stability, attraction of the motion for perturbed impulsive systems of ordinary differential equations.

In the sequel, we first recall the theory of the integro-sum inequalities (3.1.9), dwelling only on the most important results. We then formulate the conditions of solvability for Chaplygin's problem for the integro-sum inequality of Wendroff's type

$$u(x) \leq \phi(x) + \int\int_{G_n} H(y, u(y))dy + \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} \beta_j(x)u(x)d\mu_{\phi_j}, \tag{3.1.10}$$

where $x = (x_1, x_2)$, $G_n \subset \mathbb{R}^2$, μ is some measure concentrated on the curves $\{\Gamma_j\}$, $j = 1, 2, \cdots, \infty$.

Moreover, we introduce a new integro-sum inequality of the Bellman-Bihari type (also with retardation).

As an application, we investigate a hyperbolic differential equation with impulse influence on hypersurfaces, described by $\{\Gamma_j\}$; it will be obtained in some exact estimates of solutions such as the equations and the boundedness conditions of solutions.

We now introduce the main results on Gronwall-Bellman-Bihari type integral inequalities for discontinuous functions.

Theorem 3.1.1 (Samoilenko-Perestyuk [571]) *Let a non-negative piecewise continuous function $u(t)$ satisfy the inequality for all $t \geq t_0$,*

$$u(t) \leq C + \int_{t_0}^{t} V(\tau)u(\tau)d\tau + \sum_{t_0 < \tau_i < t} \beta_i u(\tau_i), \tag{3.1.11}$$

where $C \geq 0, \beta_i \geq 0, V(\tau) > 0$, and τ_i are the first kind discontinuity points of the function $u(t)$. Then the following estimate holds for all $t \geq t_0$,

$$u(t) \leq C \prod_{t_0<\tau_i<t} (1+\beta_i) \exp\Big(\int_{t_0}^{t} V(\tau) d\tau \Big). \tag{3.1.12}$$

The next result may be viewed as a linear generalization with discrete terms.

Theorem 3.1.2 (Samoilenko-Perestyuk [571]) *Let $u(t), g(t)$ be non-negative continuous on $\mathbb{R}_+$, $\{t_n\}_{n=0}^{+\infty}$ be increasing to $+\infty$, $t_n \geq 0$. If*

$$u(t) \leq C + \int_{t_0}^{t} g(s)u(s)ds + \sum_{t_0<t_k<t} \beta_k u(t_k), \tag{3.1.13}$$

where $t \geq t_0, C, \beta_k \geq 0$ are constants, then

$$u(t) \leq C \prod_{t_0<t_k<t} (1+\beta_k) \exp\left(\int_{t_0}^{t} g(s) ds \right). \tag{3.1.14}$$

Proof The proof is left to the reader as an exercise. □

Remark 3.1.1 We formulate the result of Theorem 3.1.1 as in the last work [571, Lemma 1, p. 12].

Remark 3.1.2 For applications of the results of Theorem 3.1.1, we refer to [571].

Theorem 3.1.3 (Borysenko [98]) *Let $V(t)$ be a non-negative piecewise continuous function at all $t \geq t_0$, with first kind discontinuities at the points t_i, and satisfying the integral inequality,*

$$V(t) \leq C + \int_{t_0}^{t} P(\tau)V^m(\tau)d\tau + \sum_{t_0<t_i<t} \beta_i V(t_i - 0),\ m > 0,\ m \neq 1, \tag{3.1.15}$$

where $t_1 < t_2 < \cdots, \lim_{i\to+\infty} t_i = +\infty$, $C \geq 0, \beta_i \geq 0, P(t) \geq 0$, for all $t \geq t_0$. Then the following estimates hold for all $t \geq t_0$,

$$V(t) \leq \prod_{t_0<t_i<t} (1+\beta_i)[C^{1-m} + (1-m) \int_{t_0}^{t} P(\tau)d(\tau)]^{1/(1-m)}, \tag{3.1.16}$$

if $0 < m < 1$, for all $t \geq t_0$,

$$V(t) \leq C \prod_{t_0<t_i<t} (1+\beta_i) \Big[1-(m-1)C^{m-1} \prod_{t_0<t_i<t}^{m-1} (1+\beta_i) \int_{t_0}^{t} P(\tau)d(\tau) \Big]^{-1/(m-1)}, \tag{3.1.17}$$

if $m > 1$, *for all* $t \in [t_0, +\infty]$,

$$\int_{t_0}^{t} P(\tau)d(\tau) < \frac{C^{1-m}}{(m-1)\prod_{t_0<t_i<t}^{m-1}(1+\beta_i)}.$$

Remark 3.1.3 For applications of Theorem 3.1.3, we may refer to [98, 569].

Theorem 3.1.4 (Perestyuk-Chernikova [569]) *Let $u(t)$ be a non-negative piecewise continuous function at all $t \geq t_0$, with first kind discontinuities at all $t = \tau_i$ and satisfying the inequality for all $t \geq t_0$,*

$$u(t) \leq C + \int_{t_0}^{t} V(\tau)\Phi(u(\tau))d(\tau) + \sum_{t_0<\tau_i<t} \beta_i u(\tau_i), \tag{3.1.18}$$

where constants $C \geq 0, \beta_i \geq 0, V(t)$ is a positive continuous function, $\Phi(u)$ is a positive continuous non-decreasing function for all $0 < u < \overline{u}$ $(\overline{u} < +\infty)$. Then the function $u(t)$ satisfies for all $\tau_i < t \leq \tau_i + 1$,

$$u(t) \leq \Psi_i^{-1}(\int_{\tau_i}^{t} V(\tau)d\tau), \tag{3.1.19}$$

if

$$\int_{\tau_i}^{t} V(\tau)d\tau < \psi_i^{-1}(\overline{u}-0),$$

where

$$\begin{cases} \Psi_i^{-1}(u) = \displaystyle\int_{c_i}^{u} \frac{du}{\Phi(u)},\ c_i = (1+\beta_i)\Psi_i^{-1}(\int_{\tau_i-1}^{\tau} V(\tau)d\tau), \\ \Psi_0(u) = \displaystyle\int_{c}^{u} \frac{du}{\Phi(u)},\ i = 1, 2, \cdots, \tau_0 = t_0. \end{cases}$$

Remark 3.1.4 For the applications of Theorem 3.1.4 with $\Phi(u) = u^m,\ m = 1$, we may refer to [568, 571].

Theorem 3.1.5 (Borysenko [100]) *Consider the integro-sum equation of the following form*

$$\sigma(t) = \phi(t) + \int_{t_0}^{t} K(t, s, \sigma(s))ds + \sum_{t_0<\tau_k<t} \Psi(t, t_k)\mu_k(\sigma(t_k - 0)), \tag{3.1.20}$$

where $\sigma(t), \phi(t), \Psi(t, t_k)$ are continuous non-negative functions $(k = 1, 2, \cdots)$ for all $t \geq t_0$, except for $\sigma(t)$, which has first kind discontinuities at the points t_k and

$$t_0 < t_1 < \cdots,\ \lim_{i\to+\infty} t_i = +\infty.$$

The function $K(t,s,u)$, which is non-negative at all $t \geq s \geq t_0$, is determined in the domain $\{(t,s,u) : t \geq s \geq t_0,\ |u| \leq k\}$ and at fixed t and s, it is non-decreasing with respect to u; the functions $\mu_k(\sigma)$ are continuous non-negative and non-decreasing with respect to σ. Then, for an arbitrary $t \in [t_0, +\infty]$, the estimate $u(t) \leq \sigma_\phi(t)$ exists where $\sigma_\phi(t)$ is some solution of Eq. (3.1.20), continuous in each interval $[t_k, t_{k+1}]$, $k = 0, 1, \cdots$; $u(t)$ is a piecewise continuous function with first kind discontinuities at t_i points; this function satisfies the integro-sum inequality:

$$u(t) = \phi(t) + \int_{t_0}^{t} K(t,s,\sigma(s))ds + \sum_{t_0<\tau_k<t} \Psi(t,t_k)\mu_k(\sigma(t_k-0)), \tag{3.1.21}$$

where $\sigma(t_k - 0) = \lim_{t\to t_k-0} \sigma(t)$.

Corollary 3.1.1 *Consider a non-negative piecewise continuous function $V(t)$, at $t \geq t_0$ with first kind discontinuities at the points t_i, and satisfying the integrosum inequality*

$$V(t) \leq \Psi(t) + \int_0^t P(\tau)V^m(\tau)d\tau + \sum_{t_0<\tau_k<t} \beta_i V(t_i - 0),\ m > 1,$$

where $t_1 < t_2 < \cdots, \lim_{i\to+\infty} t_i = +\infty, \Psi(t)$ is a positive monotonously non-decreasing function at $t \geq t_0, \beta_i \geq 0, P(t) \geq 0$. Then the following estimates hold for $0 < m < 1$, for all $t \geq t_0$,

$$V(t) \leq \Psi(t) \prod_{t_0<t_i<t} (1+\beta_i)\Big[1 + (1-m)\int_{t_0}^{t} \Psi^{m-1}(\tau)P(\tau)d\tau\Big]^{1/(1-m)};$$

for all $t \geq t_0$,

$$V(t) \leq \Psi(t) \prod_{t_0<t_i<t} (1+\beta_i) \exp\Big(\int_{t_0}^{t} P(\tau)d\tau\Big) \text{ for } m = 1;$$

if $m > 1$, for all $t \geq t_0$,

$$V(t) \leq \Psi(t) \textstyle\prod_{t_0<t_i<t} \times\Big[1 - (m-1) \prod_{t_0<t_i<t}^{m-1} (1+\beta_i) \int_0^t \Psi^{m-1}(\tau)P(\tau)d\tau\Big]^{1/(1-m)},$$

with

$$\int_0^t \Psi^{m-1}(\tau)P(\tau)d\tau < \Big[(m-1) \prod_{t_0<t_i<t}^{m-1} (1+\beta_i)\Big]^{-1}.$$

Theorem 3.1.6 (Borysenko [100]) *Let $u(t,x)$ be a non-negative function which is determined in the domain*

$$D = \Big\{ \bigcup_{k,j>1} D_{kj}, D_{kj} = ((t,x) : t \in [t_{k-1}, t_k], x \in [x_{j-1}, x_j], k = 1,2,\cdots, j = 1,2,\cdots) \Big\}.$$

Moreover, let $u(t,x)$ be continuous in D, with the exception of the points $\{t_i, x_i\}$ of finite jumps: $u(t_i-0, x_i-0) \neq u(t_i+0, x_i+0)$ and satisfy the integro-sum inequality

$$\begin{aligned} V(t) \le \Psi(t,x) + q(t,x) \textstyle\int_{t_0}^t \int_{x_0}^x f(\xi,\eta) u^m(\xi,\eta) d\xi d\eta \\ + \textstyle\sum_{(t_0,x_0)<(t_i,x_i)<(t,x)} \beta_i u(t_i - 0, x_i - 0), \end{aligned} \tag{3.1.22}$$

$q(t_0,x_0) = 1$*, where $\Psi(t,x) > 0$, for all $(t,x) \in D$, is non-decreasing with respect to (t,x): for all $p \le P$, $q \le Q$, $\Psi(p,q) \le \Phi(P,Q)$ at $(p,q) \in D$, $(P,Q) \in D$; $q(t,x) \le 1$, for all$(t,x) \in D$, the values $\beta_i \ge 0$ for all $i \in \mathbb{N}$, the function f is non-negative, where $f(\xi,\eta) = 0$, $(\xi,\eta) \in D_{lp}$, for $l \ne p$, for arbitrary $l = 1,2,\cdots, p = 1,2,\cdots$. Here $(t_i,x_i) < (t_{i+1},x_{i+1})$, if $t_i < t_{i+1}, x_i < x_{i+1}$, for all $i = 1,2,\cdots$, where* $\lim_{i\to+\infty} t_i = +\infty, \lim_{i\to+\infty} x_i = +\infty$. *Then the following estimates hold:*

(1) if $0 < m < 1$, for all $(t,x) \in D$,

$$\begin{aligned} u(x,t) \le \psi(t,x) q(x,t) \prod(t_0,x) \\ \times \Big[1 + (1-m) \int_{t_0}^t \int_{x_0}^x \psi^{m-1}(\xi,\eta) q^m(\xi,\eta) f(\xi,\eta) d\xi d\eta \Big]^{1/(1-m)}; \end{aligned}$$

(2) if $m = 1$, for all $(t,x) \in D$,

$$u(x,t) \le \psi(t,x) q(x,t) \prod(t_0,x) \exp\Big[\int_{t_0}^t \int_{x_0}^x f(\xi,\eta) q(\xi,\eta) d\xi d\eta \Big]; \tag{3.1.23}$$

(3) if $m > 1$, for all $(t,x) \in D$,

$$\begin{aligned} u(x,t) \le \psi(t,x) q(x,t) \Big[1 - (m-1) \Big(\prod(t_0,x) \Big)^{m-1} \Big] \\ \times \Big[\int_{t_0}^t \int_{x_0}^x \psi^{m-1}(\xi,\eta) q^m(\xi,\eta) f(\xi,\eta) d\xi d\eta \Big]^{1/(1-m)}; \end{aligned}$$

with

$$\int_{t_0}^t \int_{x_0}^x \psi^{m-1}(\xi,\eta) q^m(\xi,\eta) f(\xi,\eta) d\xi d\eta < \Big[(m-1) \Big(\prod(t_0,x) \Big)^{m-1} \Big]^{-1},$$

where $\prod(t_0,x) := \prod_{(t_0,x_0)<(t_i,x_i)<(t,x)} \Big(1 + \beta_i q(t_i,x_i) \Big)$.

Now consider the Euclidean space $\mathbb{R}^n$ with points $x = (x^1, x^2, \cdots, x^n)$, $x^0 = (x^{10}, \cdots, x^{n0})$ with the order $x^0 \le x$ $(x^{i0} \le x^i)$, *for all* $i = 1, \cdots, n$.

Define

$$\int_{x^0}^{x} \cdots du = \int_{x^{n0}}^{x^1} \cdots du_1 \cdots du_n, \sum_{x^0 < x_k < x} \alpha_k = \sum_{x^0 < x_{k1} < x^1, \cdots, x^{n0} < x_{kn} < x^n} \alpha_k,$$

then we may introduce a space of F-continuous functions $F : \mathbb{R}^n \to \mathbb{R}^n$ such that

(A) $F(x) = (F_1(x), F_2(x), \cdots, F_n(x))$, where $F_j : \mathbb{R}^n \to \mathbb{R}^n$, $j = 1, 2, \cdots, n$;
(B) $F(x) \le x$;
(C) $\lim_{|x| \to +\infty} F_j(x) \le +\infty$, *for all* $j = 1, 2, \cdots, n$.

We consider the domain $D \subset \mathbb{R}^n$:

$$D = D_{k_1, \cdots, k_n} = \Big\{ x : x^1 \in [x_{k_1 - 1}, x_{k_1}], \cdots, x^n \in [x_{k_n - 1}, x_{k_n}], k_j = 1, 2, \cdots, n \Big\}.$$

We denote by $\{x_k\} = \{x_{k1}, \cdots, x_{kn}\}$ the points of finite jumps of the function $u(x)$: $u(x_i - 0) \ne u(x_i + 0)$, for all $i \in \mathbb{N}$. Let us define as F^* the space of functions $f(x) : f \ge 0; f = 0$ only if $x \in D_{k_1, \cdots, k_n}$ at $k_i \ne k_j, i, j = 1, 2, \cdots, n$.

Theorem 3.1.7 (Borysenko [102]) *Assume that a non-negative function* $u(x)$ *determined in the domain D satisfies the inequality*

$$u(x, t) \le \psi(t, x) q(x, t) \Big[\int_{x_0}^{x} f(\tau) u^m(p(\tau)) d\tau + \int_{x_0}^{x} f(s) \Big(\int_{x_0}^{x} g(\tau) u^m(\sigma(\tau)) d\tau \Big) ds \Big] + \sum_{x^0 < x_i < x} \beta_i u(x_i - 0), \tag{3.1.24}$$

with $m > 0$*, and where* $p(t), \sigma(t) \in F$*,* $\{x_k\}$ *are the points of finite jumps of* $u(x)$*,* $\psi(x)$ *is a non-decreasing function,* $\psi(x) > 0, f \in F^*, q(x) \ge 1, g(x) \ge 0, \beta_i \ge 0$*. Then the following estimates hold*

(A) if $0 < m < 1$*, for all* $x \in D$*,*

$$u(x) \le \psi(x) q(x) \prod(x^0, x) \Big\{ 1 + (1 - m) \int_{x_0}^{x} f(t) \times \Big[\psi^{m-1}(t) q^m(p(t)) + \int_{t_0}^{t} g(\tau) \psi^{m-1}(\tau) q^m(\sigma(\tau)) \times \Big(\frac{\psi(\sigma(\tau))}{\psi(\tau)} \Big)^m d\tau \Big] dt \Big\}^{1/(1-m)}; \tag{3.1.25}$$

(B) if $m = 1$, *for all* $x \in D$,

$$u(x,t) \le \psi(x)q(x)\prod(x^0,x)\exp\Big(\int_{x_0}^{x} Q(\tau)d\tau\Big); \tag{3.1.26}$$

(C) if $m > 1$, *for all* $(t,x) \in D$,

$$\begin{aligned}u(x,t) \le{}& \psi(x)q(x)\prod(x^0,x)\Big\{1 + (1-m)\Big(\prod(x^0,x)\Big)^{m-1}\int_{x_0}^{x} f(t)\\ &\times\Big[\psi^{m-1}(x)q^m(p(t)) + \int_{t_0}^{t} g(\tau)\psi^{m-1}(\tau)q^m(\sigma(\tau))\\ &\times\Big(\frac{\psi(\sigma(\tau))}{\psi(\tau)}\Big)^m d\tau\Big]dt\Big\}^{1/(1-m)};\end{aligned} \tag{3.1.27}$$

with

$$\begin{aligned}&\int_{x_0}^{x} f(t)\Big[\psi^{m-1}(x)q^m(p(t)) + \int_{t_0}^{t} g(\tau)\psi^{m-1}(\tau)q^m(\sigma(\tau))\Big(\frac{\psi(\sigma(\tau))}{\psi(\tau)}\Big)^m d\tau\Big]dt\\ &< \Big[(m-1)\Big(\prod(t_0,x)\Big)^{m-1}\Big]^{-1}\frac{\prod^{m-1}(x^0,x)}{m-1},\end{aligned} \tag{3.1.28}$$

where

$$\begin{cases}\prod(x^0,x) := \prod\limits_{x^0<x_i<x}(1+\beta_i q(x_i)), & (3.1.29)\\[2ex] Q(t) = \dfrac{f(t)q(p(t))\psi(p(t)) + g(t)q(\sigma(t))\psi(\sigma(t))}{\psi(p(t))}. & (3.1.30)\end{cases}$$

Now we introduce the class of functions F.

Definition 3.1.1 $W \in F$, a class of functions, if and only if

(a) $W(\alpha\beta) \le W(\alpha)W(\beta)$;
(b) $W : [0,+\infty] \to [0,+\infty], W(0) = 0$;
(c) W is non-decreasing.

Theorem 3.1.8 (Borysenko [102]) *Assume that* $v(x)$ *is a piecewise continuous function with first kind discontinuities at the points* $x_i : x_0 < x_1 < \cdots, \lim_{n\to+\infty} x_n = +\infty$, $f(x) \ge 0$, *for all* $x \ge x_0$, *satisfying the following integro-sum functional inequality*

$$v(x) \le \varphi(x) + q(x)\int_{x_0}^{x} f(\tau)W(v(p(\tau)))d\tau + \sum_{x_0<x_i<x}\beta_i v(x_i - 0), \tag{3.1.31}$$

where $x \geq x_0, q(x) \geq 1, \varphi(x)$ *is positive non-decreasing,* $\beta_i = const. \geq 0$*, and* $p(s) \in F$*. Then for all* $x \in [x_0, T],\ T \leq +\infty$*, the following inequality holds*

$$v(x) \leq \varphi(x)q(x)\Phi_i^{-1}\Big\{\int_{x_i}^{x} \frac{f(t)}{\varphi(t)} W^*(t)dt\Big\}, \tag{3.1.32}$$

where for all $x \in [x_i, x_{i+1}]$,

$$\begin{cases} \displaystyle\int_{x_i}^{x} f(t)\varphi^{-1}(t)W^*(t)dt \in Dom\ (\Phi_i^{-1}),\ \Phi_0(\xi) = \int_1^{\xi} \frac{d\eta}{W(\eta)}, & (3.1.33) \\ \displaystyle\Phi_i(\xi) = \int_{l_i}^{\xi} \frac{d\eta}{W(\eta)},\ i = 1, 2, \cdots & (3.1.34) \\ \displaystyle l_i = (1 + \beta_i q(x_i))\Phi_{i-1}^{-1}(\int_{x_i-1}^{x_i} f(y)\varphi^{-1}(y)W^*(y)dy),\ i = 1, 2, \cdots, & (3.1.35) \end{cases}$$

and $W^*(v) = W(g(p(v)\varphi(p(v))))$.

The next concept is another class of functions.

Definition 3.1.2 The function $f \in F_1$, a class of functions, if and only if

(a) $f(x)$ is positive, continuous, and non-decreasing for all $x > 0$;
(b) for any $t > 1, u \geq 0$ implies $t^{-1}f(u) \leq f(t^{-1}u)$;
(c) $f(0) = 0$.

The following theorem holds for the class of functions F_1.

Theorem 3.1.9 (Borysenko [100]) *Let the piecewise continuous non-negative function* $v(x)$ *with first kind discontinuities at the points* $\{x_i\}$ *satisfy the inequality (3.1.31), where* $\varphi, q, p,$ *satisfy the conditions of Theorem 3.1.8, function* $W \in F_1$*. Then, for an arbitrary* $x \in [x_0, x^*]$*, we have, for all* $x \in [x_i, x_{i+1}]$,

$$v(x) \leq \varphi(x)q(x)\Phi_i^{-1}\Big(\int_{x_i}^{x} G(\tau)d\tau\Big),\ i = 0, 1, 2, \cdots, \tag{3.1.36}$$

where

$$\begin{cases} \displaystyle\Phi_0 = \int_1^{\xi} W^{-1}(\sigma)d\sigma,\ \Phi_i = \int_{l_i}^{\xi} W^{-1}(\sigma)d\sigma,\ i = 1, 2, \cdots, \\ \displaystyle l_i = \Big(1 + \beta_i q(x_i)\Big)\Phi_{i-1}^{-1}\Big(\int_{x_i-1}^{x_i} G(\tau)d\tau,\ G(t)f(t)q(p(t)\Big), \\ \displaystyle x^* = \sup\Big\{x : \int_{x_i-1}^{x} G(\tau)d\tau \in Dom\ \ (\Phi_{i-1}^{-1}),\ i = 1, 2, \cdots\Big\}. \end{cases}$$

Theorem 3.1.10 (Samoilenko-Borysenko [594]) *Consider the integro-sum inequality in the following form:*

$$u(x) \leq u_0 + q_1(x) \int_{x_0}^{x} f(s) W_1(u(p(s)))ds + q_2(x) \int_{x_0}^{x} g(s) W_2(u(\sigma(s)))ds$$
$$+ \sum_{x_0 < x_i < x} \beta_i u(x_i - 0), \tag{3.1.37}$$

where $f(x), g(x), p(x), \sigma(x)$ are non-negative continuous functions, for all $x \geq x_0, p(x) \leq x, \sigma(x) \leq x, q_1(x) \geq 1, q_2(x) \geq 1; W_1(x) \in F, W_2(x) \in F_1; u_0 = const. \geq 1, \beta_i = const. \geq 0, u(x)$ is a piecewise continuous non-negative function with first kind discontinuities at the points $\{x_i\}$, which satisfies conditions of Theorem 3.1.8. Then the following estimate holds for all $x \geq x_0$,

$$u(x) \geq q_1(x) q_2(x) S_i(x) F^{-1}\Big(\int_{x_0}^{x} \Psi(\bar{x}) d\bar{x}\Big), \tag{3.1.38}$$

where $S_i = G_i^{*-1}\Big(\int_{x_i}^{x} f(s) q_1(p(s))ds\Big), G_i^*(\xi) = \int_{l_i}^{\xi} W_1^{-1}(\sigma)d\sigma, \quad i = 1, 2, \cdots; \quad G_0^*(\xi) = \int_{u_0}^{\xi} W_1^{-1}(v)dv, \ l_i = (1 + \beta_i) S_{i-1}(x_i), \ F(\eta) = \int_{u_0}^{\eta} W_2^{-1}(s)ds,$ and F^{-1}, G_i^{*-1} are the inverse of the functions F and G_i^*, respectively, and

$$\int_{x_i}^{x} f(\tau) q_1(p(\tau)) d\tau \in Dom\ (G_i^{*-1}), \quad i = 1, 2, \cdots, \int_{x_0}^{x} \Psi(\tau) d\tau \in Dom\ (F^{-1}),$$

with $\Psi(x) = g(x) W_2\Big(q_1(\sigma(x)) q_2(\sigma(x))\Big) S_i(\sigma(x))$.

Theorem 3.1.11 (Borysenko [105]) *Assume that a non-negative piecewise continuous function $V(t)$ on following $J = [t_0, +\infty]$, with first kind discontinuities at points $\{t_i\} : t_1 < t_2 < \cdots, \lim_{i \to +\infty} t_i = +\infty$, satisfies the integro-sum inequality*

$$V(t) \leq \psi(t) + \int_{t_0}^{t} q(\tau) V(\tau) d\tau + \sum_{t_0 < t_i < t} a_i V^m(t_i - 0),$$

where $\psi(t)$ is a positive monotonously non-decreasing function on J, $q(t) \geq 0, a_i \geq 0, m > 0$. Then the following estimates hold, for all $t \geq t_0$,

$$\begin{cases} V(t) \leq \psi(t) \prod_{t_0 < t_i < t} \Big(1 + a_i \psi^{m-1}(t_i)\Big) \exp\Big[\int_{t_0}^{t} q(s)ds\Big], \ \text{if } 0 < m < 1, & (3.1.39) \\ V(t) \leq \psi(t) \prod_{t_0 < t_i < t} \Big(1 + a_i \psi^{m-1}(t_i)\Big) \exp\Big[m \int_{t_0}^{t} q(s)ds\Big], \ \text{if } m \geq 1. & (3.1.40) \end{cases}$$

Remark 3.1.5 In fact, Theorem 3.1.11 generalizes the fundamental results for discontinuous functions obtained by Bellman.

Theorem 3.1.12 (Borysenko [105]) *Assume that a non-negative piecewise continuous function V on $J = [t_0, +\infty]$ with first kind discontinuities at the points $\{t_i\}$ satisfies the inequality*

$$\overline{V}(t) \le \overline{\psi}(t) + \int_{t_0}^{t} \overline{q}(\tau)\overline{V}^m(\tau)d\tau + \sum_{t_0<t_i<t} \overline{V}^m(t_i - 0), \tag{3.1.41}$$

where $\overline{\psi}(t) > 0, \overline{q}(t) \ge 0, \overline{a}_i \ge 0, m > 0, m \ne 1, \overline{\psi}(t)$ is non-decreasing on J. Then we have, for all $t \ge t_0$,

$$\overline{V}(t) \le \overline{\psi}(t) \prod_{t_0<t_i<t} \left(1 + \overline{a}_i m\overline{\psi}^{m-1}(t_i)\right)\left[1 - (m-1)\int_{t_0}^{t} \overline{\psi}^{m-1}(\tau)\overline{q}(\tau)d\tau\right]^{1/(m-1)},$$
$$\text{if } 0 < m < 1, \tag{3.1.42}$$

$$\overline{V}(t) \le \overline{\psi}(t) \prod_{t_0<t_i<t} \left(1 + \overline{a}_i m\overline{\psi}^{m-1}(t_i)\right)\Big[1 - (m-1)[\prod_{t_0<t_i<t} (1 + \overline{a}_i m\overline{\psi}^{m-1}(t_i))]^{m-1}$$
$$\times \int_{t_0}^{t} \overline{q}(\tau)\overline{\psi}^{m-1}(\tau)d\tau\Big]^{-1/(m-1)}, \text{ if } m > 1, \tag{3.1.43}$$

with

$$\begin{cases} \displaystyle\int_{t_0}^{t} \overline{q}(\tau)\overline{\psi}(\tau)d\tau \le \frac{1}{m}, & (3.1.44) \\ \displaystyle\prod_{t_0<t_i<t} \left(1 + \overline{a}_i m\overline{\psi}^{m-1}(t_i)\right) < \left(\frac{m}{m-1}\right)^{1/(m-1)}. & (3.1.45) \end{cases}$$

Remark 3.1.6 Theorem 3.1.12 generalizes the result for discontinuous functions obtained by Bihari [82].

Theorem 3.1.13 (Borysenko [105]) *Assume that a non-negative function $\varphi(t)$, with first kind discontinuities at points $\{t_i\} : t_1 < t_2 < \cdots, \lim_{i\to+\infty} t_i = +\infty$, satisfies the following integro-sum inequality*

$$\varphi(t) \le C + \int_{t_0}^{t} q(s)\varphi(s)ds + \int_{t_0}^{t} q(s)\int_{t_0}^{t} q(\sigma)\varphi^m(\sigma)d\sigma ds$$
$$+ \sum_{t_0<t_i<t} \beta_i\varphi(t_i - 0), \quad \text{if } \ m > 0, \tag{3.1.46}$$

where a constant $C \geq 0, q(t) \geq 0, g(t) \geq 0, \beta = const. \geq 0$. *Then the following estimates hold all* $t \geq t_0$,

$$\varphi(t) \leq \exp\Big(\int_{t_0}^{t} q(\tau)d\tau\Big)\Big[\{C \prod_{t_0<t_i<t} (1+\beta_i)\} + (1-m)\int_{t_0}^{t} g(s)\exp\Big((m-1)\int_{t_0}^{s} g(\sigma)d\sigma\Big)ds\Big]^{1/(1-m)}, \quad \text{if } 0 < m < 1; \tag{3.1.47}$$

$$\varphi(t) \leq C \prod_{t_0<t_i<t} (1+\beta_i)\exp\Big(\int_{t_0}^{t}(q(\tau)+g(\tau))d\tau\Big), \quad \text{if } m = 1; \tag{3.1.48}$$

$$\varphi(t) \leq C \prod_{t_0<t_i<t} (1+\beta_i)\exp\Big[\int_{t_0}^{t} q(\tau)d\tau\Big] \times\Big[1-(m-1)\prod_{t_0<t_i<t}(1+\beta_i)^{m-1}C^{m-1} \times\int_{t_0}^{t} g(s)\exp\Big((m-1)\int_{t_0}^{s} g(\sigma)d\sigma\Big)ds\Big]^{-1/(1-m)}, \quad \text{if } m > 1, \tag{3.1.49}$$

with

$$\int_{t_0}^{t} g(s)\exp\Big((m-1)\int_{t_0}^{s} g(\sigma)d\sigma\Big)ds < \Big[(m-1)\prod_{t_0<t_i<t}(1+\beta_i)^{m-1}C^{m-1}\Big]^{-1}. \tag{3.1.50}$$

Using the inductive method and the methodology of the integral inequalities theory, we may prove Theorems 3.1.2–3.1.13. We shall illustrate this method only by proving Theorem 3.1.13.

Proof of Theorem 3.1.13 Suppose that $t \in [t_0, t_1]$. Then

$$\varphi(t) \leq C + \int_{t_0}^{t} q(s)\varphi(s)ds + \int_{t_0}^{t} q(s)(\int_{t_0}^{t} g(\sigma)\varphi^m(\sigma)d\sigma)ds. \tag{3.1.51}$$

Define $V(t) := C + \int_{t_0}^{t} q(s)\varphi(s)ds + \int_{t_0}^{t} q(s)(\int_{t_0}^{s} g(\sigma)\varphi^m(\sigma)d\sigma)ds$.

Obviously, $\varphi(t_0) \leq V(t_0) = C, \varphi(t) \leq V(t)$, for all $t \geq t_0$. Thus

$$\begin{aligned}\tfrac{dV}{dt} &= q(t)\varphi(t) + q(t)\textstyle\int_{t_0}^{t} g(\sigma)\varphi^m(\sigma)d\sigma \\ &\leq q(t)[V(t) + \textstyle\int_{t_0}^{t} g(\sigma)V^m(\sigma)d\sigma].\end{aligned}$$

Let $W(t) = V(t) + \int_{t_0}^t g(\sigma)V^m(\sigma)d\sigma$. Then $W(t_0) = V(t_0) = C$, $V(t) \le W(t)$, for all $t \ge t_0$. It is easy to verify that

$$\frac{dW}{dt} \le q(t)W(t) + g(t)W^m(t),$$

which yields that for all $t \ge t_0$, if $0 < m < 1$, then

$$\varphi(t) \le \exp\Big(\int_{t_0}^t q(\tau)d\tau\Big)\Big[C^{1-m} + (1-m)\int_{t_0}^t g(s) \times \exp\Big((m-1)\int_{t_0}^s g(\sigma)d\sigma\Big)ds\Big]^{1/(1-m)}; \tag{3.1.52}$$

or if $m = 1$, then

$$\varphi(t) \le C\exp\Big(\int_{t_0}^t (q(\tau) + g(\tau))d\tau\Big), \tag{3.1.53}$$

or if $m > 1$, then

$$\varphi(t) \le C\exp\Big(\int_{t_0}^t q(\tau)d\tau\Big)\Big[C^{1-m} + (1-m)\int_{t_0}^t g(s) \times \exp\Big((m-1)\int_{t_0}^s g(\sigma)d\sigma\Big)ds\Big]^{-1/(m-1)}; \tag{3.1.54}$$

satisfying

$$\int_{t_0}^t g(\tau)\exp\Big((m-1)\int_{t_0}^t g(\sigma)d\sigma\Big)d\tau < \Big((m-1)C^{m-1}\Big)^{-1}. \tag{3.1.55}$$

From (3.1.49)–(3.1.50) and (3.1.52)–(3.1.53), it follows that for all $t \in [t_0, t_1]$, $\varphi(t)$ satisfies the inequalities (3.1.47)–(3.1.49).

Using the scheme described in [98, 568, 571] for interval $[t_k, t_{k+1}]$, $k = 1, 2, \cdots$, and the estimates for the function $\varphi(t)$ on the interval $[t_{k-1}, t_k]$, we may obtain the estimates (3.1.47)–(3.1.49) on all of interval J by using inductive method. □

Now we introduce some new integro-functional inequalities of the Bellman-Bihari type. To this end, suppose that $\tau(s) \in E$ is a class of continuous functions $\tau : \mathbb{R} \to \mathbb{R}$, such that $\tau(s) \le s$, $\lim_{|s|\to+\infty} \tau(s) = +\infty$. The main result is due to Iovane [302].

Theorem 3.1.14 (Iovane [302]) *Assume that a non-negative, piecewise continuous function $\varphi(t)$, for all $t \ge t_0$, with first kind discontinuities at the points $\{t_i\}$ ($t_0 < t_1 < t_2 < \cdots, \lim_{i\to+\infty} t_i(s) = +\infty$), satisfies the following integro-sum*

inequality for all $t \geq t_0$,

$$\varphi(t) \leq n(t) + \int_{t_0}^{t} g(s)\varphi(\tau(s))ds + \sum_{t_0<t_i<t} \beta_i \varphi^m(t_i - 0), \tag{3.1.56}$$

where $n(t)$ *is a positive non-decreasing function for all* $t \geq t_0, g(s) \geq 0$, *parameter* $m > 0, \beta_i = const. \geq 0$, *then the function* $\varphi(t)$ *satisfies, for all* $t \geq t_0$,

$$\varphi(t) \leq n(t) \prod_{t_0<t_i<t} \Big(1 + \beta_i n^{m-1}(t_i)\Big) \exp\Big(\int_{t_0}^{t} g(s)\frac{n(\tau(s))}{n(s)}ds\Big), \quad \textit{if } 0 < m < 1, \tag{3.1.57}$$

$$\varphi(t) \leq n(t) \prod_{t_0<t_i<t} \Big(1 + \beta_i n^{m-1}(t_i)\Big) \exp\Big(m\int_{t_0}^{t} g(s)\frac{n(\tau(s))}{n(s)}ds\Big), \quad \textit{if } m \geq 1, \tag{3.1.58}$$

where $\varphi(t_i - 0) = \lim_{t\to t_i^{-0}} \varphi(t)$.

Remark 3.1.7 If $n(t) = c = const. > 0, \tau(s) = s, \beta_i = 0$, then a classical result of Gronwall and Bellman in [66] follows from (3.1.57)–(3.1.58). If $\beta_i = 0$, then the results in [24] are obtained; if $m = 1, n(t) = c = const.$, $\tau(s) = s$, then the results in [571] can be obtained; if $m = 1$, then the result in [568] is obtained; if $\tau(s) = s$, then the results in [105] are obtained. For the discrete case, when $n(t) = c, g(s) = 0, m = 1$, the results in [10] are obtained.

Theorem 3.1.15 (Iovane [302]) *Assume that* $\tau(s) \in E$ *and a non-negative function* $\varphi(t)$ *satisfies the inequality, for all* $t \geq t_0$,

$$\varphi(t) \leq \psi(t) + q(t)\int_{t_0}^{t} g(s)\varphi^m(\tau(s))ds + \sum_{t_0<t_i<t} \beta_i \varphi^m(t_i - 0), \tag{3.1.59}$$

where $\{t_i\}$, *satisfying the conditions of Theorem 3.1.14, are first kind discontinuity points of the function* $\varphi(t)$; $\psi(t)$ *is a positive non-decreasing function at all* $t \geq t_0$ *and* $q(t) \geq 1, g(t) \geq 0$, *for all* $t \geq t_0$, *the parameter* $m > 0, m \neq 1$, *and* $\beta_i = const. \geq 0$. *Then the function* $\varphi(t)$ *satisfies, for all* $t \geq t_0$, *if* $0 < m < 1$, *then*

$$\begin{aligned}\varphi(t) \leq{}& \psi(t) \prod_{t_0<t_i<t} (1 + \beta_i \psi^{m-1}(t_i) q^m(t_i)) \\ &\times\Big[1 + (1-m)\int_{t_0}^{t} g(s)\psi^{m-1}(s)q^m(\tau(s))\Big(\frac{\psi(\tau(s))}{\tau(s)}ds\Big)^m ds\Big]^{1/(1-m)};\end{aligned} \tag{3.1.60}$$

or if $m \geq 1$*, then*

$$\varphi(t) \leq \psi(t) \prod_{t_0<t_i<t} \Big(1 + \beta_i m \psi^{m-1}(t_i) q^m(t_i)\Big)$$
$$\times \Big\{1 + (1-m)\Big(\prod_{t_0<t_i<t} (1 + \beta_i \psi^{m-1}(t_i) q^m(t_i))\Big)^{m-1}$$
$$\times \int_{t_0}^{t} g(s)\psi^{m-1}(s) q^m(\tau(s))\Big(\frac{\psi(\tau(s))}{\tau(s)}\Big)^m ds\Big\}^{1/(1-m)}, \qquad (3.1.61)$$

with

$$\begin{cases} \displaystyle\int_{t_0}^{t} g(s)\psi^{m-1}(s) q^m(\tau(s))\Big(\frac{\psi(\tau(s))}{\tau(s)}\Big)^m ds \leq \frac{1}{m}, & (3.1.62) \\ \displaystyle\prod_{t_0<t_i<t} \Big(1 + \beta_i \psi^{m-1}(t_i) q^m(t_i)\Big) < \Big(1 + \frac{1}{m-1}\Big)^{1/m-1}. & (3.1.63) \end{cases}$$

Remark 3.1.8 If $\beta_i = 0, \psi(t) = c = const. > 0, q(t) = 1, \tau(s) = s$, then the result given by Bihari in [82] follows from Theorem 3.1.15; if $\beta_i = 0$, then the result of Theorem 3.1.15 coincides with the result given by Akinyele in [24]; if $q(t) = 1, \tau(s) = s$ in Theorem 3.1.15, then the result given by Borysenko in [105] follows.

Theorem 3.1.16 (Iovane [302]) *Assume that* $\varphi(t)$ *is a non-negative piecewise continuous function, with first kind discontinuities at the points* $\{t_i\} : t_1 < t_2 < \cdots, \lim_{i\to+\infty} t_i = +\infty$*, satisfying the following integro-sum inequality*

$$\varphi(t) \leq n(t) + q(t)\Big[\int_{t_0}^{t} f(s)\varphi(\sigma(s))ds$$
$$+ \int_{t_0}^{t} f(s)\Big(\int_{t_0}^{s} g(t)\varphi(\tau(t))dt\Big)ds + \sum_{t_0<t_i<t} \beta_i \varphi^m(t_i - 0)\Big], \quad (3.1.64)$$

where a constant $m > 0$*,* $n(t)$ *is a non-decreasing function,* $n(t) > 0$*,* $q(t) \geq 1$*,* $f(s) \geq 0$*,* $\sigma(t) \in E$*,* $g(t) \geq 0$*,* $\beta_i \geq 0, i = 1, 2, \cdots$*. Then the following estimates hold: if* $0 < m \leq 1$*, then*

$$\varphi(t) \leq n(t)q(t) \prod_{t_0<t_i<t} \Big(1 + \beta_i q^m(t_i) n^{m-1}(t_i)\Big)$$
$$\times \exp\Big(\int_{t_0}^{t} \frac{f(\xi)g(\sigma(\xi))n(\sigma(\xi)) + g(\xi)q(\tau(\xi))n(\tau(\xi))}{n(\sigma(\xi))} d\xi\Big); \qquad (3.1.65)$$

and if $m \geq 1$*, then*

$$\varphi(t) \leq n(t)q(t) \prod_{t_0<t_i<t} \left(1 + q^m(t_i)n^{m-1}(t_i)\right)$$
$$\times\left[m \int_{t_0}^{t} \frac{f(\xi)g(\sigma(\xi))n(\sigma(\xi)) + g(\xi)q(\tau(\xi))n(\tau(\xi))}{n(\sigma(\xi))} d\xi\right]. \quad (3.1.66)$$

Before stating more general results, we give some definitions.

Definition 3.1.3 A function $\psi : Q \to \mathbb{R}^n$ is said to be

i) non-decreasing in Q if $u, v \in Q$ and $u \leq v$ imply $\psi(u) \leq \psi(v)$;
ii) monotonically increasing in Q if $u, v \in Q$ and $u \leq v$ imply $\psi(u) \leq \psi(v)$, while $u < v$ implies $\psi(u) \leq \psi(v)$.

Definition 3.1.4 A function $F : \mathbb{R}_+ \times Q \to \mathbb{R}^n$ is said to be quasi-monotonically increasing in $\mathbb{R}_+ \times Q$ if for two arbitrary pairs of points (t, u), $(t, v) \in \mathbb{R}_+ \times Q$ and any $i = 1, \cdots, n$, we have that $u \leq v$ and $u_i = v_i$ imply $F_i(t, u) \leq F_i(t, v)$.

Let $0 < c < +\infty$, $B_c = \{u \in \mathbb{R}^n\ |u| < c\}$, and let $K(t, s, u)$ be a function defined for all $\alpha \leq s \leq t \leq \beta \leq +\infty$, $u \in B_c$.

For $K(t, s, u)$, we introduce the following so-called *Carathéodory conditions*:

(1) $K(t, s, u)$ is continuous with respect to $u \in B_c$ for all t and almost all s, and it is measurable with respect to s for all t and u;
(2) $K(t, s, u)$ is non-decreasing in B_c for all t and almost all s;
(3) for all $d \in (0, c)$, there are function $\mu_d(t, s)$ and $\nu_d(\tau, t, s)$, summable with respect to $s \in [\alpha, t]$ $(\alpha \leq s \leq t \leq \tau < \beta)$ such that

$$\sup_{|u|\leq d} |K(t, s, u)| \leq \mu_d(t, s), \qquad \sup_{|u|\leq d} |K(\tau, s, u) - K(t, s, u)| \leq \nu_d(\tau, t, s);$$

(4) $\lim_{\tau-t\to 0_+} (\int_\tau^t \mu_d(\tau, s)ds) = 0$ for either t or τ fixed;
(5) $\lim_{\tau-t\to 0_+} (\int_\tau^t \nu_d(\tau, t, s)ds) = 0$ for either t or τ fixed;
(6) $\lim_{\tau-t\to 0_+} (\int_\alpha^t \nu_d(\tau, s)ds) = 0$ for either $t \neq \tau_k$ or $\tau \neq \tau_k$ fixed, where the sequence $\{\tau_k\}$ is such that $\alpha < \tau_1 < \tau_2 < \ldots$, $\lim_{k\to+\infty} \tau_k \geq \beta$.

After we have introduced the above concepts, we can prove the following lemmas hold.

Lemma 3.1.1 (Azbelev-Tsalyuk [37]) *Suppose that the following conditions hold:*

1) *conditions (1)–(5) hold;*
2) $a : [\alpha, \beta) \to \mathbb{R}^n$ *is a continuous function, and* $|a(\alpha)| < c$*;*
3) $v : [\alpha, \beta) \to B_c$ *is the maximal (minimal) non-continuable solution of the equation, for all* $t \in [\alpha, \beta)$*,*

$$v(t) = a(t) + \int_\alpha^t K(t, s, v(s))ds; \quad (3.1.67)$$

4) *$u : [\alpha, \beta) \to B_c$ is a continuous function satisfying the integral inequality, for all $t \in [\alpha, \beta)$,*

$$\Delta(t) \equiv u(t) - a(t) - \int_\alpha^t K(t, s, u(s))ds \le 0, \quad \textit{respectively} \quad (0 \le \Delta(t)). \tag{3.1.68}$$

Then for all $t \in [\alpha, \beta)$,

$$u(t) \le v(t), \quad (\textit{respectively} \quad v(t) \le u(t)). \tag{3.1.69}$$

Remark 3.1.9 Lemma 3.1.1 still holds if we modify its conditions to:

i) condition (5) is replaced by (6);
ii) the functions $a, u, v : [\alpha, \beta) \to \mathbb{R}^n$ (with $|a(\alpha)| < c$) are piecewise continuous, and their discontinuity points from a subsequence of the sequence $\{\tau_k\}$ of condition (6);
iii) for all $\gamma \in (\alpha, \beta)$,

$$\sup_{t \in [\alpha, \gamma]} |u(t)| < c, \qquad \sup_{t \in [\alpha, \gamma]} |v(t)| < c.$$

Using this remark we can readily prove the following comparison lemma.

Lemma 3.1.2 (Simeonov-Bainov [593]) *Suppose that the following conditions are fulfilled:*

1) *conditions (1)–(4) and (6) hold;*
2) *$\{t_k\}_1^{+\infty}$ is a sequence such that $\alpha < t_1 < t_2 < \ldots, \lim_{k \to +\infty} t_k \ge \beta$;*
3) *$a : [\alpha, \beta) \to \mathbb{R}^n$ is a piecewise continuous function, and $|a(\alpha)| < c$;*
4) *$\psi_k : [\alpha, \beta] \times [\alpha, \beta] \times B_c \to \mathbb{R}^n$, $(t, s, u) \to \psi_k(t, s, u)$, $k \in \mathbb{N}_1$, are functions which are non-decreasing in B_c for all t, s fixed, and piecewise continuous with respect to t for all s, u fixed;*
5) *$v : [\alpha, \beta) \to \mathbb{R}^n$ is the maximal (minimal) non-continuable solution of the equation, for all $t \in [\alpha, \beta)$,*

$$v(t) = a(t) + \int_\alpha^t K(t, s, v(s))ds + \sum_{\alpha < t_k < t} \psi_k(t, t_k, v(t_k)); \tag{3.1.70}$$

6) *$u : [\alpha, \beta) \to \mathbb{R}^n$ is a piecewise continuous function satisfying the inequality for all $t \in [\alpha, \beta)$,*

$$\begin{aligned}\Delta(t) \equiv u(t) - a(t) - \int_\alpha^t K(t, s, u(s))ds - \sum_{\alpha < t_k < t} \psi_k(t, t_k, u(t_k))\\ \le 0 \quad (\textit{respectively} \quad 0 \le \Delta(t));\end{aligned} \tag{3.1.71}$$

7) *for all* $\gamma \in (\alpha, \beta)$,

$$\sup_{t\in[\alpha,\gamma]} |u(t)| < c, \quad \sup_{t\in[\alpha,\gamma]} |v(t)| < c;$$

8) *the sequence* $\{t_k\}$ *and the sequence of discontinuity points of the functions* a, u, v *form a subsequence of the sequence* $\{\tau_k\}$.
Then, for all $t \in [\alpha, \beta)$,

$$u(t) \leq v(t), \quad (\textit{respectively} \ \ v(t) \leq u(t)). \tag{3.1.72}$$

Remark 3.1.10 Lemma 3.1.2 still holds if in (3.1.70) and (3.1.71), the values $v(t_k)$ and $u(t_k)$ are replaced by $v(t_k^-)$ and $u(t_k^-)$, respectively.

In estimating solutions of equations of the form (3.1.67) and (3.1.70), the following lemma is sometimes useful.

Lemma 3.1.3 (Simeonov-Bainov [594]) *Suppose that the following conditions hold:*

1) *F:* $\mathbb{R}_+ \times Q \to \mathbb{R}^n$ *is a quasi-monotonically increasing function which is continuous inside the sets* $(\tau_{k-1}, \tau_k] \times Q$, $k \in \mathbb{N}_1$, *and for all* $k \in \mathbb{N}_1$, *and* $v \in Q$, *the limit* $\lim_{(t,u)\to(\tau_k,v),t>\tau_k} F(t,u)$ *exists;*
2) $\psi_k : Q \to \mathbb{R}^n$, $k \in \mathbb{N}_1$, *are non-decreasing functions in Q;*
3) $v : (t_0, \omega) \to \mathbb{R}^n$ *is the maximal (minimal) solution of the equation for all* $t \neq \tau_k$,

$$\begin{cases} v' = F(t, v), \\ v(\tau_k^+) = \psi_k = \psi_k(v(\tau_k)), \end{cases} \tag{3.1.73}$$

such that $v(t_0^+) = v_0$, $(t_0, v_0) \in \mathbb{R}_+ \times Q$, *and* $v(\tau^+) \in Q$ *if* $\tau_k \in (t_0, \omega)$;
4) $u : (t_0, \tilde{\omega}) \to \mathbb{R}^n$ *is a continuous function for all* $t \in (t_0, \tilde{\omega})$, $t \neq \tau_k$, *which is left-continuous at the points* τ_k *verifying*

   i) $u(t) \in Q$ *for all* $t \in (t_0, \tilde{\omega})$, *and* $u(\tau_k^+) \in Q$ *if* $\tau_k \in (t_0, \tilde{\omega})$;
   ii) $u(t_0^+) \leq v_0$ (respectively $v_0 \leq u(t_0^+)$);
   iii) $Du(t) \leq F(t, u(t))$ (respectively $F(t, u(t)) \leq Du(t)$) *for all* $t \in (t_0, \tilde{\omega})$, $t \neq \tau_k$;
   iv) $u(\tau_k^+) \leq \psi_k(u(\tau_k))$ (respectively $\psi_k(u(\tau_k)) \leq u(\tau_k^+)$), *where* $Du(t)$ *is any Dini derivative of u.*
   Then $u(t) \leq v(t)$, (respectively $v(t) \leq u(t)$) *for all* $t \in (t_0, \omega) \bigcap (t_0, \tilde{\omega})$.

By $PC(\mathbb{R}_+, \mathbb{R}^n)$, we denote the class of piecewise continuous functions $u : \mathbb{R}_+ \to \mathbb{R}^n$ that have first kind discontinuities at the points $t = t_k, k = 1, 2, \ldots,$ only and that are left-continuous at $t = t_k$.

Corollary 3.1.2 (Bainov-Simeonov [42]) *Let* $t_0 \geq 0$, $J = [t_0, +\infty)$, $u \in PC(J, \mathbb{R}^n)$, *and for all* $t > t_0$, $t \neq \tau_k$, $\tau_k > t_0$,

$$\begin{cases} Du(t) \leq A(t)u(t) + f(t), \\ u(\tau_k^+) \leq B_k u(\tau_k) + f_k, \end{cases} \tag{3.1.74}$$

where $f \in PC(J, \mathbb{R}^n)$, $A = (a_{ij}) \in PC(J, \mathbb{R}^{n\times n})$, $B_k = (b_{ij}^{(k)}) \in \mathbb{R}^{n\times n}$, $f_k \in \mathbb{R}^n$, *and for all* $t > t_0$, $i \neq j$, $1 \leq i, j \leq n$, $\tau_k > t_0$,

$$\begin{cases} a_{ij} \geq 0, \\ b_{ij}^{(k)} \geq 0. \end{cases} \tag{3.1.75}$$

If $u(t_0^+) \leq v_0$, *then for all* $t > t_0$,

$$u(t) \leq W(t, t_0^+)v_0 + \int_{t_0}^{t} W(t, s)f(s)ds + \sum_{t_0<\tau_k<t} w(t, \tau_k^+)f_k, \tag{3.1.76}$$

where $W(t, s)$ *is the Cauchy matrix of the impulse differential equation, for all* $t \neq \tau_k$, $\tau_k > t_0$,

$$\begin{cases} v'(t) = A(t)v(t), \\ v(\tau_k^+) = B_k v(\tau_k) \geq 0. \end{cases} \tag{3.1.77}$$

Remark 3.1.11 The right-hand side of (3.1.76) coincides with the solution of the comparison differential equation corresponding to (3.1.74).

Corollary 3.1.3 (Bainov-Simeonov [42]) *Let* u, a, b, f_i, $i = 1, \cdots, n$, *and* g_j, $j = 1, \cdots, n-1$, *be non-negative continuous functions in* $J = [\alpha, \beta]$, *and suppose that for all* $t \in J$,

$$\begin{aligned} u(t) \leq a(t) + b(t)\Big[&\int_{\alpha}^{t} f_1(t_1)u(t_1)dt_1 + \int_{\alpha}^{t} g_1(t_1)\left(\int_{\alpha}^{t_1} f_2(t_2)u(t_2)dt_2\right)dt_1 \\ &+ \int_{\alpha}^{t} g_1(t_1)\left(g_2(t_2)\cdots\left(\int_{\alpha}^{t_{n-2}} g_{n-1}(t_{n-1})\left(\int_{\alpha}^{t_{n-1}} f_n(t_n)u(t_n)dt_n\right)\cdots\right)\right)dt_1\Big]. \end{aligned} \tag{3.1.78}$$

Then for all $t \in J$,

$$u(t) \leq a(t) + b(t)v_1(t), \tag{3.1.79}$$

where v_1 *is the first component of the n-vector function*

$$V(t) = \int_{\alpha}^{t} Y(t)Y^{-1}(s)B(s)ds$$

and $Y(t)$ *is a fundamental matrix of the system*

$$Y'(t) = A(t)Y(t) \tag{3.1.80}$$

and

$$A = \begin{pmatrix} f_1 b & g_1 & 0 & \cdot & 0 \\ f_2 b & 0 & g_2 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & g_{n-1} \\ f_n b & 0 & 0 & \cdot & 0 \end{pmatrix}, \tag{3.1.81}$$

$$B = \left(f_1 a \, f_2 a \cdots f_n a \right). \tag{3.1.82}$$

Proof Define the functions $z_k(t), \quad k = 1, \cdots, n$, by

$$\begin{aligned} z_k(t) = & \int_\alpha^t f_k(t_k) u(t_k) dt_k + \int_\alpha^t g_k(t_k) \left(f_{k+1}(t_{k+1}) u(t_{k+1}) dt_{k+1} \right) dt_k \\ & + \int_\alpha^t g_k(t_k) \Big(\int_\alpha^{t_k} g_{k+1}(t_{k+1}) \cdots \Big(\int_\alpha^{t_{n-2}} g_{n-1}(t_{n-1}) \\ & \times \left(\int_\alpha^{t_{n-1}} f_n(t_n) u(t_n) dt_n \right) \cdots \Big) \Big) dt_k. \end{aligned} \tag{3.1.83}$$

With the above notation (3.1.83), (3.1.78) is equivalent to the system

$$\begin{cases} u \le a + b z_1, & (3.1.84) \\ z_k(\alpha) = 0, \quad k = 1, \cdots, n, & (3.1.85) \\ z_k' = f_k u + g_k z_{k+1} \le f_k b z_1 + g_k z_{k+1} + f_k a, \quad k = 1, \cdots, n-1, & (3.1.86) \\ z_n' = f_n u \le f_n b z_n + f_n a. & (3.1.87) \end{cases}$$

In the matrix notation, the relations (3.1.84)–(3.1.87) take the form

$$Z'(t) \le A(t) Z(t) + B(t), \quad Z(\alpha) = 0, \tag{3.1.88}$$

where $Z = col[z_1, \cdots, z_n]$ and A, B are defined by (3.1.81) and (3.1.82), respectively. By Corollary 18.1 in [42], with $B_k \equiv 0, Z(t) \le V(t)$, where $V(t)$ is the solution of the initial value problem

$$V'(t) = A(t) V(t) + B(t), \quad V(\alpha) = 0, \tag{3.1.89}$$

that is,

$$Z(t) \le V(t) = \int_\alpha^t Y(t) Y^{-1}(s) B(s) ds$$

where $Y(t)$ is a fundamental matrix of (3.1.89). □

Remark 3.1.12 Theorem 2 in Young [680] is the particular case of Corollary 3.1.3 when (3.1.79) coincides with (1.2.362) in Theorem 1.2.52, i.e., for $g_k = f_k, \quad k = 1, \cdots, n-1$, and $b \equiv 1$.

3.1.2 Projected Gronwall-Bellman's Inequalities for Integral Functions

As we have known, integrability is more important than continuity when we consider the equivalent integral equation of a differential equation. A question is to give an estimate for $u(t)$ which satisfies the inequality (1.2.145) where the functions $a(t), b(t), c(t)$, and $u(t)$ may not be continuous. It is interesting to extend Theorem 1.2.24 to integrable functions. Without continuity of functions and monotonicity of $a(t)$ in Theorem 1.2.24, we may discuss (1.2.145) with only integrability. In the proof of the following result, we need some techniques from [688] to overcome difficulties caused by lack of continuity and monotonicity.

Before giving the next theorem, we first note that the following simple lemma.

Lemma 3.1.4 *Suppose that $b(t)$ is monotonically non-increasing from $\mathbb{R}_+$ into $\mathbb{R}_+$ and that $\int_0^{+\infty} b(s)ds < +\infty$. Then $b(t) \to 0$ as $t \to +\infty$.*

Proof Note that $0 \le b(t) \le b(0)$ and $b(t)$ converges to a non-negative constant as $t \to +\infty$, since $b(t)$ is monotonically non-increasing and non-negative. It is certain that the constant equals to 0; otherwise, the integral $\int_0^{+\infty} b(s)ds = +\infty$. □

The following result is due to Zhang [687].

Theorem 3.1.17 (Zhang [687]) *Suppose that $a(t), b(t)$, and $c(t)$ are functions defined on $\mathbb{R}_+$ and valued in $\mathbb{R}_+$, and that $u : \mathbb{R}_+ \to \mathbb{R}_+$ is a bounded solution of inequality (1.2.145). If*

(i) $a(t)$ *is bounded,* $a^* := \lim sup_{t\to+\infty} a(t) \le a(t)$, *for all* $t \ge 0$,
(ii) $b(t)$ *is monotonically non-increasing,* $b(t) \le b(0)e^{\delta t}$, *for all* $t \ge 0$, *where* δ *is a real constant, and* $\int_0^{+\infty} b(s)ds < +\infty$,
(iii) $c(t)$ *is integrable on* $[0, +\infty)$, *i.e.,* $\int_0^{+\infty} c(s)ds < +\infty$, *and*
(iv) $\beta := \int_0^{+\infty} b(s)ds + \int_0^{+\infty} c(s)ds < 1$, *then for all* $t \ge 0$,

$$u(t) \le \frac{\tilde{a}(t)}{1-\beta} + \frac{b(0)}{(1-\beta)^2} \int_0^t (\tilde{a} - a^*)(s) \exp\left((\delta + \frac{b(0)}{1-\beta})(t-s) \right) ds, \tag{3.1.90}$$

where $\tilde{a}(t) := \sup_{s \ge t}\{a(s)\}$.

Proof Clearly, $\tilde{a}(t) := \sup_{s\ge t}\{a(s)\}$ is monotonically non-increasing and $\tilde{a}(t) \ge a(t)$.

Let

$$S = \left\{ t \in \mathbb{R}_+ : u(t) > \frac{a^*}{1-\beta} \right\}. \tag{3.1.91}$$

Note that S may be empty in some cases. Obviously, the inequality (3.1.90) holds for all $t \in \mathbb{R}_+ \backslash S$, by noting that $u(t) > a^*/(1-\beta)$ for all $t \in \mathbb{R}_+ \backslash S$, $a^* \leq a(t) \leq \tilde{a}(t)$ for all $t \geq 0$, and for all $t \geq 0$, we have

$$\frac{b(0)}{(1-\beta)^2} \int_0^t (\tilde{a} - a^*)(s) \exp\left((\delta + \frac{b(0)}{1-\beta})(t-s) \right) ds \geq 0.$$

It suffices to prove (3.1.90) holds for all $t \in S$.

Define a function

$$v(t) = \begin{cases} u(t) - \dfrac{a^*}{1-\beta}, & \textit{for all } t \in S, \\ 0, & \textit{for all } t \in \mathbb{R}_+ \backslash S. \end{cases} \tag{3.1.92}$$

Clearly, $v(t)$ is bounded, non-negative and the integrals $\int_0^t b(t-s)v(s)ds$ and $\int_0^{+\infty} c(s)v(t+s)ds$ exist for all $t \geq 0$. Also, for all $t \geq 0$,

$$u(t) \leq v(t) + \frac{a^*}{1-\beta}, \tag{3.1.93}$$

since for all $t \in S$, the equivalence of (3.1.93) holds and for all $t \in \mathbb{R}_+ \backslash S$, we have $u(t) \leq a^*/(1-\beta)$ and $v(t) = 0$ by definition. Substituting (3.1.93) into (1.2.141), we obtain, for all $t \geq 0$,

$$u(t) \leq a(t) + \int_0^t b(t-s) \left(v(s) + \frac{a^*}{1-\beta} \right) ds + \int_0^{+\infty} c(s) \left(v(t+s) + \frac{a^*}{1-\beta} \right) ds. \tag{3.1.94}$$

In particular, for all $t \in S$, the equivalence of (3.1.93) holds and from (3.1.94) it follows

$$\begin{aligned} v(t) + \frac{a^*}{1-\beta} \leq a(t) &+ \int_0^t b(t-s) \left(v(s) + \frac{a^*}{1-\beta} \right) ds \\ &+ \int_0^{+\infty} c(s) \left(v(t+s) + \frac{a^*}{1-\beta} \right) ds \\ &\leq a(t) + \int_0^t b(t-s)v(s)ds + \int_0^{+\infty} c(s)v(t+s)ds \end{aligned}$$

$$+\frac{a^*}{1-\beta}\left(\int_0^t b(t-s)ds+\int_0^{+\infty} c(s)ds\right)$$
$$\leq a(t)+\int_0^t b(t-s)v(s)ds+\int_0^{+\infty} c(s)v(t+s)ds+\frac{a^*\beta}{1-\beta},$$

that is, for all $t \in S$,

$$v(t) \leq a(t)-a^*+\int_0^t b(t-s)v(s)ds+\int_0^{+\infty} c(s)v(t+s)ds. \tag{3.1.95}$$

Indeed, (3.1.95) also holds for all $t \in \mathbb{R}_+ \backslash S$ because $v(t)=0$ for all $t \in \mathbb{R}_+ \backslash S$.

Note that

$$\lim_{t\to+\infty} v(t)=0. \tag{3.1.96}$$

In fact, with the assumption that $\gamma := \limsup_{t\to+\infty} v(t) > 0$, for all arbitrarily fixed $\theta \subset (\beta, 1)$, we have $\limsup_{t\to+\infty} v(t) < \theta^{-1}\gamma$, that is, there exists a $t_0 \geq 0$ such that $v(t) < \theta^{-1}\gamma$ for all $t \geq t_0$.

From (3.1.95), we derive for all $t \geq t_0$,

$$v(t) \leq a(t)-a^*+\int_0^{t_0} b(t-s)v(s)ds+\theta^{-1}\gamma\left(\int_0^{t-t_0} b(s)ds+\int_0^{+\infty} c(s)ds\right).$$

It follows by taking $t \to +\infty$ and by Lemma 3.1.4 that

$$\gamma \leq \theta^{-1}\gamma\left(\int_0^{+\infty} b(s)ds+\int_0^{+\infty} c(s)ds\right)=\theta^{-1}\gamma\beta<\gamma$$

which contradicts the assumption.

Furthermore, let

$$f(t) := \sup_{s\geq t} v(s). \tag{3.1.97}$$

Clearly, $f(t)$ is monotonically non-increasing and bounded. By definition and (3.1.96), we know that $f(t) \geq v(t)$ and $\lim_{t\to+\infty} f(t) = 0$. Thus, $f(t)$ is integrable on every compact interval and for every $t \in [0, +\infty)$, there exists a $t_1 \geq t$ such that

$$f(s)=\begin{cases} f(t)=v(t_1), & t \leq s \leq t_1, \\ < f(t_1), & s > t_1. \end{cases} \tag{3.1.98}$$

It follows from (3.1.95) and monotonicity of $\tilde{a}(t)$ and $b(t)$ that

$$\begin{aligned}
f(t) &= v(t_1) \\
&\leq a(t_1) - a^* + \int_0^t b(t_1 - s)f(s)ds + \int_t^{t_1} b(t_1 - s)f(s)ds + \int_0^{+\infty} c(s)f(t_1 + s)ds \\
&\leq a(t_1) - a^* + \int_0^t b(t_1 - s)f(s)ds + f(t)\left(\int_t^{t_1} b(t_1 - s)ds + \int_0^{+\infty} c(s)ds\right) \\
&\leq \tilde{a}(t_1) - a^* + \int_0^t b(t_1 - s)f(s)ds + f(t)\beta \\
&\leq \tilde{a}(t_1) - a^* + \int_0^t b(t - s)f(s)ds + f(t)\beta,
\end{aligned}$$

that is, for all $t \geq 0$,

$$\begin{aligned}
f(t) &\leq \frac{\tilde{a}(t) - a^*}{1 - \beta} + \frac{1}{1 - \beta}\int_0^t b(t - s)f(s)ds \\
&\leq \frac{\tilde{a}(t) - a^*}{1 - \beta} + \frac{b(0)}{1 - \beta}\int_0^t e^{\delta(t-s)}f(s)ds, \qquad (3.1.99)
\end{aligned}$$

or equivalently, for all $t \geq 0$,

$$e^{-\delta t}f(t) \leq \frac{\tilde{a}(t) - a^*}{1 - \beta}e^{-\delta t} + \frac{b(0)}{1 - \beta}\int_0^t e^{-\delta s}f(s)ds. \qquad (3.1.100)$$

Let

$$R(t) = \int_0^t e^{-\delta s}f(s)ds, \qquad (3.1.101)$$

whose derivative clearly exists except for a set of measure zero. Then from (3.1.100) it follows that for a. e. $t \geq 0$,

$$\frac{d}{dt}R(t) \leq \frac{\tilde{a}(t) - a^*}{1 - \beta}e^{-\delta t} + \frac{b(0)}{1 - \beta}R(t), \qquad (3.1.102)$$

that is, a. e. $t \geq 0$,

$$\frac{d}{dt}\left(R(t)\exp\left(-\frac{b(0)}{1 - \beta}t\right)\right) \leq \frac{\tilde{a}(t) - a^*}{1 - \beta}e^{-\delta t}\exp\left(-\frac{b(0)}{1 - \beta}t\right), \qquad (3.1.103)$$

where the monotonically bounded function $\tilde{a}(t)$ is integrable. Integrating from 0 to t, we obtain, for all $t \geq 0$,

$$R(t)\exp\left(-\frac{b(0)}{1-\beta}t\right) \leq \int_0^t \frac{\tilde{a}(s)-a^*}{1-\beta}\exp\left(-(\delta\frac{b(0)}{1-\beta})s\right)ds, \tag{3.1.104}$$

whence for all $t \geq 0$,

$$R(t) \leq \int_0^t \frac{\tilde{a}(s)-a^*}{1-\beta}\exp\left(-\delta s+\frac{b(0)}{1-\beta}(t-s)\right)ds. \tag{3.1.105}$$

Therefore, it follows from (3.1.100) that for all $t \geq 0$,

$$\begin{aligned} v(t) &\leq f(t) \leq \frac{\tilde{a}(t)-a^*}{1-\beta}e^{-\delta t}+\frac{b(0)}{1-\beta}e^{\delta t}R(t) \\ &\leq \frac{\tilde{a}(t)-a^*}{1-\beta}e^{-\delta t}+\frac{b(0)}{(1-\beta)^2}\int_0^t(\tilde{a}(t)-a^*)\exp\left((\delta+\frac{b(0)}{1-\beta})(t-s)\right)ds. \end{aligned} \tag{3.1.106}$$

In particular, from the definition of $v(t)$ in (3.1.92), we derive that for all $t \in S$,

$$\begin{aligned} u(t) &= v(t)+\frac{a^*}{1-\beta} \\ &\leq \frac{\tilde{a}(t)}{1-\beta}+\frac{b(0)}{(1-\beta)^2}\int_0^t(\tilde{a}(t)-a^*)\exp\left((\delta+\frac{b(0)}{1-\beta})(t-s)\right)ds, \end{aligned} \tag{3.1.107}$$

which implies that (3.1.65) holds both for all $t \in \mathbb{R}_+\backslash S$ and for all $t \in S$. This completes the proof. □

Clearly, (1.2.144) is in a special form of (1.2.145) where $a(t) = a + \exp(-\alpha t)\sum_{i=0}^m a_i t^i, b(t) = b\exp(-\alpha t)$, and $c(t) = c\exp(-\gamma t)$. In this case, these functions possess continuity and monotonicity. In fact, we have the following result.

Lemma 3.1.5 (Zhang-Deng [688]) *If* $\alpha > 0, a, a_i, i = 0, 1, \cdots, m$, *are non-negative, and* $a_{k+1} \leq \alpha a_k/(k+1), k = 0, 1, \cdots, m-1$, *then* $a(t) = a + \exp(-\alpha t)\sum_{i=0}^m a_i t^i$ *is monotonically non-increasing.*

Proof An easy computation yields, for all $t \geq 0$,

$$\begin{aligned} a'(t) &= \exp(-\alpha t)\left(-\sum_{i=0}^m \alpha a_i t^i+\sum_{i=1}^m i a_i t^{i-1}\right) \\ &= \exp(-\alpha t)\left(-\sum_{i=0}^m \alpha a_i t^i+\sum_{i=0}^{m-1}(i+1)a_{i+1}t^i\right) \end{aligned}$$

$$
\begin{aligned}
&= \exp(-\alpha t)\left(\sum_{i=0}^{m-1}((i+1)a_{i+1}-\alpha a_i)t^i - \alpha a_m t^m\right) \\
&\leq 0. \qquad (3.1.108)
\end{aligned}
$$

This proves $a(t)$ is monotonically non-increasing. □

The next corollary deals with inequality (1.2.144) for the cases with continuity and monotonicity.

Corollary 3.1.4 (Zhang-Deng [688]) *Suppose that the bounded function $u : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies inequality (1.2.144), where $\alpha > 0, \gamma > 0, a, a_i, b$, and c are non-negative , $i = 0, 1, \ldots, m, a_{k+1} \leq \alpha a_k/(k+1), k = 0, 1, \ldots, m-1$, and $\beta := b/\alpha + c/\gamma < 1$. Then for all $t \geq 0$, we have*

$$
\begin{aligned}
u(t) \leq (1-\beta)^{-1}&\left(a + \exp(-\alpha t)(\sum_{i=0}^{m} a_i t^i)\right) \\
&-(1-\beta)^{-2} b \exp(-\alpha t)\sum_{i=0}^{m} a_i \sum_{j=0}^{i}(\frac{b}{1-\beta})^{-i-1}\frac{i!}{j!}\left(\frac{bt}{1-\beta}\right)^j \\
&+(1-\beta)^{-2} b \sum_{i=0}^{m} a_i i!(\frac{b}{1-\beta})^{-i-1}\exp\left((-\alpha + \frac{b}{1-\beta})t\right).
\end{aligned}
\qquad (3.1.109)
$$

Proof By Lemma 3.1.5, $\tilde{a}(t) := \sup_{s \geq t} a(s) = a(t)$ and $a^* := \limsup_{t\to+\infty} a(t) = a$. By Theorem 3.1.17, it follows that for all $t \geq 0$,

$$
\begin{aligned}
u(t) &\leq (1-\beta)^{-1}\left(a + \exp(-\alpha t)(\sum_{i=0}^{m} a_i t^i)\right) \\
&\quad + \frac{b}{(1-\beta)^2}\int_0^t \exp(-\alpha s)\sum_{i=0}^{m} a_i s^i \exp\left((-\alpha + \frac{b}{1-\beta})(t-s)\right) ds \\
&\leq (1-\beta)^{-1}\left(a + \exp(-\alpha t)(\sum_{i=0}^{m} a_i t^i)\right) \\
&\quad + \frac{b}{(1-\beta)^2}\exp\left((-\alpha + \frac{b}{1-\beta})t\right)\sum_{i=0}^{m} a_i \int_0^t s^i \exp\left(-\frac{b}{1-\beta}s\right) ds.
\end{aligned}
\qquad (3.1.110)
$$

We can prove by induction that for all $n \in \mathbb{Z}_+$,

$$\int_0^t s^n \exp(-\phi s)ds = -\frac{1}{\phi^{n+1}} \exp(-\phi t) \sum_{i=0}^{m} \frac{n!}{i!}(\phi t)^i + \frac{1}{\phi^{n+1}} n!. \tag{3.1.111}$$

Thus (3.1.110) follows from (3.1.109). □

Now we can apply the above theorem to deal with the case where not all terms include an exponential function.

Corollary 3.1.5 (Zhang-Deng [688]) *Suppose that the bounded function* $u : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies inequality (1.2.145), where* $a(t) = a+a_0(t+\sigma_1)^{-n}, b(t) = b(t+\sigma_2)^{-m}, c(t) = ct\exp(-t^2)\sin^2 t^2, a, a_0, b,$ *and* c *are non-negative,* $\sigma_1 > 0, \sigma_2 > 0, n$ *and* m *are positive integers, and* $\beta := b/(m-1)\sigma_2^{1-m} + c/5 < 1$. *Then for all* $t \geq 0$, *there holds that*

$$\begin{aligned} u(t) \leq{}& \frac{a + a_0(t+\sigma_1)^{-n}}{1-\beta} + \frac{ba_0\sigma_2^{-m}}{(1-\beta)^2} \exp\left(\xi(t+\sigma_1)\right) \\ &\times\Bigg\{ \sum_{j=1}^{n-1} \frac{(-\xi)^{j-1}(n-j-1)!}{(n-1)!} \Big(-(t+\sigma_1)^{j-n} \exp\left(-\xi(t+\sigma_1)\right) + \sigma_1^{j-n} \exp(-\xi\sigma_1) \Big) \\ &+ \frac{(-\xi)^{n-1}}{(n-1)!} \Big(Ei(-\xi(t+\sigma_1)) - Ei(-\xi\sigma_1) \Big) \Bigg\}, \end{aligned} \tag{3.1.112}$$

where $\xi := b\sigma_2^{-m}/(1-\beta), Ei(x) := \int_{-\infty}^{x} t^{-1}\exp(t)dt$.

Proof Obviously, $a(t)$ and $b(t)$ are monotonically non-increasing, and the derivative $b'(t) = -mb(t+\sigma_2)^{-m-1} \leq 0$. Hence, condition (ii) of Theorem 3.1.17 is satisfied for $\delta = 0$. Moreover,

$$\begin{aligned} \int_0^{+\infty} c(t)dt &= \int_0^{+\infty} ct\exp(-t^2)\sin^2 t^2 dt = \frac{c}{2}\int_0^{+\infty} \exp(-t)\sin^2 t dt \\ &= \frac{c}{4}\int_0^{+\infty} \exp(-t)(1-\cos 2t)dt = \frac{c}{5}. \end{aligned} \tag{3.1.113}$$

It is easy to verify that $\int_0^{+\infty} b(s)ds + \int_0^{+\infty} c(s)ds = (b/(m-1))\sigma_2^{1-m} + c/5 < 1$. By Theorem 3.1.17, for all $t \geq 0$,

$$\begin{aligned} u(t) &\leq \frac{a + a_0(t+\sigma_1)^{-n}}{1-\beta} + \frac{b\sigma_2^{-m}}{(1-\beta)^2} \int_0^t a_0(s+\sigma_1)^{-n} \exp\left(\frac{b\sigma_2^{-m}}{1-\beta}(t-s)\right) ds \\ &\leq \frac{a + a_0(t+\sigma_1)^{-n}}{1-\beta} + \frac{ba_0\sigma_2^{-m}}{(1-\beta)^2} \exp\Big(\xi(t+\sigma_1)\Big) \int_{\sigma_1}^{t+\sigma_1} s^{-n}\exp(-\xi s)ds. \end{aligned}$$

Note that by induction,

$$\int_0^{+\infty} s^{-n}\exp(\phi s)ds = -\exp(\phi s)\sum_{i=1}^{n-1}\frac{\phi^{i-1}(n-1-i)!}{(n-1)!}x^{i-n} + \frac{\phi^{n-1}Ei(\phi x)}{(n-1)!} + C, \quad n > 1,$$

where $C > 0$ is a constant. Then we can obtain (3.1.112). □

The above theorem can be applied to deal with inequality (1.2.145) where $a(t)$ does not possess monotonicity.

Corollary 3.1.6 (Zhang-Deng [688]) *Suppose that the bounded function* $u : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies inequality (1.2.145) with*

$$a(t) = \begin{cases} a, & 0 \le t < 1 \\ \dfrac{2(t-n)}{n+a}, & n \le t < n + \dfrac{1}{2}, \quad n = 1, 2, \cdots, \\ -\dfrac{2(t-n-1)}{n+a}, & n + \dfrac{1}{2} \le t < n+1, \quad n = 1, 2, \cdots \end{cases} \tag{3.1.114}$$

and $b(t) = b\exp(-\alpha t), c(t) = c\exp(-\gamma t), \alpha > 0, \gamma > 0, a, b, c \ge 0$, *and* $\beta := b/\alpha + c/\gamma < 1$. *Then*

(a) when $0 \le t \le 1$,

$$u(t) \le \frac{1+a}{1-\beta} + \frac{b}{(1-\beta)^2\rho}\Big(\exp(\rho t) - 1\Big); \tag{3.1.115}$$

(b) when $n \le t < n + 1/2$,

$$u(t) \le \frac{1+an}{n(1-\beta)} + \frac{b}{(1-\beta)^2\rho}\Big(C_0 + S_1(t)\Big)\exp(\rho t); \tag{3.1.116}$$

(c) when $n + 1/2 \le t < n + 1 - n/(2n+2)$,

$$u(t) \le \frac{2(n+1-t)+an}{n(1-\beta)} + \frac{b}{(1-\beta)^2\rho}\left(C_0 + S_1(n+\frac{1}{2}) + S_2(t)\right)\exp(\rho t); \tag{3.1.117}$$

(d) when $n + 1 - n/(2n+2) \le t < n+1$,

$$\begin{aligned} u(t) \le{}& \frac{1+a(n+1)}{(n+1)(1-\beta)} \\ &+\frac{b}{(1-\beta)^2\rho}\Big(C_0 + S_1(\frac{n+1}{2}) + S_2(\frac{n+1-n}{2n+2}) + S_3(t)\Big)\exp(\rho t), \end{aligned} \tag{3.1.118}$$

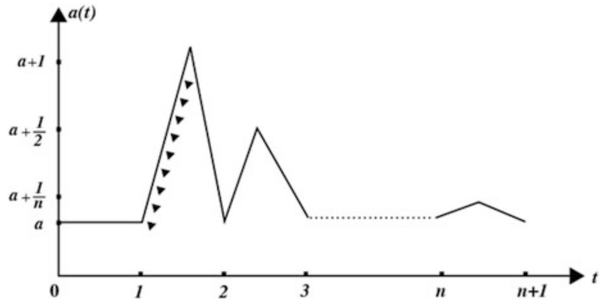

Fig. 3.1 Graph of *a(t)* without monotonicity

where $n = 1, 2, \cdots$ *(Fig. 3.1),*

$$\begin{cases} \rho = -\alpha + b/(1-\beta), \quad C_0 = -\exp(-\rho) + 1 - \sum_{i=1}^{n-1} (1/i)(\exp(-\rho(i+1)) - \exp(-\rho i)), \\ S_1(t) = \Big(\exp(-\rho n) - \exp(-\rho t)\Big)/n, \\ S_2(t) = (2/\rho n)(\rho t - \rho n - \rho + 1)\exp(-\rho t) + (1/\rho n)(\rho - 2)\exp\Big(-\rho(n+1/2)\Big), \\ S_3(t) = -(1/(n+1))\Big(\exp(-\rho t) - \exp\Big(-\rho(n+1-n/(2n+2))\Big)\Big). \end{cases}$$

Proof When $0 \le t \le 1, \tilde{a}(t) = \sup_{s \ge t} a(s) = a + 1$. When $t > 1$, we can easily calculate

$$\tilde{a}(t) = \begin{cases} \dfrac{a+1}{n}, & n \le t \le \dfrac{n+1}{2} \\ \dfrac{a - 2(t-n-1)}{n}, & \dfrac{n+1}{2} \le t < \dfrac{n+1-n}{(2n+2)}, \\ \dfrac{a+1}{(n+1)}, & \dfrac{n+1-n}{(2n+2)} \le t < n+1, \end{cases} \tag{3.1.119}$$

where $n = 1, 2, \cdots$, and $a^* := \limsup_{t \to +\infty} a(t) = a \le a(t)$, for all $t \ge 0$. By Theorem 3.1.17, we obtain the following estimates:

(1) When $0 \le t \le 1$, we get

$$\begin{aligned} u(t) &\le \frac{1+a}{1-\beta} + \frac{b}{(1-\beta)^2} \int_0^t \exp\left(-(\alpha - \frac{b}{1-\beta})(t-s)\right) ds \\ &= \frac{1+a}{1-\beta} + \frac{b}{(1-\beta)^2} \left(\exp(\rho t) - 1\right). \end{aligned} \tag{3.1.120}$$

(2) When $n \le t < n + 1/2$, we obtain

$$
\begin{aligned}
u(t) &\le \frac{1+an}{n(1-\beta)} + \frac{b}{(1-\beta)^2}\int_0^t (\tilde{a}(s) - a^*)\exp\left(-(\alpha - \frac{b}{1-\beta})(t-s)\right) ds \\
&\le \frac{1+an}{n(1-\beta)} + \frac{b}{(1-\beta)^2}\left(\int_0^1 \exp(\rho(t-s))ds \sum_{i=1}^{n-1}\frac{1}{i}\int_i^{i+1}\exp(\rho(t-s))ds\right. \\
&\quad \left. + \frac{1}{n}\int_n^t \exp(\rho(t-s))ds\right) \\
&\le \frac{1+an}{n(1-\beta)} + \frac{b}{(1-\beta)^2\rho}(C_0 + S_1(t))\exp(\rho t).
\end{aligned} \tag{3.1.121}
$$

(3) When $n + 1/2 \le t < n + 1 - n/(2n+2)$, we have

$$
\begin{aligned}
u(t) &\le \frac{2(n+1-t)+an}{n(1-\beta)} + \frac{b}{(1-\beta)^2}\int_0^t (\tilde{a}(s) - a^*)\exp\left(-(\alpha - \frac{b}{1-\beta})(t-s)\right) ds \\
&\le \frac{2(n+1-t)+an}{n(1-\beta)} + \frac{b}{(1-\beta)^2}\Big(\int_0^1 \exp(\rho(t-s))ds \\
&\quad + \sum_{i=1}^{n-1}\frac{1}{i}\int_i^{i+1}\exp(\rho(t-s))ds + \frac{1}{n}\int_n^{n+1/2}\exp(\rho(t-s))ds \\
&\quad + \frac{2(n+1-s)}{n}\int_{n+1/2}^t \exp(\rho(t-s))ds\Big) \\
&\le \frac{2(n+1-t)+an}{n(1-\beta)} + \frac{b}{(1-\beta)^2\rho}\left(C_0 + S_1(\frac{n+1}{2}) + S_2(t)\right)\exp(\rho t).
\end{aligned} \tag{3.1.122}
$$

(4) When $n + 1 - n/(2n+2) \le t < n + 1$, we conclude

$$
\begin{aligned}
u(t) &\le \frac{1+a(n+1)}{(n+1)(1-\beta)} + \frac{b}{(1-\beta)^2}\int_0^t (\tilde{a}(s) - a^*)\exp(-(\alpha - \frac{b}{1-\beta})(t-s))ds \\
&\le \frac{1+a(n+1)}{(n+1)(1-\beta)} + \frac{b}{(1-\beta)^2}\Big(\int_0^1 \exp(\rho(t-s))ds + \sum_{i=1}^{n-1}\frac{1}{i}\int_i^{i+1}\exp(\rho(t-s))ds \\
&\quad + \frac{1}{n}\int_n^{n+1/2}\exp(\rho(t-s))ds + \frac{2(n+1-s)}{n}\int_{n+1/2}^{n+1-n/(2n+2)}\exp(\rho(t-s))ds \\
&\quad + \frac{1}{n+1}\int_{n+1-n/(2n+2)}^t \exp(\rho(t-s))ds\Big)
\end{aligned}
$$

$$\leq \frac{1+a(n+1)}{(n+1)(1-\beta)} + \frac{b}{(1-\beta)^2\rho}\Big\{C_0 + S_1(\frac{n+1}{2})$$
$$+S_2\left(\frac{n+1-n}{(2n+2)}\right) + S_3(t)\Big\}\exp(\rho t). \tag{3.1.123}$$

The proof is thus complete. □

We note the above theorem can be also applied to deal with inequality (1.2.145) where functions $u(t), a(t), b(t)$, and $c(t)$ are not continuous (see, Figs. 3.2 and 3.3).

Corollary 3.1.7 (Zhang-Deng [688]) *Suppose that the bounded function* $u : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies inequality (1.2.145), where*

$$\begin{cases} a(t) = \dfrac{1}{n^2+1} + a, \qquad t \in [n, n+1), \qquad n = 0, 1, 2, \cdots, & (3.1.124) \\ b(t) = \begin{cases} b, & t = 0 \\ a\exp(-n\alpha), & t \in (n-1, n], \qquad n = 1, 2, \cdots, \end{cases} & (3.1.125) \\ c(t) = \begin{cases} c(t+\sigma_3)^{-k}, & t \in (m, m+1), \ \ or \ \ t \geq k+1, \quad m = 0, 1, \cdots, k, \\ 0, & t = m, \qquad m = 0, 1, \cdots, k, \end{cases} \end{cases} \tag{3.1.126}$$

and $\alpha > 0, \sigma_3 > 0, k \in \mathbb{Z}, k > 1, a, b, c \geq 0$*, and* $\beta := b/\alpha + (c/(k-1))\sigma_3^{1-k} < 1$. *Then*

$$u(t) \leq \frac{1+a(n^2+1)}{(n^2+1)(1-\beta)} - \frac{b}{\rho(n^2+1)(1-\beta)^2} + \frac{bC_1}{\rho(1-\beta)^2}\exp(\rho t),$$

for all $t \in [n, n+1), n = 0, 1, 2, \cdots$*, where* $\rho = -\alpha + b/(1-\beta)$*, and*

$$C_1 = \sum_{i=0}^{n-1} \frac{\exp(-\rho i) - \exp(-\rho(i+1))}{i^2+1} + \frac{\exp(-\rho n)}{n^2+1}.$$

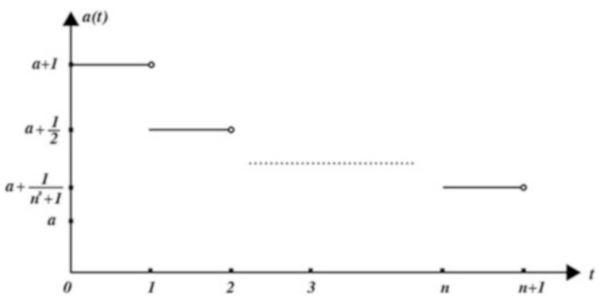

Fig. 3.2 Graph of *a(t)* without continuity

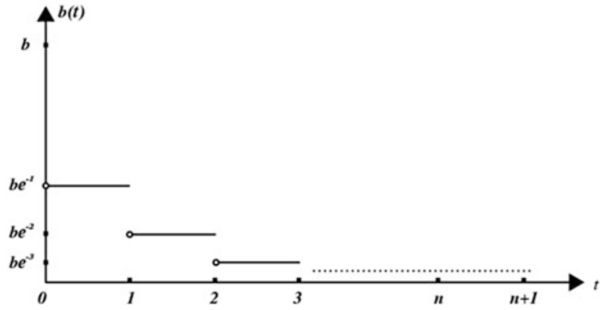

Fig. 3.3 Graph of *b(t)* without continuity

Proof Note that $b(t) = B\exp(-\alpha t)$, $\int_0^{+\infty} c(t)dt = \int_0^{+\infty} c(t+\sigma_3)^{-k}dt = (c/(k-1))\sigma_3^{1-k}$, and $a^* := \limsup_{t\to+\infty} a(t) = a \le a(t)$, for all $t \ge 0$. Obviously, $\beta < 1$ when b and c are chosen small enough or when α and σ_3 are chosen large enough. By Theorem 3.1.17, for all $t \in [n, n+1), n = 0, 1, 2, \cdots$,

$$\begin{aligned} u(t) &\le \frac{1 + a(n^2+1)}{(n^2+1)(1-\beta)} + \frac{b}{(1-\beta)^2}\int_0^t (\tilde{a}(s) - a^*)\exp\Big((\alpha - \frac{b}{1-\beta})(t-s)\Big)ds \\ &\le \frac{1 + a(n^2+1)}{(n^2+1)(1-\beta)} + \frac{b}{(1-\beta)^2}\Big\{\int_i^{i+1}\frac{1}{i^2+1}\exp(\rho(t-s))ds \\ &\quad + \sum_{i=1}^{n-1}\frac{1}{i}\int_i^{i+1}\exp(\rho(t-s))ds + \int_n^t \frac{1}{n^2+1}\exp(\rho(t-s))ds\Big\} \\ &\le \frac{1 + a(n^2+1)}{(n^2+1)(1-\beta)} - \frac{b}{\rho(n^2+1)(1-\beta)^2} + \frac{bC_1}{\rho(1-\beta)^2}\exp(\rho t). \end{aligned}$$

□

3.1.3 The Gronwall Inequalities for Modified Stieltjes Integrals

In this section, we shall introduce the Gronwall inequalities for modified Stieltjes integrals. Now we first introduce some basic concepts.

Definition 3.1.5 A subdivision σ of an interval $[a, b]$ is defined to be a finite set of numbers $x_0, x_1, \cdots, x_n$ with $a = x_0 < x_1 < \cdots < x_n = b$. Any subdivision σ_1 is

said to be a refinement of σ if $\sigma \subset \sigma_1$. Let f and g be defined on $[a, b]$ and let σ be a subdivision of $[a, b]$. Let $S_\sigma(f, g)$ denote the sum

$$S_\sigma(f, g) = \sum_i \frac{1}{2}[f(x_i) + f(x_{i-1})][g(x_i) - g(x_{i-1})].$$

The integral $\int_a^b f(s)dg(s)$ is defined to be "limit" of $S_\sigma(f, g)$. More precisely, if there is a real number I with the property that for every $\varepsilon > 0$, there is a subdivision σ of $[a, b]$ such that for any refinement σ_1 of σ, we have $|S_{\sigma_1}(f, g) - I| < \varepsilon$, then we define I to be the integral $\int_a^b f(s)dg(s)$. This integral is known as the mean σ–integral. Another integral, the Dushnik-integral, which differs from the mean σ-integral in that the term $\frac{1}{2}[f(x_i) + f(x_{i-1})]$ is replaced by $f(\xi_i)$, where ξ_i is any point that satisfies $x_{i-1} < \xi_i < x_i$. We refer the reader to [277, 350] and [310] for further details on these two integrals. We shall use the symbol $(m)\int_a^b f(s)dg(s)$ and $(b)\int_a^b f(s)dg(s)$ to refer to the mean σ- and the Dushnik-integrals, respectively.

It is well-known that if f is a function with only discontinuities of the first kind and g is a function of bounded variation, then both the mean σ- and the Dushnik-integrals exist.

Let S denote the space of all real-valued functions defined on $[a, b]$ with right- and left-hand limits at every point in $[a, b]$. Define a norm on S by $\|f\|_\infty = \sup\{|f(s)| : a \le s \le b\}$. S is then a Banach space.

The following lemma gives a representation for bounded linear functionals on S.

Lemma 3.1.6 (Schmaedeke [578]) *Let l be a continuous linear functional on the space S. Then there exist functions of bounded variation g and h such that for every f in S, we have*

$$l(f) = (m)\int_a^b f(s)dg(s) + (b)\int_a^b f(s)dh(s).$$

The proof of this lemma is a direct consequence of a theorem of Kaltenborn (cf. [310]) and we shall omit the details.

Second, we shall recall some properties of the mean σ-integral, which include:

(i) for any functions f and g of bounded variation where g is right continuous, we have

$$\begin{cases} \lim\limits_{b \to a^+} (m)\int_a^b f(s)dg(s) = 0, \\ \lim\limits_{a \to b^-} (m)\int_a^b f(s)dg(s) = \frac{1}{2}[f(b) + f(b^-)]\Delta g(b) \end{cases} \tag{3.1.127}$$

where $\Delta g(b) = g(b) - g(b^-)$. The first limit shows that the integral $(m)\int_a^b f(s)dg(s)$ is right continuous at t.

(ii) another property is

$$(m)\int_a^b f(s)dg(s) \leq \sup\{|f(s)| : a \leq s \leq b\}V[g; a, b]$$

where $V[g; a, b]$ is the total variation of g on the interval $[a, b]$.

Before proving the next Theorem 3.1.18, let us show that T' is maximal in the sense indicated. Let g be an increasing function with $\Delta g(\tau) = b \geq 2$, where $0 < \tau \leq T$ and let

$$f(t) = \begin{cases} 0, \ 0 \leq t < \tau, \\ 1, \ \tau \leq t \leq T. \end{cases}$$

Since g is increasing, we have $g(t) \geq g(\tau)$ for all $t \geq \tau$. By a direct computation, we have

$$(m)\int_0^t f(s)dg(s) = \begin{cases} 0, & 0 \leq t < \tau, \\ g(t) - g(\tau) + b/2, & \tau \leq t \leq T. \end{cases}$$

Hence $f(t) \leq (m)\int_0^t f(s)dg(s)$, that is, (3.1.128) below satisfied with $\varepsilon = 0$, however, (3.1.129) holds only for $0 \leq t < \tau$.

The following result is about functions of bounded variation.

Theorem 3.1.18 (Schmaedeke-Sell [580]) *Let f and g be functions of bounded variation on $[0, T]$ and let $\varepsilon \geq 0$. Assume further that f and g are right continuous and that $f \geq 0$ and g is increasing. If for all $1 \leq t \leq T$,*

$$f(t) \leq \varepsilon + (m)\int_0^t f(s)dg(s), \tag{3.1.128}$$

then there exist constants $T' > 0$ and $K > 0$, depending on g but not on f, such that $0 < T' \leq T, 0 \leq K$ and for all $0 \leq t < T'$,

$$f(t) \leq K\varepsilon. \tag{3.1.129}$$

Furthermore, T' is the maximal in the sense that either $T' = T$ or $\Delta g(T') \geq 2$.

Proof Since f and g are functions of bounded variation, there exist constants B and M such that $|f(t)| \leq B$ on $[0, T]$ and $V[g; 0, T] = M$.

We shall assume that the hypotheses of the above theorem hold throughout. First define T' to be either the first point (in the natural ordering) for which $\Delta g(T') \geq 2$, or if no such point exists, set $T' = T$. In the interval $[0, T']$, list all the points $T_1, T_2, \cdots, T_n$ for which $1 \leq \Delta g(T_i) < 2, i = 1, 2, \cdots, n$. (Since g has bounded variation, there are at most a finite number of these.) □

Corollary 3.1.8 *Assume that the hypotheses of Theorem 3.1.18 are satisfied and* $V[g; 0, T] < 2$. *Then* $T' = T$ *and (3.1.129) holds on* $0 \leq t \leq T$.

Corollary 3.1.9 *Assume that the hypotheses of Theorem 3.1.18 are satisfied and that* f *is continuous. Then* $T' = T$ *and (3.1.129) holds on* $0 \leq t \leq T$.

Lemma 3.1.7 *If* $f(t) \leq K\varepsilon$ *for all* $0 \leq t \leq t_1$, *then there is a* $t_2 > t_1$ *and a constant* $K' > 0$ *such that* $f(t) \leq K'\varepsilon$ *for all* $0 \leq t \leq t_2$.

Proof Since g is right continuous, we can find a $t_2 > t_1$ such that $V[g; t_1, t_2] = p < 1$. Then for all $t_1 \leq t \leq t_2$, (3.1.128) reduces to

$$\begin{aligned} f(t) &\leq \varepsilon + (m)\int_0^{t_1} f(s)dg(s) + (m)\int_{t_1}^{t_2} f(s)dg(s) \\ &\leq \varepsilon + K\varepsilon M + B\rho. \end{aligned} \tag{3.1.130}$$

By replacing the bounded B with $(KM + 1)\varepsilon$ in (3.1.130), we get

$$f(t) \leq (KM + 1)\varepsilon(1 + \rho) + B\rho^2.$$

By reiterating this, we conclude that for all $t_1 \leq t \leq t_2$,

$$f(t) \leq (KM + 1)(1 - \rho)^{-1}\varepsilon, \tag{3.1.131}$$

which completes the proof. □

Lemma 3.1.8 *If* $f(t) \leq K\varepsilon$ *for all* $0 \leq t \leq t_1$ *and* $V[g; t_1, t] \leq \rho < 1$ *for all* $t_1 \leq t < t_2$, *then there is a constant* $K' > 0$ *such that* $f(t) \leq K'\varepsilon$ *for all* $0 \leq t < t_2$.

Proof This follows from the observation that the estimates in Lemma 3.1.7 are uniform for $t_1 \leq t < t_2$. Since g is right continuous, we should note that the hypotheses on the variation in Lemma 3.1.8 can be satisfied even if $V[g; t_1, t_2] \geq 1$. □

Lemma 3.1.9 *If* $f(t) \leq K\varepsilon$ *for all* $0 \leq t < t_1$ *and* $\Delta g(t_1) < 2$, *then there is a constant* $K' > 0$ *such that* $f(t) \leq K'\varepsilon$ *for all* $0 \leq t \leq t_1$.

Proof Indeed, we need to compute a bound for $f(t_1)$. If $0 \leq \tau < t_1$, then we have

$$\begin{aligned} f(t_1) &\leq \varepsilon + (m)\int_0^{t_1} f(s)dg(s) \\ &= \varepsilon + (m)\int_0^{\tau} f(s)dg(s) + (m)\int_{\tau}^{t_1} f(s)dg(s) \\ &\leq \varepsilon + K\varepsilon V[g; 0, \tau] + (m)\int_{\tau}^{t_1} f(s)dg(s). \end{aligned}$$

Now let $\tau \to t_1^-$. From (3.1.125) it follows that

$$f(t_1) \leq \varepsilon + KM\varepsilon + \frac{1}{2}[f(t_1) + f(t_1^-)]\Delta g(t_1). \tag{3.1.132}$$

Since $f(t_1^-) \leq K\varepsilon$, (3.1.128) reduces to

$$f(t_1) \leq (KM + K + 1)\varepsilon + \frac{1}{2}f(t_1)\Delta g(t_1),$$

or

$$f(t_1) \leq \frac{KM + K + 1}{1 - \frac{1}{2}\Delta g(t_1)}\varepsilon. \tag{3.1.133}$$

The proof is thus complete. □

Now consider the interval $[0, T_1]$. Since g has bounded variation, we can find a real number ρ and a partition of the interval $[0, T_1], 0 = t_0 < t_1 < \cdots < t_m < T_1$ so that $0 < \rho < 1$ and

$$V[g; t_{i-1}, t_i] \leq \rho,\ i = 1, 2, \cdots, m,$$
$$V[g; t_m, t] \leq \rho,\ t_m \leq t < T_1.$$

Now apply Lemma 3.1.7 successively to each of the interval $[t_{i-1}, t_i]$ and Lemma 3.1.8 to the interval $[t_m, T_1)$. We can conduce $f(t) \leq K\varepsilon$ on all $0 \leq t < T_1$. Now applying Lemma 3.1.9, we get $f(t) \leq K'\varepsilon$ on $0 \leq t \leq T_1$. By repeating this process for each of the intervals $[T_1, T_2], \cdots, [T_n, T')$, we complete the proof of the theorem. □

We should note that the final constant $K > 0$ is finite and it can be computed directly by applying (3.1.131) and (3.1.133) a finite number of times.

It should be clear from the above that if $T' = T$ and $\Delta g(T) < 2$, then the inequality (3.1.129) can be satisfied on the closed interval $0 \leq t \leq T$.

Proofs of Corollaries 3.1.8–3.1.9 The first corollary is obvious since for every $\tau, 0 \leq t \leq T$, we have $\Delta g(\tau) \leq V[g; 0, T]$. To prove the second corollary, we simply refer back to Lemma 3.1.9 and note that if f is continuous, we do not need the assumption that $\Delta g(t_1) < 2$, to get the desired conclusion. □

Remark 3.1.13 The topology that occurs when $\Delta g(T') \geq 2$ for the mean σ-integral does not arise for the Dushnik-integral. We can show that if, for all $1 \leq t \leq T$,

$$f(t) \leq \varepsilon + (b)\int_0^t f(s)dg(s),$$

where f, g and ε are as given in Theorem 3.1.18, then there is a constant $K > 0$ such that $f(t) \leq K\varepsilon,\ 0 \leq t \leq T$.

The argument for the Dushnik-integral differs in only one point from that given above, i.e., inequality (3.1.132) now takes the form

$$f(t_1) \leq \varepsilon + KM\varepsilon + f(t_1^-)\Delta g(t_1),$$

which hence implies that

$$f(t_1) \leq (KM + K\Delta g(t_1) + 1)\varepsilon, \tag{3.1.134}$$

regardless of the size of $\Delta g(t_1)$.

Remark 3.1.14 If f satisfies an inequality of the form, for all $1 \leq t \leq T$,

$$|f(t)| \leq \varepsilon + (m)\int_0^t |f(s)|k(s)dg(s), \tag{3.1.135}$$

then this reduces to case studied in Theorem 3.1.18 by replacing (3.1.128) by

$$|f(t)| \leq \varepsilon + (m)\int_0^t |f(s)|dv(s),$$

where $v(t) = (m)\int_0^t k(s)dg(s)$.

Remark 3.1.15 The failure of Gronwall's inequality when $\Delta g(T') \geq 2$ is related to the eigenvalue problem

$$\lambda f(t) = (m)\int_0^t f(s)dg(s),$$

that is, if we define the operator, for all $0 \leq t \leq T$,

$$U : f \to (m)\int_0^t f(s)dg(s),$$

then we can show that $\lambda > 0$ is an eigenvalue of U if and only if there is a $T', 0 < T' \leq T$, such that $\Delta u(T') = 2\lambda$. This subject was further studied in [579].

3.1.4 Linear One-Dimensional Gronwall-Bellman Integral Inequalities for Volterra-Stieltjes Integrals

We note that Hildebrandt [278] studied linear differentio-Stieltjes integral equations involving Wall's theory of harmonic matrices [635] and the concept of the Young integral [277, 676]. This type of integral allows us to integrate any function of bounded variation with respect to another and to distinguish between the value of

a function at some point and the right and the left-hand limit at this point (e.g., if we regard the integral as a function of the upper limit). Thus Hildebrandt [278] derived a necessary and sufficient condition for the existence and uniqueness to both homogeneous and non-homogeneous equations

$$\begin{cases} Y(x) &= Y_0 + \int_a^x dA(s)Y(s), \\ Y(x) &= U(x) + \int_a^x dA(s)Y(s) \end{cases}$$

where x varies in the closed interval $[a, b]$, Y and U are n-dimensional vector functions, A is an $n \times n$ matrix function, defined also on $[a, b]$.

In this section, we introduce the results due to Groh [238] which deals with the case of only one dimension. Clearly, in this case Hildebrandt's results have an especially simple and explicit representation. Using this, Groh [238] was able to solve a nonlinear differentio-Stieltjes integral equation

$$y(x) = y_0 + \int_a^x f(s, y(s))dm(s), \quad x \in [a, b] \tag{3.1.136}$$

with a continuous Lipschitzian function f and a function m, of bounded variation on $[a, b]$. Because of the discontinuities of m, the usual proof of existence and uniqueness for the classical explicit first-order differential equations via Banach's fixed point principle is not applicable; in general the corresponding operator is not contractive. Indeed, this problem is solved by introducing an appropriately weighted norm in generalization of the well-known very effective norm $\|f\| = \sup\{e^{-\alpha x}|f(x)|; x \in [a, b]\}$, first introduced by Morgenstrn [417], which allows us to obtain the global solution in the case $m(x) = x$, (see, e.g., [638]). Another approach to solve Eq. (3.1.136) was given in Das and Sharma [165] in which an assertion about local existence was proved. In [164], they investigated applications in deterministic control theory, see, e.g., Rishel [558].

MacNerney [367] extended the work of Wall [634, 635] about harmonic matrices to a more abstract setting by using of product integral methods. This approach allows also to establish a nonlinear integral operation [377]. We refer to Neuberger [428], Cox [158], Ingram [301], Helton [264], Herod [273], Bitzer [82], Lovelady [367], Helton [268], Hinton [284], Reneke [557] and Gibson [229] for further development in various directions. For Stieltjes integral equations based on various types of integrals, we consult MacNerney and Herod [273, 376], Helton and Stuck [270]. Using another integral concept, Kurzweil had developed in the fiftieth also a general theory of differential equations with possibly left continuous solutions, see [325, 326], also [581] and the further work of Schwabik. In this case, there are connections between Kurzweil's and the present approach, compared to the local existence theorem [326].

To prove the continuous dependence of the solution to (3.1.111) on the initial data, the classical Gronwall lemma [270], (see also Theorem 3.1.19), is not applicable. Groh [238] replaced the exponential function by a suitable discontinuous but “harmonic” [637] function and derived in this manner an appropriate generalized Gronwall inequality of the type described by Herod [274] for general linear Stieltjes integrals. As for other types of integrals, see also Hinton [284], Schmaedeke and Sell [579], Schwabik [581], Helton [264], Wright, Klasi and Kennebeck [652], and Kroll and Smith [322]. For the purely discontinuous case, we refer to Jones [304], Willett and Wong [648], and Chandra and Fleishman [129].

From a detailed analysis of the above linear and nonlinear equations (which allow right- and left-hand discontinuous), we point out that there are some defects: the existence and uniqueness of solutions depend on sometime troublesome conditions for the function m or the functions m and f, respectively. This gap will be omitted if we use another version of these equations. Thus we can solve uniquely the equation

$$y(x) = y_0 + \int_a^x y(s-0)dm(s), \quad \textit{for all } x \in [a,b],$$

without any conditions, and the existence and uniqueness of a solution to the nonlinear equation

$$y(x) = y_0 + \int_a^x f(s, y(s-0))dm(s), \quad \textit{for all } x \in [a,b],$$

requires only the usual Lipschitz condition on the function f. We also note that equations of this often type occur in the theory of stochastic equations, based on the integral calculations of Itô, see, e.g., Doléans-Dade [189], Gihman and Skorohod [246].

If we first use the linear equations only with right continuous functions instead of arbitrary functions of bounded variation, the situation is much simpler and some proofs are more elegant. For example, we may give a very short proof of Gronwall's lemma in this case (see, e.g., Theorem 3.1.19) which is due to Groh [238] and also the connections between the solutions of both versions of the homogeneous equation are more transparent.

Concerning mechanical interpretations of some Stieltjes integral equations, we refer to the monograph of Gantmacher and Kreǐn [223] and the nice appendix of the Russian translation of Atkinson's monograph, written by Kac and Kreǐn [306], and Langer [351] and Reid [553], Denny [178] and Hönig [289]. For classical nonlinear Volterra integral equations, see Miller [402], Mamedov and Aširov [380].

We shall only deal with real-valued bounded functions and especially with functions of bounded variation on the closed interval $[a,b]$. For such a function g of bounded variation and all points $x \in [a,b]$, we denote the differences $g(x) - g(x-0)$, $g(x+0) - g(x)$, and $g(x+0) - g(x-0)$ by $\Delta^- g(x)$, $\Delta^+ g(x)$, and

$\Delta_{-}^{+}g(x)$, respectively. We make the convention $g(a-0) = g(a)$ and $g(b+0) = g(b)$. Clearly, the Δ^{-}, Δ^{+} and Δ_{-}^{+} are linear operators. Furthermore, let $|g|(x)$ be the total variation of g on the segment $[a, x]$.

Finally, let m be a fixed function of bounded variation on $[a, b]$. Without loss of generality, we always assume $m(a) = 0$.

We assume additionally that the function m is non-decreasing on the interval $[a, b]$. Consequently, $1 + \Delta^{+}m(x) > 0$, but we suppose, in addition, that $1 - \Delta^{-}m(x) > 0$ for all $x \in [a, b]$. Then the equation

$$h(x) = 1 + \int_a^x h(s)dm(s), \quad \textit{for all } x \in [a, b], \tag{3.1.137}$$

has a unique positive solution. With the help of this function, we may formulate an appropriate analogue of Gronwall's lemma, (see also, Theorem 3.1.20), due to Groh [238].

Theorem 3.1.19 (Groh [238]) *Let c be a non-negative constant and y a function of bounded variation on* $[a, b]$ *satisfying for all* $x \in [a, b]$*,*

$$0 \le y(t) \le c + \int_a^x y(s)dm(s). \tag{3.1.138}$$

Then for all $x \in [a, b]$*,*

$$y(x) \le c \cdot h(x). \tag{3.1.139}$$

Proof Obviously, here the function h plays the role of the exponential function in the classical case. For the proof, we can use the same method as in the argument solving the homogeneous equation

$$y(t) = y_0 + \int_a^x y(s)dm(s), \qquad \textit{for all } x \in [a, b], \tag{3.1.140}$$

with respect to an arbitrary real initial value y_0.

After consideration of a continuous m, we have to deal with functions m, which have finitely many discontinuities. The general case can be treated by approximation. We omit this procedure, but we only give a very simple proof with respect to the special case of both right continuous functions y and m.

We simplify our considerations. To this end, we assume the right continuity of the function m on the interval $[a, b]$. Also we assume $1 - \Delta^{-}m(x) > 0,\ 1 + \Delta^{-}m(x) > 0$ for all $x \in [a, b]$. Especially, we are interested in an analogous form of the relation $e^x e^{-x} = 1$, where we interpret e^x and e^{-x} as solutions to the equations $y(x) =$

$1 + \int_0^x y(s)ds$ and $\overline{y}(x) = 1 - \int_0^x y(s)ds$, respectively. For this matter, we consider the equations

$$\begin{cases} y(x) = \eta + \displaystyle\int_a^x y(s)dm(s), & (3.1.141) \\ \overline{y}(x) = \eta - \displaystyle\int_a^x \overline{y}(s)dm(s), & (3.1.142) \\ w(x) = \eta + \displaystyle\int_a^x w(s-0)dm(s), & (3.1.143) \\ \overline{w}(x) = \eta - \displaystyle\int_a^x \overline{w}(s-0)dm(s), & (3.1.144) \end{cases}$$

with the common initial value $\eta \in \mathbb{R}$. Applying the above results, we obtain the (positive) solutions to these equations for all $x \in [a, b]$ in the following form:

$$\begin{cases} y(x) = \eta e^{m(x)} / \displaystyle\prod_{\tau \leq x} [1 - \Delta^- m(\tau)] e^{\Delta - m(\tau)}, & (3.1.145) \\ \overline{y}(x) = \eta e^{-m(x)} / \displaystyle\prod_{\tau \leq x} [1 + \Delta^- m(\tau)] e^{-\Delta - m(\tau)}, & (3.1.146) \\ w(x) = \eta e^{m(x)} \displaystyle\prod_{\tau \leq x} [1 + \Delta^- m(\tau)] e^{-\Delta - m(\tau)}, & (3.1.147) \\ \overline{w}(x) = \eta e^{-m(x)} \displaystyle\prod_{\tau \leq x} [1 - \Delta^- m(\tau)] e^{\Delta - m(\tau)}. & (3.1.148) \end{cases}$$

From this it follows that $y(x) = \overline{w}(x)^{-1}, w(x) = \overline{y}(x)^{-1}$ $(x \in [a, b])$. For that reason, Eqs. (3.1.141) and (3.1.144) as well as (3.1.143) and (3.1.142) are in a natural way adjoint to each other (compare with [278, Sect. 12, pp. 370–1]).

As a nice application we give as indicated previously a very simple proof of the Gronwall lemma above with respect to right continuous functions m and y (m non-decreasing).

Let h and $\overline{k}$ be the solutions to the equations

$$h(x) = 1 + \int_a^x h(s)dm(s) \quad (\textit{for all } x \in [a, b])$$

and

$$\overline{k}(x) = 1 - \int_a^x \overline{k}(s-0)dm(s) \quad (\textit{for all } x \in [a, b])$$

respectively. For a non-negative constant c and all $x \in [a, b]$, we assume that $0 \leq y(x) \leq c + \int_a^x y(s)dm(s)$. Setting $z(x) = \int_a^x y(s)dm(s)$, we have $y(x) - z(x) \leq c$. We

multiply by the integrating factor $\overline{k}(x-0)$ and integrate both sides, and obtain

$$\int_a^x y(s)\overline{k}(s-0)dm(s) - \int_a^x z(s)\overline{k}(s-0)dm(s) \leq c\int_a^x \overline{k}(s-0)dm(s),$$

where we have used the fact that m is non-decreasing. It thus follows

$$\int_a^x \overline{k}(s-0)dz(s) + \int_a^x z(s)d\overline{k}(s) \leq c(1-\overline{k}(x)).$$

In this case, the integration by parts is very simple (compare for example with [Sztz 20.9, p. 132][284]). Using $z(a)=0$, we have

$$\overline{k}(x)z(x) \leq c(1-k(x)), y(x)-c \leq z(x) \leq c(\overline{k}(x)^{-1}-1) = c(h(x)-1),$$

and, finally, the desired result

$$y(x) \leq c \cdot h(x) \quad (\textit{for all } x \in [a,b]).$$

With the help of solutions to (3.1.143) and (3.1.142) for $\eta = 1$, we can verify the version of Gronwall's lemma from above in the same fashion. □

The result of Hildebrandt [278] has the following form.

Lemma 3.1.10 ([278]) *Equation (3.1.140) has a unique solution if and only if the relation* $1-\Delta^- m(x) \neq 0$ *holds for all* $x \in [a,b]$. *Then the solution can be expressed by, for all* $x \in [a,b]$,

$$y(x) = y_0 e^{m(x)} \prod_{\tau<x}[1+\Delta^+ m(\tau)]e^{-\Delta^+ m(\tau)} / \prod_{\tau\leq x}[1-\Delta^- m(\tau)]e^{\Delta - m(\tau)}. \tag{3.1.149}$$

Clearly, in case of $1-\Delta^- m(x)$, $1+\Delta^+ m(x) > 0$, for all $x \in [a,b]$, the solution y is (strictly) positive on the whole interval $[a,b]$ whenever $y_0 > 0$. Because this assertion is one of the crucial points of the following considerations, we repeat here some ideas of Hildebrandt's proof [278]. First, let us consider Eq. (3.1.117) with a continuous weight function m. Employing the formula

$$\int_c^d m^n(s)dm(s) = (m^{n+1}(d) - m^{n+1}(c))/n+1,$$

$$(a \leq c < d \leq b; \; n = 0, 1, 2, \ldots), \tag{3.1.150}$$

we obtain, by a term by term integration, that $x \mapsto y_0 e^{m(x)}$, for all $x \in [a,b]$, is the solution to (3.1.140), (compare with [634, p. 74]). Now we consider the case of only

one discontinuity of m at $a < x_1 < b$. On the interval $[a, x_1)$, we have $y(x) = y_0 e^{m(x)}$ again, while the point x_1, we have

$$\begin{aligned} y(t) &= y_0 + \int_a^x y(s)dm(s) \\ &= y_0 + \int_a^{x_1-0} y(s)dm(s) + \int_{x_1-0}^{x_1} y(s)dm(s) \\ &= y(x_1 - 0) + y(x_1)\Delta^- m(x_1). \end{aligned}$$

Thus, it follows that the value $y(x_1)$ is determined uniquely if and only if the term $1 - \Delta^- m(x_1)$ is non-vanishing. In this case,

$$\begin{aligned} y(x_1) &= y_0 e^{m(x_1-0)}/[1 - \Delta^- m(x_1)] \\ &= y_0 e^{m(x_1)}/[1 - \Delta^- m(x_1)]e^{\Delta - m(x_1)}. \end{aligned}$$

Furthermore,

$$\begin{aligned} y(x_1 + 0) &= y_0 + \int_a^{x_1} y(s)dm(s) + y(x_1)\Delta^+ m(x_1) \\ &= [1 + \Delta^+ m(x_1)]y(x_1) \\ &= y_0 e^{m(x_1)}/[1 + \Delta^+ m(x_1)]e^{\Delta + m(x_1)}/[1 - \Delta^- m(x_1)]e^{\Delta - m(x_1)} \end{aligned}$$

without any new condition. Thus for $x_1 < x \le b$, we have

$$y(x) = y_0(x_1 + 0) + \int_{x_1+0}^x y(s)dm(s).$$

Noting that the continuity of m in $(x_1, b]$, we can derive

$$\begin{aligned} y(x) &= y(x_1 + 0)e^{m(x) - m(x_1+0)} \\ &= y_0 e^{m(x_1)}/[1 + \Delta^+ m(x_1)]e^{\Delta + m(x_1)}/[1 - \Delta^- m(x_1)]e^{\Delta - m(x_1)}. \end{aligned}$$

In the case of finitely many discontinuities of m, the assertion for (3.1.140) can be proved step by step. In general, we approximate the function m by suitable functions m_k ($k = 1, 2, \ldots$) with the same continuous part as m and with those discontinuities of m, which are greater than $1/k$. Now we solve the equations

$$y_k(x) = y_0 + \int_a^x y_k(s)dm_k(s), \quad \textit{for all } x \in [a, b] \tag{3.1.151}$$

and show that the limit of the sequence $\{y_k\}$ ($k = 1, 2, \ldots$) is a solution to Eq. (3.1.140).

The uniqueness of this solution can be shown with the step of some corresponding non-homogeneous equations, which have weight functions with only finitely many discontinuities. But for this as well as for the proof of the following general assertion about the non-homogeneous equation

$$y(x) = u(x) + \int_a^x y(s)dm(s), \quad \textit{for all } x \in [a, b] \tag{3.1.152}$$

with arbitrary functions m and u of bounded variation, we refer to Hilderandt [278]. For convenience, we assume that $1 - \Delta^- m(x) > 0$, $1 + \Delta^+ m(x) > 0$, for all $x \in [a, b]$. Then the equation

$$h(x) = 1 + \int_a^x h(s)dm(s), \quad \text{for all } x \in [a, b], \tag{3.1.153}$$

has a unique positive solution h and we can formulate the following assertion: Eq. (3.1.121) has a unique solution y, defined by the formula, for all $x \in [a, b]$,

$$y(x) = h(x)[u(a) + \int_a^x h(s)^{-1}du(s) + \sum_{a<\tau\leq x} h(\tau - 0)^{-1}\Delta^- m(\tau)\Delta^- u(\tau)$$
$$- \sum_{a\leq\tau<x} h(\tau + 0)^{-1}\Delta^+ m(\tau)\Delta^+ u(\tau)].$$

If we interpret the interval $[a, b]$ as the time scale of some system, which is described, for instance, by the homogeneous equation (3.1.140), because of the relation $y(x) = y(x - 0) + y(x)\Delta^- m(x)$, we can say that this system is anticipative. At the points x with $\Delta^- m(x) \neq 0$, we need for the further evolution of y information about the near future.

After consideration of a continuous m, we shall deal with function m, which have finitely many discontinuities. The general case can be treated by approximation. We omit this procedure, but we refer to Herod [274] in a more general setting. Next, we only give a very simple proof of Theorem 3.1.19 with respect to the special case of both right continuous functions y and m.

Proof of Theorem 3.1.19 Let h and $\overline{k}$ be the solutions to the following two equations

$$h(x) = 1 + \int_a^x h(s)dm(s), \quad \textit{for all } x \in [a, b]$$

and

$$\overline{k}(x) = 1 - \int_a^x \overline{k}(s - 0)dm(s), \quad \textit{for all } x \in [a, b]$$

respectively. For a non-negative constant c and all $x \in [a,b]$, we assume that $0 \leq y(x) \leq c + \int_a^x y(s)dm(s)$, i.e., (3.1.138) holds. Setting $z(x) = \int_a^x y(s)dm(s)$, we have $y(x) - z(x) \leq c$. Multiplying by the integrating factor $\overline{k}(x-0)$ and integrating both sides, we obtain

$$\int_a^x y(s)\overline{k}(s-0)dm(s) - \int_a^x z(s)\overline{k}(s-0)dm(s) \leq c\int_a^x \overline{k}(s-0)dm(s),$$

where we have used the fact that m is non-decreasing. Thus it follows that

$$\int_a^x \overline{k}(s-0)dz(s) + \int_a^x z(s)d\overline{k}(s) \leq c(1-\overline{k}(x)).$$

In this case, the integration by parts is very simple (compare, for example, with [283]). Using $z(a) = 0$, we have

$$\overline{k}(x)z(x) \leq c(1-k(x)), y(x) - c \leq z(x) \leq c(\overline{k}(x)^{-1} - 1) = c(h(x) - 1),$$

which implies the desired result (3.1.139).

With the help of the solutions to (3.1.156) and (3.1.166) for $\eta = 1$ (see below), we can complete the proof. □

Let m be a function of bounded variation again. The function f, which is defined and continuous on the set $[a,b] \times \mathbb{R}$, satisfies a uniform Lipschitz condition: for all$x \in [a,b]$, y_1, $y_2 \in \mathbb{R}$,

$$|f(x,y_1) - f(x,y_2)| \leq L|y_1 - y_2|,$$

with some positive constant L.

Theorem 3.1.20 (Groh [238]) *Assume that there exists a positive constant α such that for all points $x \in [a,b]$ with $\Delta^- m(x) \neq 0$,*

$$|\Delta^- m(x)|^{-1} > \alpha > L. \tag{3.1.154}$$

Then for every real y_0 and for all $x \in [a,b]$, the equation

$$y(t) = y_0 + \int_a^x f(s,y(s))dm(s), \tag{3.1.155}$$

has a unique solution y, which depends continuously on the initial data $y(a) = y_0$.

Proof We fix first the solution h_α to the equation

$$h_\alpha(x) = 1 + \alpha\int_a^x h_\alpha(s)d|m|(s), \quad \textit{for all } x \in [a,b],$$

which is positive due to condition (3.1.154) and the relation $\Delta^{-}|m|(x) = |\Delta^{-}m(x)|$, for all $x \in [a, b]$. Furthermore, it is clear that we need to seek a solution of (3.1.155) in the set of all functions g which satisfy the conditions, for all points $x \in [a, b]$,

$$m(x) = m(x-0) \Rightarrow g(x) = g(x-0),$$
$$m(x) = m(x+0) \Rightarrow g(x) = g(x+0).$$

This means that every point of left continuity of m is also one of the function g and similarly with respect to right continuity. To define a space of such functions, let us introduce a new metric ρ_m on the set $[a, b]$ by, for all$x,\ y \in [a, b]$,

$$\rho_m(x, y) = |x - y| + |m(x) - m(y)|.$$

We should note that the matric space $([a, b],\ \rho_m)$ is not complete and consequently not compact. Let us consider, for example, a point $x \in [a, b]$ with $\Delta^{+}m(x) > 0,\ \Delta^{-}m(x) = 0$, and a sequence $x_n \to 0,\ x_n > x$ which converges in the usual sense from the right to the point x. Then this sequence is not convergent in the space $([a, b],\ \rho_m)$ because there exist neighborhoods $(x - \delta,\ x]$ $(0 < \delta \leq \Delta^{+}m(x))$ not containing any point of the sequence $\{x_n\}$.

We denote by $C_m[a, b]$ the space of all real-valued bounded functions which are continuous with respect to the topology obtained by the metric ρ_m. Because of the completeness of the real axis $\mathbb{R}$, the space $C_m[a,\ b]$ with the supremum norm is complete itself (cf. [422]). But for considerations, we have to introduce another norm, which is defined for every $y \in C_m[a, b]$ by

$$\|y\|_{\alpha} = \sup\{h_{\alpha}(x)^{-1}|y(x)|;\ x \in [a,\ b]\}.$$

Because of the positivity and finiteness of h_{α}, this norm is equivalent to the supremum norm (see, e.g., [638]). Consequently, $C_m[a, b]$ with the norm $\|\cdot\|_{\alpha}$ is also a Banach space and we can apply Banach's fixed point principle. To this end, we show that the operator T, defined for every $y \in C_m[a,\ b]$ by

$$(Ty)(x) = y_0 + \int_a^x f(s, y(s))dm(s), \quad \textit{for all } x \in [a, b],$$

is contractive on the space $(C_m[a,\ b],\ \|\cdot\|_{\alpha})$.

Indeed, let y and z be arbitrary functions in $C_m[a,\ b]$. Then, because of the Lipschitz condition and the definition of h_{α} for every $x \in [a, b]$, we have

$$\begin{aligned}|(Ty)(x) - (Tz)(x)| &= \left|\int_a^x [f(s, y(s)) - f(s, z(s))]dm(s)\right| \\ &\leq \int_a^x |f(s, y(s)) - f(s, z(s))|d|m|(s)\end{aligned}$$

$$\begin{aligned}&\le L\int_a^x |y(s)-z(s)|d|m|(s)\\&\le L\|y-z\|_\alpha \int_a^x h_\alpha(s)d|m|(s)\\&= \alpha^{-1}L\|y-z\|_\alpha(h_\alpha(x)-1)\\&\le \alpha^{-1}L\|y-z\|_\alpha h_\alpha(x).\end{aligned}$$

Consequently, we have

$$h_\alpha^{-1}(x)|(Ty)(x)-(Tz)(x)| \le \alpha^{-1}L\|y-z\|_\alpha,$$

which means $\|Ty-Tz\|_\alpha \le \alpha^{-1}L\|y-z\|_\alpha$. Because of (3.1.154), we have $\alpha^{-1}L<1$ and the operator T is strict contractive. It follows that there exists a unique solution to Eq. (3.1.155).

Finally, let y_0 and z_0 be two initial values for Eq. (3.1.155) and y and z the corresponding solutions. By the Lipschitz condition, then for every $x \in [a,b]$, we have

$$|y(x)-z(x)| \le |y_0-z_0| + L\int_a^x |y(s)-z(s)|d|m|(s).$$

The solution of Eq. (3.1.155) is of bounded variation and we can apply the Gronwall inequality Theorem 3.1.19 to obtain for all $x \in [a,b]$,

$$|y(x)-z(x)| \le |y_0-z_0|h_L(x),$$

where the function h_L is the solution to the equation

$$h_L(x) = 1 + L\int_a^x h_L(s)d|m|(s), \quad \textit{for all } x \in [a,b].$$

The function h_L is bounded and therefore the solution y to Eq. (3.1.155) depends continuously on its initial value y_0. □

In order to give a very simple proof of the Gronwall inequality Theorem 3.1.19, we need to study other types of equations.

$$\begin{cases} w(x) = w_0 + \int_a^x w(s-0)dm(s), \quad \textit{for all } x \in [a,b], & (3.1.156)\\ w(x) = u(x) + \int_a^x w(s-0)dm(s), \quad \textit{for all } x \in [a,b], & (3.1.157)\\ w(x) = w_0 + \int_a^x f(s,w(s-0))dm(s), \quad \textit{for all } x \in [a,b]. & (3.1.158)\end{cases}$$

where the functions m and u are of bounded variation on $[a,b]$, w_0 is a real and the function f is continuous and satisfies the above Lipschitz condition with the same Lipschitz constant L. Equation (3.1.156) has a unique solution w, which can be expressed by the formula

$$w(x) = w_0 e^{m(x)}[1 + \Delta^- m(x)]e^{-\Delta^- m(x)} \prod_{\tau<x}[1 + \Delta_-^+ m(\tau)]e^{\Delta_-^+ m(\tau)}, \quad \textit{for all } x \in [a,b]. \tag{3.1.159}$$

The convergence of the (possibly) infinite products follows from the absolute convergence of the series $\sum_{\tau<x} \Delta_-^+ m(\tau)$ (c.f. [537] or [315]). In the sequel, we consider only the essential difference between Eqs. (3.1.156) and (3.1.140) which occurs naturally at points x with $\Delta^- m(x) \neq 0$. At such points, we have

$$w(x) = [1 + \Delta^- m(x)]w(x-0), \quad w(x+0) = [1 + \Delta_-^+ m(x)]w(x-0). \tag{3.1.160}$$

But to solve uniquely this equation (3.1.160), we need no further condition, contrary to the case of Eq. (3.1.140) in which the equations

$$y(x) = [1 - \Delta^- m(x)]^{-1} y(x-0), \quad y(x+0) = [1 + \Delta^+ m(x)]y(x)$$

must hold. To simplify the discuss, we consider the solution of (3.1.157) in the case where

$$1 + \Delta^- m(x),\ 1 + \Delta^+ m(x) > 0, \quad \text{for all } x \in [a,b].$$

In this case, the homogeneous equation

$$k(x) = 1 + \int_a^x k(s-0)dm(s), \quad \text{for all } x \in [a,b], \tag{3.1.161}$$

has a positive solution k and we can write the unique solution to the non-homogeneous equation (3.1.157) in the form

$$\begin{aligned} w(x) = k(x)[u(a) &+ \int_a^x k(s-0)^{-1} du(s) - k(a+0)^{-1}\Delta^+ m(a)\Delta^+ u(a) \\ &- \sum_{a<\tau<x} k(\tau+0)^{-1}\Delta_-^+ m(\tau)\Delta_-^+ u(\tau)] - \Delta^- m(x)\Delta^- u(x), \quad \textit{for all } x \in [a,b]. \end{aligned}$$

The proof of this assertion is similar to the argument in [278], which verifies the solution of (3.1.152), using the Dirichlet formula.

To show the continuous dependence of solutions to Eq. (3.1.158) on the initial value w_0, we need the following version of Gronwall's lemma.

Theorem 3.1.21 (Groh [238]) *Let $c \geq 0$ and w be a function of bounded variation on $[a, b]$ satisfying the inequality, for all $x \in [a, b]$,*

$$0 \leq w(x) \leq c + \int_a^x w(s-0)d|m|(s). \tag{3.1.162}$$

Then we have for all $x \in [a, b]$,

$$w(x) \leq c \cdot k(x), \tag{3.1.163}$$

where k is the positive solution to Eq. (3.1.164) with $\alpha = 1$.

Proof The proof is similar to that of Theorem 3.1.20. □

Now we turn to the last Eq. (3.1.158) and have the flowing conclusion.

Theorem 3.1.22 (Groh [238]) *The nonlinear equation (3.1.158) has a unique solution w, which depends continuously on the initial value w_0.*

Proof We may choose here on the set $C_m[a, b]$ with the weighted norm

$$\|w\|_\alpha = \sup\left\{k_\alpha(x)^{-1}|w(x)| : \ x \in [a, b]\right\},$$

where the weighted function k_α is the positive and bounded solution to the equation

$$k_\alpha(x) = 1 + \alpha \int_a^x k_\alpha(s-0)d|m|(s), \quad \textit{for all } x \in [a, b] \tag{3.1.164}$$

with $\alpha = 2L$. We shall show that the operator U, defined by,

$$(Uw)(x) = w_0 + \int_a^x f(s, w(s-0))dm(s), \quad \textit{for all } w \in C_m[a, b], \ x \in [a, b],$$

is contractive. In fact, for all $w, \ v \in C_m[a, b]$ and all $x \in [a, b]$, we have

$$\begin{aligned}
|(Uw)(x) - (Uw)(x)| &\leq \int_a^x |f(s, w(s-0)) - f(s, v(s-0))|d|m|(s) \\
&\leq L\|w - v\|_\alpha \int_a^x k_\alpha(s-0)d|m|(s) \\
&\leq 2^{-1}\|w - v\|_\alpha k_\alpha(x),
\end{aligned}$$

which implies $\|Uw - Uv\|_\alpha \leq 2^{-1}\|w - v\|_\alpha$.

We consider the case of right continuous solutions and hence we assume the right continuity of the function m on the interval $[a, b]$. Also we assume $1 - \Delta^- m(x) > 0$, $1 + \Delta^- m(x) > 0$ for all $x \in [a, b]$. Especially, we are interested in an analogous form of the relation $e^x e^{-x} = 1$, where we interpret e^x and e^{-x} as solutions to the

equations $y(x) = 1 + \int_0^x y(s)ds$ and $\overline{y}(x) = 1 - \int_0^x \overline{y}(s)ds$, respectively. For this purpose, we may consider the equations

$$\begin{cases} y(x) = \eta + \displaystyle\int_a^x y(s)dm(s), & (3.1.165) \\ \overline{y}(x) = \eta - \displaystyle\int_a^x \overline{y}(s)dm(s), & (3.1.166) \\ w(x) = \eta + \displaystyle\int_a^x w(s-0)dm(s), & (3.1.167) \\ \overline{w}(x) = \eta - \displaystyle\int_a^x \overline{w}(s-0)dm(s), & (3.1.168) \end{cases}$$

with the common initial value $\eta \in \mathbb{R}$. Applying the results on Eqs. (3.1.140) and (3.1.156)–(3.1.161), we obtain the positive solutions to these equations for all $x \in [a, b]$ in the following form:

$$\begin{cases} y(x) = \eta e^{m(x)} \displaystyle\prod_{\tau \le x} [1 - \Delta^- m(\tau)] e^{\Delta - m(\tau)}, \\ \overline{y}(x) = \eta e^{-m(x)} \displaystyle\prod_{\tau \le x} [1 + \Delta^- m(\tau)] e^{-\Delta - m(\tau)}, \\ w(x) = \eta e^{m(x)} \displaystyle\prod_{\tau \le x} [1 + \Delta^- m(\tau)] e^{-\Delta - m(\tau)}, \\ \overline{w}(x) = \eta e^{-m(x)} \displaystyle\prod_{\tau \le x} [1 - \Delta^- m(\tau)] e^{\Delta - m(\tau)}, \end{cases}$$

which thus implies that $y(x) = \overline{w}(x)^{-1}$, for all $x \in [a, b]$. For that reason, Eqs. (3.1.140) and (3.1.168) as well as (3.1.166) and (3.1.167) are in a natural way adjoint to each other (cf. [278]).

In the same manner as the proof of Theorem 3.1.19 with respect to right continuous functions m and y (m non-decreasing), we can complete the proof. $\square$

3.1.5 Linear One-Dimensional Gronwall-Bellman Integral Inequalities for Distributions

In this section, we introduce some results, due to Rao [548], on linear one-dimensional integral inequalities of Gronwall-Bellman type for distributions. We need some lemmas.

Lemma 3.1.11 *Let f and g be two real-valued functions on the real line $\mathbb{R}$ such that both are of bounded variation on every compact subinterval of $\mathbb{R}$. Then fg defines*

a distribution, and the derivative of fg in the sense of distributions is equal to the locally summable function $(fg)'$ given by, for almost all x,

$$f'(x)g(x) + f(x)g'(x) \tag{3.1.169}$$

that is,

$$D(fg) = (Df)g + f(Dg), \tag{3.1.170}$$

where Df and Dg denote derivatives of the functions f and g, respectively, in the sense of distributions.

Proof If T is a distribution on $\mathbb{R}$ and φ is a C^∞ function with compact support on $\mathbb{R}$, by $\langle T, \varphi\rangle$ we denote the action of T on φ.

Since f and g are functions of bounded variation on every compact subinterval of $\mathbb{R}$, so is fg. Furthermore, fg is measurable and bounded in every compact interval and, hence, is locally integrable. Thus fg defines a distribution on $\mathbb{R}$.

Let $D(fg)$ be the derivative of fg in the sense of distributions. Let φ be a C^∞ function with compact support contained in an interval $[a, b]$. Then, by definition

$$\langle D(fg), \varphi\rangle = -\langle fg, \varphi'\rangle = -\int_{-\infty}^{+\infty} (fg)(x)\varphi'(x)dx = -\int_a^b (fg)(x)\varphi'(x)dx. \tag{3.1.171}$$

Now for almost all x,

$$(fg\varphi)'(x) = (fg)(x)\varphi'(x) + \Big(f'(x)g(x) + f(x)g'(x)\Big)\varphi(x). \tag{3.1.172}$$

Hence

$$-\int_a^b (fg)(x)\varphi'(x)dx = \int_a^b (f'(x)g(x) + f(x)g'(x))\varphi(x)dx - \int_a^b (fg\varphi)'(x)dx. \tag{3.1.173}$$

Since $\varphi(b) = \varphi(a) = 0$ and noting that the support of φ is contained in $[a, b]$, we have

$$\int_a^b (fg\varphi)'(x)dx = (fg\varphi)(b) - (fg\varphi)(a) = 0. \tag{3.1.174}$$

Hence

$$-\int_a^b (fg)(x)\varphi'(x)dx = \int_{-\infty}^{+\infty} (f'(x)g(x) + f(x)g'(x))\varphi(x)dx,$$

i.e.,

$$\langle D(fg), \varphi\rangle = \langle f'g + fg', \varphi\rangle.$$

Since the function $f'(x)g(x) + f(x)g'(x)$ is a locally integrable function, it defines a distribution and its action on a C^{∞} function φ with compact support is given by

$$\int_{-\infty}^{+\infty} (f'(x)g(x) + f(x)g'(x))\varphi(x)dx.$$

Hence

$$D(fg) = f'g + fg' = (Df)g + f(Dg)$$

in the sense of distributions. This completes the proof. □

The next result is due to Rao [547].

Theorem 3.1.23 (Rao [547]) *Let $y(t)$ and $u(t)$ be non-negative functions of bounded variation with $u(t)$ increasing and let $K(t)$ be a non-negative function integrable with respect to $u(t)$ on $0 \le t \le T$, satisfying the inequality for all $0 \le t \le T$,*

$$y(t) \le C + \int_0^t K(s)y(s)du(s), \tag{3.1.175}$$

where $C \ge 0$ is a constant. Then for all $0 \le t \le T$,

$$y(t) \le C\Big[1 + \int_0^t K(s)\exp\Big(\int_s^t K(\eta)du(\eta)\Big)du(s)\Big]. \tag{3.1.176}$$

Proof Let $x(t) = \int_0^t K(s)y(s)du(s)$. Clearly, $x(t)$ is a function of bounded variation and from (3.1.175) it follows

$$y(t) \le C + x(t), \qquad Dx = K(t)y(t)Du.$$

Hence

$$Dx \le K(t)(C + x(t))Du = K(t)x(t)Du + CK(t)Du,$$

i.e.,

$$Dx - K(t)x(t)Du \le CK(t)Du. \tag{3.1.177}$$

Multiplying both sides of (3.1.177) by $\exp(-\int_0^t K(\eta)du(\eta))$ and using Lemma 3.1.11, we obtain

$$D\Big[\Big(\exp\Big(-\int_0^t K(\eta)du(\eta)\Big)\Big)x\Big] \le CK(t)\Big[\exp\Big(-\int_0^t K(\eta)du(\eta)\Big)\Big]Du.$$

Integrating the above inequality with respect to t between 0 and t, we get

$$\Big[\exp\Big(-\int_0^t K(\eta)du(\eta)\Big)\Big]x(t) - x(0) \le C\int_0^t K(s)\Big[\exp\Big(-\int_0^s K(\eta)du(\eta)\Big)\Big]du(s),$$

i.e.,

$$x(t) \le C\int_0^t K(s)\Big[\exp\Big(\int_s^t K(\eta)du(\eta)\Big)\Big]du(s),$$

since $x(0) = 0$.

Using the fact that $y(t) \le C + x(t)$, we conclude that for all $t \in [0, T]$,

$$y(t) \le C\Big[1 + \int_0^t K(s)\Big[\exp\Big(\int_s^t K(\eta)du(\eta)\Big)\Big]du(s)\Big],$$

which completes the proof. □

Theorem 3.1.24 (Rao [547]) *Let $y(t), f(t)$ and $u(t)$ be functions of bounded variation with $u(t)$ increasing and let $K(t)$ be a non-negative function integrable with respect to $u(t)$ on $0 \le t \le T$. Furthermore, if the following inequality holds for all $t \in [0, T]$,*

$$y(t) \le f(t) + \int_0^t K(s)y(s)du(s), \tag{3.1.178}$$

then for all $t \in [0, T]$,

$$y(t) \le f(t) + \int_0^t K(s)f(s)\Big[\exp\Big(\int_s^t K(\eta)du(\eta)\Big)\Big]du(s). \tag{3.1.179}$$

Proof Let

$$x(t) = \int_0^t K(s)y(s)du(s), \quad x(0) = 0. \tag{3.1.180}$$

In view of (3.1.178), we have

$$y(t) \le f(t) + x(t). \tag{3.1.181}$$

From (3.1.179) it follows

$$D(x) = K(t)y(t)Du \le K(t)(f(t) + x(t))Du,$$

which gives,

$$Dx - K(t)x(t)Du \leq K(t)f(t)Du.$$

Multiplying both sides of the above inequality by $\exp\left[-\int_0^t K(\eta)du(\eta)\right]$, integrating with respect to t between 0 and t, and using (3.1.181), we may obtain the required result (3.1.179). □

Now we suppose that u is absolutely continuous on $[0, T]$, where $u'(t)$, which exists a.e., is non-negative or in particular, $u(t) \equiv t$. Then we have the following remarks (see also Rao [547]).

Remark 3.1.16 If the functions y, f and g are piecewise continuous on $0 \leq t \leq T$ and g is non-negative, then Theorem 3.1.24 reduces to a theorem of [305].

Remark 3.1.17 In Theorem 3.1.23, setting $K(t) \equiv L > 0$, a constant, $f(t)$ and $y(t)$ are non-negative integrable functions on $0 \leq t \leq T$, we see that Lemma 1 of [45] follows. In addition to the above, if $f(t)$ has bounded variation on $[0, T]$, then Theorem 3.1.24 reduces to Lemma 2 of [45].

Remark 3.1.18 The choice that $K(t)$ and $y(t)$ are non-negative continuous functions on $0 \leq t \leq T$ and $f(t)$ is any continuous function on $0 \leq t \leq T$ reduces Theorem 3.1.24 to Corollary 1.9.1 of [338], which is a generalized version of the celebrated integral inequality of Gronwall-Bellman type.

Next, we shall introduce the integral inequalities of Gronwall-Bellman type with multi-distributions due to Guan and Liu [243]. The integral inequalities in the sense of Lebesgue-Stieltjes for the functions of bounded variations enable us to study the properties of measure differential large scale systems with multi-distributional derivatives (cf. [233]).

We always assume that the function $u_j(t)$ is right continuous at $t = 0, j = 1, \ldots, m$. Let $BV(I)$ denote the set of all functions of bounded variation defined on $I \subseteq \mathbb{R}$ and taking values in $\mathbb{R}$.

Theorem 3.1.25 (Guan-Liu [243]) *Suppose that, for $j = 1, \ldots, m$ and for all $t \in [0, T]$,*

(i) $g_j(t) \geq 0, y(t) \geq 0$, *and* $g_j(t), y(t), f(t) \in BV([0, T])$;
(ii) $u_j(t)$ *are non-decreasing in t;*
(iii) $h_j(t)$ *are non-negative and integrable with respect to* $u_j(t)$, *and if for all* $t \in [0, T]$,

$$y(t) \leq f(t) + \sum_{j=1}^{m} g_j(t) \int_0^t h_j(s)y(s)du_j(s), \tag{3.1.182}$$

then for all $t \in [0, T]$,

$$y(t) \leq A_m(f) + A_m(g_m) \int_0^t h_m(s) A_m(f) \times \exp\left(\int_s^t h_m(\xi) A_m(g_m) du_m(\xi)\right) du_m(s), \tag{3.1.183}$$

where $A_k(v)$ *is defined inductively as follows:*

$$\begin{cases} A_1(v) = v \\ A_{k+1}(v) = A_k(v) + A_k(g_k) \int_0^t h_k(s) A_k(v) \exp\left(\int_s^t h_k(\xi) A_k(g_k) du_k(\xi)\right) du_k(s), \\ \qquad k = 1, \ldots, m-1. \end{cases} \tag{3.1.184}$$

Proof Let for all $t \in [0, T]$,

$$x_i(t) = \int_0^t h_i(s) y(s) du_i(s), \qquad i = 1, \ldots, m. \tag{3.1.185}$$

It is easy to see that $x_i(t)$ are functions of bounded variation, $x_i(0) = 0$. Hence, in view of (3.1.182), we have

$$y(t) \leq f(t) + \sum_{j=1}^{m} g_j(t) x_j(t) \tag{3.1.186}$$

and

$$Dx_i = h_i(t) y(t) Du_i. \tag{3.1.187}$$

When $i = 1$, from (3.1.186) and (3.1.187) we derive

$$Dx_1 = h_1(t) y(t) Du_1 \leq h_1(t) \left[f(t) + \sum_{j=1}^{m} g_j(t) x_j(t) \right] Du_1,$$

i.e.,

$$Dx_1 - h_1(t) g_1(t) x_1(t) Du_1 \leq h_1(t) \left[f(t) + \sum_{j=2}^{m} g_j(t) x_j(t) \right] Du_1. \tag{3.1.188}$$

Multiplying both sides of (3.1.188) by $\exp(-\int_0^t h_1(s)du_1(s))$ and using Lemma 3.1.11, we obtain

$$D\left[x_1 \exp\left(-\int_0^t h_1(s)g_1(s)du_1(s)\right)\right]$$
$$\leq h_1(t)\left[f(t)+\sum_{j=2}^{m} g_j(t)x_j(t)\right]\exp\left(-\int_0^t h_1(s)g_1(s)du_1(s)\right)Du_1.$$

Integrating with respect to t between 0 and t, we get

$$\exp\left(-\int_0^t h_1(s)g_1(s)du_1(s)\right)x_1(t)-x_1(0)$$
$$\leq \int_0^t h_1(s)\left[f(s)+\sum_{j=2}^{m} g_j(s)x_j(s)\right]\times \exp\left(-\int_0^s h_1(\xi)g_1(\xi)du_1(\xi)\right)du_1(s),$$

which yields

$$x_1(t)\leq \int_0^t h_1(s)\left[f(s)+\sum_{j=2}^{m} g_j(s)x_j(s)\right]\exp\left(\int_s^t h_1(\xi)g_1(\xi)du_1(\xi)\right)du_1(s). \tag{3.1.189}$$

Since $x_j(t)$ are non-decreasing, from (3.1.186) we derive at

$$\begin{aligned} y(t) &\leq f(t)+g_1(t)\left[f(s)+\sum_{j=2}^{m} g_j(s)x_j(s)\right] \\ &\quad\times \exp\left(-\int_s^t h_1(\xi)g_1(\xi)du_1(\xi)\right)du_1(s)+\sum_{j=2}^{m} g_j(t)x_j(t) \\ &\leq f(t)+g_1(t)\int_0^t h_1(s)f(s)\exp\left(\int_s^t h_1(\xi)g_1(\xi)du_1(\xi)\right)du_1(s) \\ &\quad+\sum_{j=2}^{m}[g_j(t)+g_1(t)\int_0^t h_1(s)g_j(s) \\ &\quad\times\left(\int_s^t h_1(\xi)g_1(\xi)du_1(\xi)\right]du_1(s)]x_j(t) \\ &= A_2(f)+\sum_{j=2}^{m} A_2(g_j)x_j(t), \end{aligned} \tag{3.1.190}$$

where $A_2(f)$ and $A_2(g_j)$ are defined as (3.1.184).

When $i = 2$, by (3.1.187) and (3.1.190), we get

$$Dx_2 = h_2(t)y(t)Du_2 \le h_2(t)\left[A_2(f) + \sum_{j=2}^{m} A_2(g_j)x_j(t)\right]Du_2$$

i.e.,

$$Dx_2 - h_2(t)A_2(g_2)x_2(t)Du_2 \le h_2(t)\left[A_2(f) + \sum_{j=3}^{m} A_2(g_j)x_j(t)\right]Du_2.$$

Multiplying both sides by $\exp\Big(-\int_0^t h_2(s)A_2(g_2)du_2(s)\Big)$ and reckoning similarly from (3.1.188) to (3.1.189), we have

$$x_2(t) \le \int_0^t h_2(s)\left[A_2(f) + \sum_{j=3}^{m} A_2(g_j)x_j(s)\right]\left(\int_s^t h_2(\xi)A_2(g_2)du_2(\xi)\right)du_2(s).$$

Using (3.1.190) and noting that $x_j(t), (j = 1, 2, \cdots, m)$ are non-decreasing, we obtain

$$\begin{aligned}
y(t) &\le A_2(f) + A_2(g_2)\int_0^t h_2(s)\left[A_2(f) + \sum_{j=3}^{m} A_2(g_j)x_j(s)\right] \\
&\quad \times \exp\left(\int_s^t h_2(\xi)A_2(g_2)du_2(\xi)\right)du_2(s) + \sum_{j=3}^{m} A_2(g_j)x_j(t) \\
&\le A_2(f) + A_2(g_2)\int_0^t h_2(s)A_2(f)vexp\left(\int_s^t h_2(\xi)A_2(g_2)du_2(\xi)\right)du_2(s) \\
&\quad + \sum_{j=3}^{m}[A_2(g_j) + A_2(g_2)\int_0^t h_2(s)A_2(g_j) \\
&\quad \times \exp\left(\int_s^t h_2(\xi)A_2(g_2)du_2(\xi)\right)du_2(s)]x_j(t) \\
&= A_3(f) + \sum_{j=3}^{m} A_3(g_j)x_j(t),
\end{aligned}$$

where $A_3(f)$ and $A_3(g_j)$ are given by (3.1.184).

When $i = m - 1$, we easily see that

$$y(t) \le A_m(f) + A_m(g_m)x_m. \tag{3.1.191}$$

Hence, for $i = m$, by (3.1.187) and (3.1.191),

$$Dx_m = h_m(t)y(t)Du_m \le h_m\left[A_m(f) + A_m(g_m)x_m(t)\right]Du_m$$

i.e.,

$$Du_m - h_m(t)A_m(g_m)x_m(t)Du_m \le h_m(t)A_m(f)Du_m.$$

Multiplying both sides by $\exp\left(-\int_0^t h_m(s)A_m(g_m)du_m(s)\right)$ and integrating with respect to t between 0 and t,

$$x_m(t) \le \int_0^t h_m(s)A_m(f)\left(\int_s^t h_m(\xi)A_m(g_m)du_m(\xi)\right)du_m(s). \qquad (3.1.192)$$

Therefore the estimate (3.1.183) now follows from (3.1.191) and (3.1.192). $\square$

Theorem 3.1.26 (Guan-Liu [243]) *Suppose that, for $j = 1, \ldots, m$ and for all $t \in [0, T]$,*

(i) $f(t) > 0, y(t) \ge 0$, and $y(t) \in BV([0, T])$; $f(t), u_j(t)$ are non-decreasing;
(ii) $g_j(t) \ge 0$ and $g_j(t)$ $(j = 2, \ldots, m)$ are non-decreasing, $g_1(t) \in BV([0, T])$;
(iii) $h_j(t) \ge 0$ and are integrable with respect to $u_j(t)$, and if the inequality (3.1.182) holds, then for all $t \in [0, T]$,

$$y(t) \le B_m(f), \qquad (3.1.193)$$

where $B_k(v)$ is defined inductively as follows:

$$\begin{cases} B_0(v) &= v \\ B_k(v) &= vB_{k-1}(g_k)\exp\left(\int_0^t h_k(s)B_{k-1}(g_k)du_k(s)\right), \quad k = 1, \ldots, m. \end{cases}$$

Proof Since $f(t) > 0$ and is non-decreasing, from (3.1.182) it follows

$$z(t) \le 1 + \sum_{j=1}^m g_j(t)\int_0^t h_j(s)z(s)du_j(s),$$

where $z(t) = y(t)/f(t)$. Noting that $g_1(t) \ge 1$, we have

$$z(t) \le g_1(t)\left[1 + \int_0^t h_1(s)z(s)du_1(s) + \sum_{j=2}^m g_j(t)\int_0^t h_j(s)z(s)du_j(s)\right].$$

Defining $x_1(t) = 1 + \int_0^t h_1(s)z(s)du_1(s)$, we derive from the above inequality

$$z(t) \le g_1(t)\left[x_1(t) + \sum_{j=2}^{m} g_j(t)\int_0^t h_j(s)z(s)du_j(s)\right]. \qquad (3.1.194)$$

Thus,

$$\begin{aligned} Dx_1 &= h_1(t)z(t)Du_1 \\ &\le h_1(t)g_1(t)\left[x_1(t) + \sum_{j=2}^{m} g_j(t)\int_0^t h_j(s)z(s)du_j(s)\right]Du_1, \end{aligned}$$

i.e.,

$$Dx_1 - h_1(t)g_1(t)x_1Du_1 \le h_1(t)g_1(t)\left(\sum_{j=2}^{m} g_j(t)\int_0^t h_j(s)z(s)du_j(s)\right)Du_1. \qquad (3.1.195)$$

Multiplying both sides of (3.1.192) by $\exp\Big(-\int_0^t h_1(s)g_1(s)du_1(s)\Big)$ and using Lemma 3.1.11, we obtain

$$\begin{aligned} &D\left[x_1\exp\left(-\int_0^t h_1(s)g_1(s)du_1(s)\right)\right] \\ &\le h_1(t)g_1(t)\exp\left(-\int_0^t h_1(s)g_1(s)du_1(s)\right)\left(\sum_{j=2}^{m} g_j(t)\int_0^t h_j(s)z(s)du_j(s)\right)Du_1. \end{aligned}$$

Integrating with respect to t between 0 and t, we get

$$\begin{aligned} x_1(t) \le \exp\left(\int_0^t h_1(s)g_1(s)du_1(s)\right) \times [1 + \int_0^t \exp\left(-\int_0^s h_1(\xi)g_1(\xi)du_1(\xi)\right)h_1(s)g_1(s) \\ \times\left(\sum_{j=2}^{m} g_j(s)\int_0^s h_j(\xi)z(\xi)du_j(\xi)\right)du_1(s)], \end{aligned} \qquad (3.1.196)$$

since $x_1(0) = 1$. In view of (3.1.196) and (3.1.194), we obtain

$$\begin{aligned} y(t) \le fg_1\Bigg\{ &\exp\left(\int_0^t h_1g_1du_1(s)\right) \\ &\times\Big[1 + \int_0^t \exp(-\int_0^s h_1(\xi)g_1(\xi)du_1(\xi))h_1(s)g_1(s) \end{aligned}$$

$$\times\left(\sum_{j=2}^{m} g_j(s)\int_0^s h_j(\xi)z(\xi)du_j(\xi)\right)du_1(s)\Big]+\sum_{j=2}^{m} g_j(t)\int_0^t h_j(s)z(s)du_j(s)\Bigg\}$$

$$=fg_1\exp\left(\int_0^t h_1g_1du_1(s)\right)\Bigg\{1+\int_0^t\exp\Big[\left(-\int_0^s h_1(\xi)g_1(\xi)du_1(\xi)\right)$$

$$\times h_1(s)g_1(s)\left(\sum_{j=2}^{m} g_j(s)\int_0^s h_j(\xi)z(\xi)du_j(\xi)\right)du_1(s)\Big]$$

$$+\exp\left(-\int_0^t h_1(s)g_1(s)du_1(s)\right)\sum_{j=2}^{m}\sum_{j=2}^{m} g_j(t)\int_0^t h_j(s)z(s)du_j(s)\Bigg\}.$$

Noting that $g_j(t)$ $(j=2,\ldots,m)$ are non-decreasing, we can get

$$y(t)\le B_1(f)\Bigg\{1+\sum_{j=2}^{m} g_j(t)\int_0^t h_j(s)z(s)du_j(s)\Big[\exp\left(-\int_0^t h_1(s)g_1(s)du_1(s)\right)$$

$$+\int_0^t\exp\left(-\int_0^s h_1(\xi)g_1(\xi)du_1(\xi)\right)h_1(s)g_1(s)du_1(s)\Big]\Bigg\}.$$

From the equality,

$$\int_0^t\exp\left(-\int_0^s h_1(\xi)g_1(\xi)du_1(\xi)\right)h_1(s)g_1(s)du_1(s)=1-\exp\left(-\int_0^t h_1(s)g_1(s)du_1(s)\right),$$

it follows that

$$y(t)\le B_1(f)\left[1+\sum_{j=2}^{m} g_j(t)\int_0^t h_j(s)z(s)du_j(s)\right]. \tag{3.1.197}$$

Similarly to the above argument, from (3.1.197) we conclude

$$z(t)\le (B_1(f)/f(t))[1+g_2(t)\int_0^t h_2(s)z(s)du_2(s)$$

$$+\sum_{j=3}^{m} g_j(t)\int_0^t h_j(s)z(s)du_j(s)]$$

$$\le (B_1(f)/f(t))g_2(t)[1+\int_0^t h_2(s)z(s)du_2(s)$$

$$+\sum_{j=3}^{m} g_j(t)\int_0^t h_j(s)z(s)du_j(s)].$$

Using the fact

$$(B_1(f)/f(t))g_2(t) = g_2 B_0(g_1)\exp\left(\int_0^t h_1 B_0(g_1)du_1(s)\right) = B_1(g_2),$$

we see that

$$z(t) \le B_1(g_2)\left[1 + \int_0^t h_2(s)z(s)du_2(s) + \sum_{j=3}^m g_j(t)\int_0^t h_j(s)z(s)du_j(s)\right]. \tag{3.1.198}$$

Let $x_2(t) = 1 + \int_0^t h_2(s)z(s)du_2(s)$. Then

$$\begin{cases} z(t) \le B_1(g_2)\left[x_2(t) + \sum_{j=3}^m g_j(t)\int_0^t h_j(s)z(s)du_j(s)\right], \\ Dx_2 = h_2(t)z(t)Du_2 \le h_2(t)[B_1(g_2)x_2 + B_1(g_2) \\ \qquad \times \sum_{j=3}^m g_j(t)\int_0^t h_j(s)z(s)]Du_2, \end{cases} \tag{3.1.199}$$

i.e.,

$$Dx_2 - h_2 B_1(g_2)x_2 Du_2 \le h_2(t)B_1(g_2)\left(\sum_{j=3}^m g_j(t)\int_0^t h_j(s)z(s)du_j(s)\right)Du_2.$$

Multiplying both sides by $\exp\left(-\int_0^t h_2 B_1(g_2)du_2(s)\right)$. Using Lemma 3.1.11, and integrating from 0 to t, we can obtain

$$\begin{aligned} x_2(t) \le \exp\left(\int_0^t h_2 B_1(g_2)du_2(s)\right)\Bigg[1 + \int_0^t \exp\left(-\int_0^s h_2 B_1(g_2)du_2(\xi)\right)h_2 B_1(g_2) \\ \times\left(\sum_{j=3}^m g_j(s)\int_0^s h_j(\xi)z(\xi)du_j(\xi)\right)du_2(s)\Bigg]. \end{aligned} \tag{3.1.200}$$

By (3.1.200), (3.1.199), and noting that $g_j(t)$ $(j = 3, \ldots, m)$ are non-decreasing, we obtain

$$\begin{aligned} y(t) \le fB_1(g_2)\Bigg\{ & \exp\left(\int_0^t h_2 B_1(h_2)du_2(s)\right) \\ & \times\Bigg[1 + \int_0^t \exp\left(-\int_0^s h_2 B_1(g_2)du_2(\xi)\right)h_2 B_1(g_2) \end{aligned}$$

$$\times \left(\sum_{j=3}^{m} g_j(s) \int_0^s h_j(\xi) z(\xi) du_j(\xi) \right) du_2(s) \Big] + \sum_{j=3}^{m} g_j(t) \int_0^t h_j(s) z(s) du_j(s) \Big\}$$

$$\le f B_1(g_2) \exp\left(\int_0^t h_2 B_1(g_2) du_2(s) \right) \Big\{ 1 + \sum_{j=3}^{m} g_j(t) \int_0^t h_j(s) z(s) du_j(s)$$

$$\times \Big[\exp\left(- \int_0^t h_2 B_1(g_2) du_2(s) \right)$$

$$+ \int_0^t \exp\left(- \int_0^s h_2 B_1(g_2) du_2(\xi) \right) h_2 B_1(g_2) du_2(s) \Big] \Big\}.$$

$$= B_2(f) \Big\{ 1 + \sum_{j=3}^{m} g_j(t) \int_0^t h_j(s) z(s) du_j(s) \Big\}. \tag{3.1.201}$$

By induction, it follows that

$$\begin{aligned} y(t) &\le B_{m-1}(f) \Big\{ 1 + g_m \int_0^t h_m(s) z(s) du_m(s) \Big\} \\ &\le B_{m-1}(f) g_m(t) \Big\{ 1 + \int_0^t h_m(s) z(s) du_m(s) \Big\} \end{aligned}$$

or

$$\begin{aligned} z(t) &\le (B_{m-1}(f)/f) g_m(t) \left[1 + v int_0^t h_m(s) z(s) du_m(s) \right] \\ &= B_{m-1}(g_m) \left[1 + v int_0^t h_m(s) z(s) du_m(s) \right]. \end{aligned}$$

Let $x_m(t) = 1 + \int_0^t h_m(s) z(s) du_m(s)$. Then

$$\begin{cases} z(t) \le B_{m-1}(g_m) x_m(t), \\ D x_m = h_m(t) z(t) D u_m \le h_m(t) B_{m-1}(g_m) x_m D u_m \end{cases} \tag{3.1.202}$$

or

$$D x_m - h_m B_{m-1}(g_m) x_m D u_m \le 0.$$

Multiplying the above inequality both sides by $\exp\left(- \int_0^t h_m B_{m-1}(g_m) du_m(s) \right)$ and integrating from 0 to t, we get

$$x_m(t) \le \exp\left(\int_0^t h_m B_{m-1}(g_m) du_m(s) \right). \tag{3.1.203}$$

Thus from (3.1.202) and (3.1.203), it follows that

$$z(t) \le B_{m-1}(g_m) \exp\left(\int_0^t h_m B_{m-1}(g_m) du_m(s)\right)$$

i.e.,

$$y(t) \le f B_{m-1}(g_m) \exp\left(\int_0^t h_m B_{m-1}(g_m) du_m(s)\right) = B_m(f),$$

which completes the proof. $\square$

Theorem 3.1.27 (Guan-Liu [243]) *Suppose that, for $j = 1, \ldots, m$ and for all $t \in [0, T]$,*

(i) $y(t), f(t) \in BV([0, T])$, $u_j(t)$ are non-decreasing;
(ii) $h_j(t)$ are non-decreasing and integrable with respect to $u_j(t)$.

Furthermore, if for all $t \in [0, T]$,

$$y(t) \le f(t) + \sum_{j=1}^{m} \int_0^t h_j(s) y(s) du_j(s), \tag{3.1.204}$$

then for all $t \in [0, T]$,

$$y(t) \le f(t) + \sum_{j=1}^{m} \int_0^t h_j(s) f(s) \times \exp\left(\sum_{j=1}^{m} \int_s^t h_j(\xi) du_j(\xi)\right) du_j(s). \tag{3.1.205}$$

Proof Let $x(t) = \sum_{j=1}^m \int_0^t h_j(s) y(s) du_j(s)$. By (3.1.204), we get

$$y(t) \le f(t) + x(t)$$

and

$$Dx = \sum_{j=1}^{m} h_j(t) y(t) Du_j \le \sum_{j=1}^{m} h_j(t) [f(t) + x(t)] Du_j$$

or

$$Dx - \sum_{j=1}^{m} h_j(t) x(t) Du_j \le \sum_{j=1}^{m} h_j(t) f(t) Du_j. \tag{3.1.206}$$

Multiplying both sides of (3.1.206) by $\exp(-\int_0^t \sum_{j=1}^m h_j(s)du_j(s))$, using Lemma 3.1.11, integrating from 0 to t, and noting that $x(0) = 0$, we get

$$D\left[\exp\left(-\int_0^t \sum_{j=1}^m h_j(s)du_j(s)\right)x\right] \le \sum_{j=1}^m h_j(t)f(t)\exp\left(-\int_0^t \sum_{j=1}^m h_j(s)du_j(s)\right)Du_j$$

and

$$x(t) \le \sum_{j=1}^m \int_0^t h_j(s)f(s)\exp\left(\int_s^t \sum_{j=1}^m h_j(\xi)du_j(\xi)\right)du_j(s).$$

Therefore,

$$\begin{aligned} y(t) &\le f(t) + x(t) \\ &\le f(t) + \sum_{j=1}^m \int_0^t h_j(s)f(s)\exp\left(\int_s^t \sum_{j=1}^m h_j(\xi)du_j(\xi)\right)du_j(s) \end{aligned}$$

which yields (3.1.205). □

Theorem 3.1.28 (Guan-Liu [243]) *Suppose that, for $j = 1, \ldots, m$ and for all $t \in [0, T]$,*

(i) $y(t) \ge 0$ and $y(t) \in BV([0, T])$;
(ii) $f(t) > 0$ and $f(t), u_j(t)$ are non-decreasing;
(iii) $h_j(t)$ are non-negative and integrable with respect to $u_j(t)$, and if the inequality (3.1.204) holds, then for all $t \in [0, T]$,

$$y(t) \le f(t)\exp\left(\int_0^t \sum_{j=1}^m h_j(s)du_j(s)\right).$$

Proof Since $f(t)$ is non-decreasing, from (3.1.204) it follows

$$z(t) \le 1 + \sum_{j=1}^m \int_0^t h_j(s)z(s)du_j(s),$$

where $z(t) = y(t)/f(t)$, which is a bounded variation function on $[0, T]$.

Letting $x(t) = 1 + \sum_{j=1}^m \int_0^t h_j(s)z(s)du_j(s)$, then $x(0) = 1$ and $z(t) \le x(t)$,

$$Dx = \sum_{j=1}^m h_j(t)z(t)Du_j \le \sum_{j=1}^m h_j(t)x(t)Du_j,$$

i.e.,

$$Dx - \sum_{j=1}^{m} h_j(t)x(t)Du_j \leq 0.$$

Multiplying both sides by $\exp(-\int_0^t \sum_{j=1}^m h_j(s)du_j(s))$, and integrating from 0 to t, we have

$$x(t) \leq \exp\left(\int_0^t \sum_{j=1}^{m} h_j(s)du_j(s)\right).$$

Hence

$$z(t) \leq x(t) \leq \exp\left(\int_0^t \sum_{j=1}^{m} h_j(s)du_j(s)\right),$$

which implies that

$$y(t) \leq f(t) \exp\left(\int_0^t \sum_{j=1}^{m} h_j(s)du_j(s)\right).$$

This therefore completes the proof. □

Remark 3.1.19 If we suppose that u_j are absolutely continuous on $[0, T]$, where $u'(t)$, which exists a.e., is non-negative or in particular $u_j(t) \equiv t$. The functions y, f, g_j and h_j are piecewise continuous on $[0, T]$, then Theorem 3.1.26 reduces to Theorem 1 of [182].

Remark 3.1.20 If the functions y, f, g_j and h_j are piecewise continuous on $[0, T]$, then Theorem 3.1.25 containing m-linear terms is a generation of the results in [547].

Remark 3.1.21 Theorem 3.1.27 contains multi-distributional derivatives, which generalized the results in [547]. If the functions $y(t), f(t)$, and $h_j(t)$ are piecewise continuous, $m = 1$, then Theorem 3.1.28 reduces to a basic Gronwall inequality.

The following result is a new Gronwall-Bellman type integral inequalities for multi-distributions due to Oguntuase [436] and are in the sense of Lebesgue-Stieltjes integral for functions of bounded variation. These inequalities generalize some results of Guan and Liu obtained in [243], i.e., Theorems 3.1.25–3.1.26.

We shall assume that the functions $u_j(t)$ is right continuous at $t = 0$, $j = 1, \ldots, m$. We shall let $BV(I)$ denote the set of all functions of bounded variation defined on $I \subseteq \mathbb{R}$ and taking values in $\mathbb{R}$.

Theorem 3.1.29 (Oguntuase [436]) *Suppose that for $j = 1, \ldots, m$ and for all $t,\ s \in [0, T]$:*

(1) $Q_j(t, s) \geq 0,\ y(t) \geq 0$ and $Q_j(t, s),\ y(t),\ f(t) \in BV[0, T]$,
(2) $u_j(t)$ are non-decreasing in t,
(3) $Q_j(t, s)$ and its partial derivatives $\frac{\partial}{\partial t}Q_j(t, s)$ are continuous and non-decreasing in its first variable and that $Q_j(t, s)$ and $\frac{\partial}{\partial t}Q_j(t, s)$ are non-negative and integrable with respect to $u_j(t)$ and if the following inequality holds,

$$y(t) \leq f(t) + \sum_{j=1}^{m} \int_0^t Q_j(t, s)y(s)du_j(s), \tag{3.1.207}$$

then for all $t,\ s,\ \tau \in [0, T]$,

$$\begin{aligned} y(t) \leq A_m(f) + A_m(1) \int_0^t &\left(Q_m(s, s)A_m(f) + \int_0^s \frac{\partial}{\partial s} Q_m(s, \tau)A_m(f) \right) \\ &\times \exp\left(\int_s^t Q_m(s, \tau)A_m(1)du_m(\tau) \right) du_m(s) \end{aligned} \tag{3.1.208}$$

where $A_k(v)$ is defined inductively as follows

$$\begin{cases} A_1(v) = v, \\ A_{k+1}(v) = A_k(v) + \displaystyle\int_0^t \left(A_k(Q_k(s, s))A_k(v) + \int_0^s \frac{\partial}{\partial s} A_k\left(Q_k(s, \tau)\right) A_k(v)du_k(\tau) \right) \\ \qquad\qquad \times \left(\displaystyle\int_s^t A_k(Q_k(s, \tau))du_k(\tau) \right) du_k(s). \end{cases} \tag{3.1.209}$$

Proof Let

$$x_i(t) = \int_0^t Q_i(t, s)y(s)du_i(s), \quad t,\ s \in [0, T], \quad i = 1, \ldots, m. \tag{3.1.210}$$

Clearly, $x_i(t)$ are functions of bounded variation. We also observed that $x_i(0) = 0$. Hence, in view of (3.1.210), inequality (3.1.207) reduces to

$$y(t) \leq f(t) + \sum_{j=0}^{m} x_j(t). \tag{3.1.211}$$

Thus

$$Dx_i(t) = Q_i(t, t)y(t)Du_i(t) + \int_0^t \frac{\partial}{\partial t} Q_i(t, s)y(s)Du_i(s). \tag{3.1.212}$$

If we put $i = 1$ in (3.1.211) and (3.1.212), we obtain

$$
\begin{aligned}
Dx_1(t) &= Q_1(t,t)y(t)Du_1(t) + \int_0^t \frac{\partial}{\partial t} Q_1(t,s)y(s)Du_1(s) \\
&\le \Big(Q_1(t,t)\Big(f(t) + \sum_{j=1}^m x_j(t)\Big) + \int_0^t \frac{\partial}{\partial t} Q_1(t,s)\Big(f(s) + \sum_{j=1}^m x_j(s)\Big)\Big)Du_1(t),
\end{aligned}
$$

which implies

$$
\begin{aligned}
&Dx_1(t) - \Big(Q_1(t,t)x_1(t) + \int_0^t \frac{\partial}{\partial t} Q_1(t,s)x_1(s)\Big)Du_1(t) \\
&\le \Big(Q_1(t,t)\Big(f(t) + \sum_{j=2}^m x_j(t)\Big)\Big) + \int_0^t \frac{\partial}{\partial t} Q_1(t,s)\Big(f(s) + \sum_{j=2}^m x_j(s)\Big)Du_1(t).
\end{aligned} \tag{3.1.213}
$$

Multiplying both sides of (3.1.213) by $\exp\Big(-\int_0^t Q_1(t,s)du_1(s)\Big)$, we have

$$
\begin{aligned}
&\Big(Dx_1(t) - \Big(Q_1(t,t)x_1(t) + \int_0^t \frac{\partial}{\partial t} Q_1(t,s)x_1(s)\Big)Du_1(t)\Big)\exp\Big(-\int_0^t Q_1(t,s)du_1(s)\Big) \\
&\le \Big(Q_1(t,t)\Big(f(t) + \sum_{j=2}^m x_j(t)\Big) \int_0^t \frac{\partial}{\partial t} Q_1(t,s)\Big(f(s) + \sum_{j=2}^m x_j(s)\Big)\Big) \\
&\quad \times \exp\Big(-\int_0^t Q_1(t,s)du_1(s)\Big)Du_1(t).
\end{aligned}
$$

By Lemma 3.1.11, we have

$$
\begin{aligned}
&D(x_1(t)\exp\Big(-\int_0^t Q_1(t,s)du_1(s)\Big) \\
&\le \Big(Q_1(t,t)\Big(f(t) + \sum_{j=2}^m x_j(t)\Big) + \int_0^t \frac{\partial}{\partial t} Q_1(t,s)\Big(f(s) \sum_{j=2}^m x_j(s)\Big)\Big) \\
&\quad \times \exp\Big(-\int_0^t Q_1(t,s)du_1(s)\Big)Du_1(t).
\end{aligned} \tag{3.1.214}
$$

Integrating (3.1.214) with respect to t from 0 to t, we have

$$
\begin{aligned}
&\Big(x_1(t) - x_1(0)\Big)\exp\Big(-\int_0^t Q_1(t,s)du_1(s)\Big) \\
&\le \int_0^t \Big(Q_1(s,s)\Big(f(s) + \sum_{j=2}^m x_j(s)\Big) + \int_0^s \frac{\partial}{\partial s} Q_1(s,\tau)\Big(f(\tau) + \sum_{j=2}^m x_j(\tau)\Big)\Big)
\end{aligned}
$$

$$\times \exp\Big(-\int_0^s Q_1(s,\tau)du_1(\tau)\Big)du_1(s). \tag{3.1.215}$$

Since $x_1(0) = 0$, we obtain

$$x_1(t) \le \int_0^t \Big(Q_1(s,s)\Big(f(s) + \sum_{j=2}^m x_j(s)\Big) + \int_0^s \frac{\partial}{\partial s} Q_1(s,\tau)\Big(f(\tau) + \sum_{j=2}^m x_j(\tau)\Big)\Big)$$
$$\times \exp\Big(-\int_s^t Q_1(s,\tau)du_1(\tau)\Big)du_1(s). \tag{3.1.216}$$

If we put (3.1.216) into (3.1.211) and use the fact that $X_j(t)$ are non-decreasing, then we obtain

$$\begin{aligned} y(t) &\le f(t) + \int_0^t \Big(Q_1(s,s)\Big(f(s) + \sum_{j=2}^m x_j(s)\Big) + \int_0^s \frac{\partial}{\partial s} Q_1(s,\tau)\Big(f(\tau) + \sum_{j=2}^m x_j(\tau)\Big)\Big) \\ &\quad \times \exp\Big(-\int_s^t Q_1(s,\tau)du_1(\tau)\Big)du_1(s) + \sum_{j=2}^m x_j(t) \\ &\le f(t) + \int_0^t \Big(Q_1(s,s)f(s) + \int_0^s \frac{\partial}{\partial s} Q_1(s,\tau)f(\tau)\Big) \exp\Big(-\int_0^s Q_1(s,\tau)du_1(\tau)\Big)du_1(s) \\ &\quad + \sum_{j=2}^m \Big(1 + \int_0^t \Big(Q_1(s,s) + \int_0^s \frac{\partial}{\partial s} Q_1(s,\tau)\Big) \exp\Big(-\int_s^t Q_1(s,\tau)du_1(\tau)\Big)du_1(s)\Big)x_j(t), \end{aligned}$$

i.e.,

$$y(t) \le A_2(f) + \sum_{j=2}^m A_2(1)x_j(t), \tag{3.1.217}$$

where $A_2(f)$ and $A_2(1)$ are as defined in (3.1.209).

When $i = 2$, it follows from inequalities (3.1.211) and (3.1.217) that

$$\begin{aligned} Dx_2(t) &= Q_2(t,t)y(t)Du_2(t) + \int_0^t \frac{\partial}{\partial t} Q_2(t,s)y(s)Du_2(s) \\ &\le \Big(Q_2(t,t)\Big(A_2(f) + \sum_{j=2}^m A_2(1)x_j(t)\Big) \\ &\quad + \int_0^t \frac{\partial}{\partial t} Q_2(t,s)\Big(A_2(f) + \sum_{j=2}^m A_2(f)x_j(s)\Big)\Big)Du_2(t), \end{aligned}$$

i.e.,

$$\begin{aligned}
&Dx_2(t) - \Big(Q_2(t,t)A_2(1)x_2(t) + \int_0^t \frac{\partial}{\partial t} Q_2(t,s)A_2(1)x_2(s)\Big)Du_2(t) \\
&\leq \Big(Q_2(t,t)(A_2(f) + \sum_{j=3}^{m} A_2(1)x_j(t)) \\
&\quad + \int_0^t \frac{\partial}{\partial t} Q_2(t,s)(A_2(f) + \sum_{j=3}^{m} A_2(1)x_j(s))\Big)Du_2(t).
\end{aligned} \tag{3.1.218}$$

Multiplying both sides of (3.1.218) by $\exp\Big(-\int_0^t Q_1(t,s)du_1(s)\Big)$, we get

$$\begin{aligned}
&\Big(Dx_2(t) - \Big(Q_2(t,t)A_2(1)x_2(t) + \int_0^t \frac{\partial}{\partial t} Q_2(t,s)A_2(1)x_2(t)\Big)Du_2(t)\Big) \\
&\times \exp\Big(-\int_0^t Q_2(t,s)A_2(1)du_2(s)\Big) \\
&\leq \Big(Q_2(t,t)\Big(A_2(f) + \sum_{j=3}^{m} A_2(1)x_j(t)\Big) \int_0^t \frac{\partial}{\partial t} Q_2(t,s)\Big(A_2(1) + \sum_{j=3}^{m} A_2(1)x_j(s)\Big)\Big) \\
&\times \exp\Big(-\int_0^t Q_2(t,s)A_2(1)du_2(s)\Big)Du_2(t).
\end{aligned}$$

By Lemma 3.1.11, we obtain

$$\begin{aligned}
&D(x_2(t)\exp\Big(-\int_0^t Q_2(t,s)A_2(1)du_2(s)\Big) \\
&\leq \Big(Q_2(t,t)\Big(A_2(f) + \sum_{j=3}^{m} A_2(1)x_j(t)\Big) + \int_0^t \frac{\partial}{\partial t} Q_2(t,s)\Big(A_2(f) + \sum_{j=3}^{m} A_2(1)x_j(s)\Big)\Big) \\
&\times \exp\Big(-\int_0^t Q_2(t,s)A_2(1)du_2(s)\Big)Du_2(t).
\end{aligned} \tag{3.1.219}$$

Integrating (3.1.219) with respect to t from 0 to t and noting that $x_2(0) = 0$, we have

$$\begin{aligned}
x_2(t) \leq &\int_0^t \Big(Q_2(s,s)\Big(A_2(f) + \sum_{j=3}^{m} A_2(1)x_j(s)\Big) + \int_0^s \frac{\partial}{\partial s} Q_2(s,\tau)\Big(A_2(f) + \sum_{j=3}^{m} A_2(1)x_j(\tau)\Big)\Big) \\
&\times \exp\Big(-\int_s^t Q_2(s,\tau)A_2(1)du_2(\tau)\Big)du_2(s).
\end{aligned} \tag{3.1.220}$$

Inserting (3.1.220) into (3.1.217) and using the fact that $x_j(t)$ are non-decreasing, we obtain

$$\begin{aligned}
y(t) \le\ & A_2(f) + A_2(1)\int_0^t \Big(Q_2(s,s)[A_2(f) + \sum_{j=3}^{m} A_2(1)x_j(s)] \\
& + \int_0^s \frac{\partial}{\partial s}Q_2(s,\tau)\Big(A_2(f) + \sum_{j=3}^{m} A_2(1)x_j(\tau)\Big)\Big) \\
& \times \exp\Big(-\int_s^t Q_2(s,\tau)A_2(1)du_2(\tau)\Big)du_2(s) + \sum_{j=3}^{m} A_2(1)x_j(t) \\
\le\ & A_2(f) + A_2(1)\int_0^t \Big(Q_2(s,s)A_2(f) + \int_0^s \frac{\partial}{\partial s}Q_2(s,\tau)A_2(f)\Big) \\
& \times \exp\Big(-\int_0^s Q_2(s,\tau)A_2(1)du_2(\tau)\Big)du_2(s) \\
& + \sum_{j=1}^{m} \Big(A_2(1) + A_2(1)\int_0^t \Big(Q_2(s,s)A_2(1) + \int_0^s \frac{\partial}{\partial s}Q_2(s,\tau)A_2(1)\Big) \\
& \times \exp\Big(-\int_0^s Q_2(s,\tau)A_2(1)du_2(\tau)\Big)du_2(s)\Big)x_j(t) \\
=\ & A_3(f) + \sum_{j=3}^{m} A_3(1)x_j(t),
\end{aligned} \tag{3.1.221}$$

where $A_3(f)$ and $A_3(1)$ are as defined in (3.1.209).

If we set $i = m-1$, then we easily obtain

$$y(t) \le A_m(f) + A_m(1)x_m(t). \tag{3.1.222}$$

Next, suppose $i = m$, then (3.1.212) and (3.1.220) imply

$$\begin{aligned}
Dx_m(t) =\ & Q_m(t,t)y(t)Du_m(t) + \int_0^t \frac{\partial}{\partial t}Q_m(t,s)y(s)Du_m(s) \\
\le\ & \Big(Q_m(t,t)\Big(A_m(f) + A_m(1)x_m(t)\Big) + \int_0^t \frac{\partial}{\partial t}Q_m(t,s)\Big(A_m(1)x_m(s)\Big)\Big)Du_m(t).
\end{aligned} \tag{3.1.223}$$

Thus,

$$\begin{aligned}
& Dx_m(t) - \Big(Q_m(t,t)A_m(1)x_m(t) + \int_0^t \frac{\partial}{\partial t}Q_m(t,s)A_m(1)x_m(s)\Big)Du_m(t) \\
\le\ & \Big(Q_m(t,t)A_m(f) + \int_0^t \frac{\partial}{\partial t}Q_m(t,s)A_m(f)\Big)Du_m(t).
\end{aligned} \tag{3.1.224}$$

Multiplying both sides of (3.1.224) by $\exp\left(-\int_0^t Q_m(t,s)A_m(1)du_m(s)\right)$ and integrating with respect to t from 0 to t, and noting that $x_m(0) = 0$, we have

$$x_m(t) \leq \int_0^t \left(Q_m(s,s)A_m(f) + \int_0^s \frac{\partial}{\partial s} Q_m(s,\tau)A_m(f) \right) \times \exp\left(-\int_s^t Q_m(t,\tau)A_m(1)du_m(\tau) \right) du_m(s). \tag{3.1.225}$$

Substituting (3.1.224) into (3.1.225) and noting that $x_m(t)$ is non-decreasing, we conclude

$$y(t) \leq A_m(f) + A_m(1) \int_0^t \left(Q_m(s,s)A_m(f) + \int_0^s \frac{\partial}{\partial s} Q_m(s,\tau)A_m(f) \right) \times \exp\left(-\int_s^t Q_m(t,\tau)A_m(1)du_m(\tau) \right) du_m(s).$$

This completes the proof. □

As an immediate consequence of the above theorem, we can observe that if we set $Q_j(t,s) = g_j(t)h_j(s),\ j = 1, 2, \ldots, m$, then Theorem 3.1.28 reduces to the next corollary.

Corollary 3.1.10 (Guan-Liu [243]) *Suppose that for $j = 1, \ldots, m$ and for all $t,\ s \in [0, T]$,*

(1) $g_j(t) \geq 0,\ y(t) \geq 0$ and $g_j(t), y(t), f(t) \in BV[0, T]$,
(2) $u_j(t)$ are non-decreasing in t,
(3) $h_j(t)$ are non-negative and integrable with respect to $u_j(t)$ and if the following inequality holds

$$y(t) \leq f(t) + \sum_{j=1}^{m} g_j(t) \int_0^t h_j(s)y(s)du_j(s). \tag{3.1.226}$$

Then for all $t \in [0, T]$,

$$y(t) \leq A_m(f) + A_m(1) \int_0^t g_m(s)h_m(s)A_m(f) \exp\left(\int_s^t g_m(s)h_m(\tau)du_m(\tau) \right) du_m(s). \tag{3.1.227}$$

Remark 3.1.22 Corollary 3.1.10 is essentially not the same as Theorem 2.1 in [243] in the sense that Corollary 3.1.10 contains Theorem 2.1 in [243] as a special case. Indeed, Corollary 3.1.10 is more general than Theorem 2.1 in [243].

3.1.6 Gronwall-Wendroff Type Inequalities with Piecewise Continuous Functions and Discrete Continuous Variables

In this section, we introduce some results, due to Blandzi et al. [378], on Gronwall-Wendroff type inequalities with piecewise continuous functions and discrete continuous variables,

In 1997, Blandzi et al. [378] proved the following two inequalities. The first inequality involves piecewise continuous functions, whereas the second is similar to Gronwall-Wendroff-type inequality and involves discrete and continuous variables.

Let $\alpha = (\alpha_1, \cdots \alpha_k)$ be multi-index. By $S_k^{n,\nu}$ we denote a set of multi-index such that $\alpha \in S_k^{n,\nu}$ if $n \geq \alpha_1 > \alpha_2 > \cdots > \alpha_k \geq \nu$. Furthermore, we assume that $n \geq \nu$ and $k \leq n - \nu + 1$.

Let two sequences of functions $\{\phi_n(x)\}_{n=1}^{\infty}$, $\{\psi_n(x)\}_{n=1}^{\infty}$ be given and a function $g(x)$ defined on some set J. Assuming that the integrals involved exist, we define the integral operators for all $x \in J$, $\alpha \in S_k^{n,\nu}$,

$$T_\alpha(\phi, \psi, y) = \phi_{\alpha_1}(x) \int_0^x \psi_{\alpha_1}(x_1)\phi_{\alpha_2}(x_1)\Big(\int_0^{x_1} \psi_{\alpha_2}(x_2)\phi_{\alpha_3}(x_2)$$
$$\times \int_0^{x_2} \cdots \Big(\int_0^{x_{k-2}} \psi_{\alpha_{k-1}}(x_{k-1})\phi_{\alpha_k}(x_{k-1}) \int_0^{x_{k-1}} \psi_{\alpha_k}(x_2)y(x_k)\,dx_k\Big)\,dx_{k-1}\cdots\Big)\,dx_1.$$

Theorem 3.1.30 (Blandzi-Popenda-Agarwal [378]) *Let $u : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous on every interval $[(n-1)h, nh)$, where h is a fixed positive constant and $n \in \mathbb{N}$. Furthermore, let $\alpha_i : [0, h] \to \mathbb{R}_+$, $i \in \mathbb{N}_0$ be continuous, and c is a positive constant. Then for every solution u of the inequality, for all $n \in \mathbb{N}_0$, $x \in [0, h)$,*

$$u(x + nh) \leq c + \int_0^x \sum_{j=0}^{n} \alpha_j(t)u(t + jh)\,dt, \tag{3.1.228}$$

the following estimate holds for all $n \in \mathbb{N}_0, x \in [0, h)$,

$$u(x + nh) \leq c\left(1 + \int_0^x r_n(t)e^{\int_0^t [\alpha_{n-1}(s)-\alpha_n(s)]\,ds}\,dt\right) e^{\int_0^x \alpha_n(t)\,dt}, \tag{3.1.229}$$

where $\{r_n\}_{n=1}^{\infty}$ is the solution of the problem

$$\begin{cases} r_{n+1}(x) = r_n(x)e^{\int_0^x [\alpha_{n-1}(t)-\alpha_n(t)]\,dt} + \alpha_n(x)\displaystyle\int_0^x r_n(t)e^{\int_0^t [\alpha_{n-1}(s)-\alpha_n(s)]\,ds}\,dt + \alpha_n(x), \quad n = 1, 2, \cdots, \\ r_1(x) = \alpha_0(x), \quad x \in [0, h). \end{cases} \tag{3.1.230}$$

Proof Corresponding to the inequality (3.1.228), we consider the equation

$$w(x+nh) = c + \int_0^x \sum_{j=0}^{n} \alpha_j(t) w(t+jh)\, dt, \quad \text{for all } n \in \mathbb{N}_0,\ x \in [0,h). \tag{3.1.231}$$

We shall show that for all $n \geq 1, x \in [0,h)$, the solution of (3.1.231) can be written as

$$w(x+nh) = c\left(1 + \int_0^x r_n(t) e^{\int_0^t [\alpha_{n-1}(s) - \alpha_n(s)]\, ds}\, dt\right) e^{\int_0^x \alpha_n(t)\, dt}. \tag{3.1.232}$$

To this end, we note that the solution of the Volterra integral equation

$$z(x) = \phi(x) + \int_0^x \psi(t) z(t)\, dt, \ \text{for all}\ \ x \geq 0,$$

where the function ϕ is differentiable, appears as

$$z(x) = \left(\phi(0) + \int_0^x \phi'(t) e^{-\int_0^t \psi(s)\, ds}\, dt\right) e^{\int_0^x \psi(t)\, dt}. \tag{3.1.233}$$

For $n = 0$, Eq. (3.1.231) reduces to

$$w(x) = c + \int_0^x \alpha_0(t) w(t)\, dt, \quad \textit{for all}\ \ x \in [0,h),$$

and in view of (3.1.233), its solution is

$$w(x) = c e^{\int_0^x \alpha_0(t)\, dt}.$$

For $n = 1$, Eq. (3.1.231) reduces to

$$w(x+h) = w(x) + \int_0^x \alpha_1(t) w(t+h)\, dt.$$

It is clear that the function $w(x)$ is differentiable and its derivative is $c\alpha_0(x) e^{\int_0^x \alpha_0(t)\, dt}$. Furthermore, since $w(nh) = c$, for all $n \in \mathbb{N}_0$, from (3.1.233), we derive

$$\begin{aligned}
w(x+nh) &= \left(c + c\int_0^x \alpha_0(t) e^{\int_0^t \alpha_0(s)\, ds} e^{-\int_0^t \alpha_1(s)\, ds}\, dt\right) e^{\int_0^x \alpha_1(t)\, dt} \\
&= c\left(1 + \int_0^x \alpha_0(t) e^{\int_0^t [\alpha_0(s) - \alpha_1(s)]\, ds}\, dt\right) e^{\int_0^x \alpha_1(t)\, dt} \\
&= c\left(1 + \int_0^x r_1(t) e^{\int_0^t [\alpha_0(s) - \alpha_1(s)]\, ds}\, dt\right) e^{\int_0^x \alpha_1(t)\, dt}.
\end{aligned}$$

Thus, (3.1.232) is true for $n = 1$. Now suppose that (3.1.232) is true for $n = k$, and $w(x + kh)$ is a differentiable function, then for $n = k + 1$, we have

$$w(x + (k + 1)h) = w(x + kh) + \int_0^x \alpha_{k+1}(t)w(t + (k + 1)h)\,dt.$$

To use (3.1.233), we substitute $\phi(x) = w(x + kh)$, and note that

$$\begin{aligned}
&\phi'(x)e^{-\int_0^x \alpha_{k+1}(t)\,dt} \\
&= \Big(cr_k(x)e^{\int_0^x[\alpha_{k-1}(t)-\alpha_k(t)]\,dt}e^{\int_0^x \alpha_1(t)\,dt} \\
&\quad +c\left[1 + \int_0^x r_k(t)e^{\int_0^t[\alpha_{k-1}(s)-\alpha_k(s)]\,ds}\,dt\right]\alpha_k(x)e^{\int_0^x \alpha_k(t)\,dt}\Big)e^{-\int_0^x \alpha_{k+1}(t)\,dt} \\
&= c\Big(r_k(x)e^{\int_0^x[\alpha_{k-1}(t)-\alpha_k(t)]\,dt}e^{\int_0^x \alpha_1(t)\,dt} \\
&\quad +\alpha_k(x)\int_0^x r_k e^{\int_0^t[\alpha_{k-1}(s)-\alpha_k(s)]\,ds}\,dt + \alpha_k(x)\Big)e^{\int_0^x[\alpha_k(t)-\alpha_{k+1}(t)]\,dt}
\end{aligned}$$

which, along with (3.1.230), yields

$$\phi'(x)e^{-\int_0^x \alpha_{k+1}(t)\,dt} = cr_{k+1}(x)e^{\int_0^x[\alpha_k(t)-\alpha_{k+1}(t)]\,dt}.$$

Using the above relation in (3.1.233), we obtain

$$w(x + (k + 1)h) = \left(c + c\int_0^x r_{k+1}(t)e^{\int_0^t[\alpha_k(s)-\alpha_{k+1}(s)]\,ds}\,dt\right)e^{\int_0^x \alpha_{k+1}(t)\,dt},$$

which is the same as (3.1.232) for $n = k + 1$.

The differentiability of the function $w(x + (k + 1)h)$ follows from its explicit representation, and the continuity of r_{k+1}, which is a consequence of the continuity of r_k and the coefficients α_n. The estimate (3.1.229) is now a consequence of the general remarks about monotonic operators. □

Remark 3.1.23 If we denote $\alpha_{-1}(x) = \alpha_0(x)$, $r_0(x) \equiv 0$, then the bound (3.1.229) also holds for $n = 0$.

Remark 3.1.24 The solution of (3.1.230) can be represented by using the operators T_α, and then an estimate of the inequality (3.1.228) may be defined only in terms of the coefficients α_i and the constant c. To see this, we multiply (3.1.230) by $e^{\int_0^x \alpha_n(t)\,dt}$ and substitute $y_n(x) = r_n(x)e^{\int_0^x \alpha_{n-1}(t)\,dt}$ to obtain for all $n \in \mathbb{N}$,

$$y_{n+1}(x) = y_n(x) + \phi_n(x)\int_0^x \beta_n(t)y_n(t)\,dt + \phi_n(x), \tag{3.1.234}$$

where

$$\phi_n(x) = \alpha_n(x)e^{\int_0^x \alpha_n(t)\,dt},\ \beta_n(x) = e^{-\int_0^x \alpha_n(t)\,dt}.$$

The solution of (3.1.234) can be written as

$$y_{n+1}(x) = y_1(x) + \sum_{i=1}^{n} \phi_i(x) + \sum_{i=1}^{n-1} \sum_{\alpha \in S_k^{n,2}} \sum_{j<\alpha_k} T_\alpha(\phi, \beta, \phi_j(x))$$

$$+ \sum_{k=1}^{n} \sum_{\alpha \in S_k^{n,1}} T_\alpha(\phi, \beta, y_1(x)).$$

The above representation follows by induction and proper grouping of the elements.

3.2 Linear One-Dimensional Discontinuous Generalizations on the Gronwall-Bellman Inequalities

In this section, we introduce some results on linear one-dimensional discontinuous generalizations on the Gronwall-Bellman inequalities.

3.2.1 Linear One-Dimensional Discontinuous Integral Inequalities of Volterra Type

In the following Theorems 3.2.1–3.2.2, all functions are assumed to be real-valued and defined on a given interval I with zero at left endpoint. The domain of $k(t, s)$ is taken to be the subset of $I \times I$ for which $t \geq s$. All functions are assumed Lebesgue measurable and all functions of one variable are assumed to be non-negative.

We call such a function $x(t)$ locally integrable on I if each $t \in I$, its Lebesgue integral $\int_0^t x(s)ds$ is finite.

Theorem 3.2.1 (Willett [647]) *Suppose that for all $t \in I$,*

$$u(t) \leq w_*(t) + w(t) \int_0^t v(s)u(s)ds, \tag{3.2.1}$$

where $vw,\ vw_*$ and vu are locally integral on I. Then for all $t \in I$,

$$u(t) \leq w_*(t) + w(t)\Big(\exp \int_0^t vwds\Big)\Big(\int_0^t vw_*ds\Big). \tag{3.2.2}$$

Proof For the proof of Theorem 3.2.1, we can refer to [647] or [137]. □

We state Theorem 3.2.1 here in order to simplify the proofs of the following theorems.

Theorem 3.2.2 (Willett [647]) *Suppose that for all $t \in I$,*

$$u(t) \leq w_0(t) + \sum_{i=1}^{n} w_i(t) \int_0^t v_i(s)u(s)ds, \tag{3.2.3}$$

where $v_i w_j$ ($i = 1, 2, \cdots, n; j = 0, 1, \cdots, n$) and $v_i u$ ($i = 1, 2, \cdots, n$) are locally integral on I. Then

$$u \leq E_n w_0, \tag{3.2.4}$$

where E_i ($i = 0, 1, \cdots, n$) is defined inductively as the composition of $i + 1$ functional operators, i.e., $E_i = D_i D_{i-1} \cdots D_0$, where

$$D_0 w = w,\ D_j w = w + (E_{j-1} w_j) \left(\exp \int_0^t v_j E_{j-1} w_j ds \right) \int_0^t v_j w ds,\ \ j = 1, 2, \cdots, n. \tag{3.2.5}$$

Proof For $n = 1$, the theorem reduces to Theorem 3.2.1, and hence is true. Suppose n is given and $n > 1$. The proof is by finite induction. Assume the following two statements (A) and (B) hold for $i = k$, where k is some integer between 0 and $n - 1$ ($0 \leq k \leq n - 1$) :

(A) $E_i w_j$ exists and $v_m E_i w_j$ is locally integrable, $j = i + 1, i + 2, \cdots, n, 0; m = i + 1, i + 2, \cdots, n$;
(B) $u \leq E_i w_0 + \sum_{m=i+1}^{n} (E_i w_m) \int_0^t v_m u ds$.

(A) and (B) hold by assumption for $i = k = 0$, and we shall prove that (A) and (B) hold for $i = k + 1$ if $k \leq n - 2$ and that the theorem is true if $k = n - 1$. In either case, the theorem follows.

Let $w = E_k w_j$. From the definition of E_{k+1}, we obtain

$$E_{k+1} w_j = D_{k+1} w = w + (E_k w_{k+1}) \left(\exp \int_0^t v_{k+1} E_k w_{k+1} ds \right) \int_0^t v_{k+1} w ds. \tag{3.2.6}$$

It follows from (A) with $i = k$ and the local integrability of $v_{k+1} w_0$ that $E_{k+1} w_j$ ($j = k + 1, k + 2, \cdots, n, 0$) exists and that $v_m E_{k+1} w_j$ ($m = k + 1, k + 2, \cdots, n; j = k + 1, k + 2, \cdots, n, 0$) is locally integrable. Hence, (A) holds for $i = k + 1$, and we may apply Theorem 3.2.1 with $v = v_{k+1}$ to (B) with $i = k$ to obtain

$$u \leq w_0^* + (E_k w_{k+1}) \left(\exp \int_0^t v_{k+1} E_k w_{k+1} ds \right) \int_0^t v_{k+1} w_0^* ds \tag{3.2.7}$$

where

$$\begin{aligned} w_0^* &= E_k w_0 + \sum_{m=k+2}^{n} (E_k w_m) \int_0^t v_m u ds, \qquad \text{if } k \leq n-2, \\ &= E_k w_0, \qquad \text{if } k = n-1. \end{aligned} \tag{3.2.8}$$

Obviously, $u(a) > 0$. Now we define a function r by the equations. Substituting from (3.2.8) for w_0^* in (3.2.7), rearranging the terms, and using the fact that $\int_0^t v_m u ds$ is a non-decreasing function of t, we can, in the case $k \leq n-2$, obtain from (3.2.7) inequality (B) with $i = k+1$. If $k = n-1$, we can get (3.2.4) from (3.2.7), which is the conclusion of the theorem. □

In the sequel, we shall introduce some inequalities, due to Willett [647], of the form

$$u(t) \leq w_0(t) + \int_0^t k(t,s)u(s)ds, \qquad \textit{for all } t > 0, \tag{3.2.9}$$

where $k(t,s)$ and $w_0(t)$ are known non-negative functions and $u(t)$ is an unknown non-negative function. For example, we can refer to Bellman [63], Coddington and Levinson [137], Willett [647] and others. In order to obtain from (3.2.9) a genuine upper bound for $u(t)$, i.e., an upper bound independent of u, it seems necessary to separate the variable t in $k(t,s)$ from the integrand involving $u(s)$. This can be done by assuming that $k(t,s)$ is directly separable, i.e., that there exist measurable functions $v_i(t)$ and $w_i(t)$ $(i = 1, \cdots, n)$ such that

$$k(t,s) \leq \sum_{i=1}^{n} w_i(t)v_i(s), \tag{3.2.10}$$

or by applying Hölder's inequality to $\int_0^t k(t,s)u(s)ds$. The latter is the so-called L^p case and is analyzed in [647].

We shall state as Theorem 3.2.1 the case when $k(t,s)$ is directly separable with $n = 1$. A special case of this result was first established in Gronwall [239], and a result of nearly the same generality of Theorem 3.2.1 appears in [137]. Actually, we need only that some derivative of $k(t,s)$ with respect to t be directly separable, but there seems to be no need to produce the details beyond the first derivative case since the procedure is clear.

In the next theorem, we shall show how an inequality of the same form as (3.2.3) can be produced in the case that $k(t,s)$ is differentiable and $\partial k(t,s)/\partial t$ rather than $k(t,s)$ is directly separable.

Theorem 3.2.3 (Willett [647]) *Suppose that inequality (3.2.9) holds and that $\partial k(t,s)/\partial t$ exists in the domain of $k(t,s)$ satisfying*

$$\frac{\partial k(t,s)}{\partial t} \leq \sum_{i=1}^{n} w_i(t)v_i(s). \tag{3.2.11}$$

Let $r(t)$ be a non-negative measurable function such that for all $t \in I$,

$$k(t,t) \leq r(t).$$

(We are assuming still that all functions of one variable are non-negative.) If $v_i u$ and w_i $(i = 1, 2, \cdots, n)$, r, and rw_0 are locally integrable on I, then for all $t \in I$,

$$u(t) \leq w_0^*(t) + \sum_{i=1}^{n} w_i^*(t) \int_0^t v_i(s)u(s)ds, \tag{3.2.12}$$

where

$$\begin{cases} w_0^*(t) = w_0(t) + \displaystyle\int_0^t r(s)w_0(s)\left(\exp\int_s^t r(\tau)d\tau\right)ds, \\ w_i^*(t) = \displaystyle\int_0^t w_i(s)\left(\exp\int_s^t r(\tau)d\tau\right)ds, \quad i = 1, 2, \cdots, n. \end{cases} \tag{3.2.13}$$

Proof Define a differentiable function $\theta(t)$ on I by

$$\theta(t) = \int_0^t k(t,s)u(s)ds.$$

Computing the derivative $\theta'(t)$ and then using Eq. (3.2.9), we obtain

$$\theta'(t) = r(t)w_0(t) + r(t)\theta(t) + \int_0^t \frac{\partial k}{\partial t}(t,s)u(s)ds.$$

Next, transposing $r(t)\theta(t)$ and multiplying the inequality by $\exp\left(-\int_0^t r(\xi)\,d\xi\right)$, we get

$$\begin{aligned} &\left(\theta\,\exp(-\int_0^t r(\xi)\,d\xi)\right)' \\ &\leq rw_0\,\exp\left(-\int_0^t r(\xi)\,d\xi\right) + \left(\exp(-\int_0^t r(\xi)\,d\xi)\right)\int_0^t \frac{\partial k}{\partial t}(t,s)u(s)ds. \end{aligned}$$

Now substituting for $\partial k(t,s)/\partial t$ from (3.2.11) and integrating between zero and an arbitrary point of I. Inequality (3.2.12) follows from the monotonicity of $\int_0^t v_i u ds$, $(i = 1, 2, \cdots, n)$, by substituting for $\theta(t)$ in (3.2.9). □

Inequality (3.2.12) is of the same form as inequality (3.2.3); hence, if the integrability assumptions of Theorem 3.2.2 are satisfied in this situation, Theorem 3.2.2 can be applied to produce a bound on $u(t)$.

In 1971, Thompson [622] gave some inequalities related to the positive solutions of a Volterra equation of the second kind in which the equation considered is

$$u(t) = f(t) + \int_0^t k(t,s)u(s)ds \tag{3.2.14}$$

where $k(t,s)$ is a Volterra type kernel, that is, $k(t,s) = 0$ for $s > t$. Unless otherwise stated, we shall assume $f \in L_+^2(\mathbb{R}_+)$ and $k(t,s) \in L_+^2(\mathbb{R}_+)$, where

$$L_+^2(\mathbb{R}_+) = \left\{h :\ h \in L^2(\mathbb{R}_+),\ h(t) \geq 0\ \textit{for all}\ t \in \mathbb{R}_+\right\}.$$

Define $k_1(t,s) = k(t,s)$ and for all $n \geq 2$,

$$k_n(t,s) = \int_s^t k(t,\sigma)k_{n-1}(\sigma,s)d\sigma.$$

By the resolvent kernel, we mean that $H(t,s) = \sum_{i=1}^n k_i(t,s)$ and this series converges in $L^2(\mathbb{R}_+ \times \mathbb{R}_+)$.

We now state the result due to Tricomi [625].

Theorem 3.2.4 (Tricomi [625]) *Let $f \in L^2(\mathbb{R}_+), k \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$, then the Eq. (3.2.14) admits a unique L^2 solution u(t) satisfying*

$$u(t) = f(t) + \int_0^t H(t,s)f(s)ds,\quad a.\,e.\ on\ \mathbb{R}_+.$$

Proof See, e.g., [625]. □

We also need the following lemma in Beesack [50].

Lemma 3.2.1 (Beesack [50]) *Let $f \in L_+^2(\mathbb{R}_+)$ and $k \in L_+^2(\mathbb{R}_+ \times \mathbb{R}_+)$ in Eq. (3.2.14), then*

$$u(t) \geq 0,\quad a.\,e.\ on\ \mathbb{R}_+.$$

Lemma 3.2.2 (Beesack [50]) *Let $v \in L_+^2(\mathbb{R}_+)$ and*

$$v(t) \leq f(t) + \int_0^t k(t,s)v(s)ds,\quad a.\,e.\ on\ \mathbb{R}_+. \tag{3.2.15}$$

If $u(t)$ is the unique L^2 solution of Eq. (3.2.14), then

$$v(t) \le u(t), \quad a.\,e. \;\; on \;\; \mathbb{R}_+. \tag{3.2.16}$$

Thompson [622] proved the following inequality.

Theorem 3.2.5 (Thompson [622]) *Let $f_i \in L^2_+(\mathbb{R}_+)$ and $k_i \in L^2_+(\mathbb{R}_+ \times \mathbb{R}_+)$, let u_i be the unique L^2 solution of*

$$u_i(t) = f_i(t) + \int_0^t k_i(t,s)u_i(s)ds, \tag{3.2.17}$$

where $i = 1, 2, \cdots, n$. If for all $t \in \mathbb{R}_+$,

$$F(t) = \sum_{i=1}^{n} f_i(t), \tag{3.2.18}$$

and for all $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$k(t,s) = \sum_{i=1}^{n} k_i(t,s), \tag{3.2.19}$$

then the unique L^2 solution u of

$$u(t) = F(t) + \int_0^t k(t,s)u(s)ds \tag{3.2.20}$$

exists, and satisfies

$$\sum_{i=1}^{n} u_i(t) \le u(t) \quad a.\,e. \;\; on \;\; \mathbb{R}_+. \tag{3.2.21}$$

Proof Since $f_i \in L^2_+(\mathbb{R}_+)$ and $k_i \in L^2_+(\mathbb{R}_+ \times \mathbb{R}_+)$, then $F \in L^2(\mathbb{R}_+)$ and $k \in L^2(\mathbb{R}_+ \times \mathbb{R}_+)$, so by Theorem 3.2.4 , $u(t)$ exists. The proof that $\Sigma_{i=1}^n u_i(t) \le u(t)$ a. e. on $\mathbb{R}_+$ is by induction on n.

The theorem is obvious for $n = 1$. Assume that it is true for $i = 1, 2, \cdots, n-1$. Let

$$v(t) = \sum_{i=1}^{n-1} f_i(t) + \int_0^t \sum_{i=1}^{n-1} k_i(t,s)v(s)ds.$$

Therefore,

$$u_n(t) + v(t) = f_n(t) + \sum_{i=1}^{n-1} f_i(t) + \int_0^t k_n(t,s)u_n(s)ds$$
$$+ \int_0^t \sum_{i=1}^{n-1} k_i(t,s)v(s)ds, \quad a.\ e.\ \ on\ \ \mathbb{R}_+.$$

It follows from Lemma 3.2.1 that $u_i(t) \geq 0$ a. e. on $\mathbb{R}_+$ for $i = 1, 2, \cdots, n$. Thus

$$u_n(t) + v(t) \leq \sum_{i=1}^{n-1} f_i(t) + \int_0^t \sum_{i=1}^{n} k_i(t,s)[u_n(s) + v(s)]ds,$$

or

$$u_n(t) + v(t) \leq F(t) + \int_0^t k(t,s)\Big(u_n(s) + v(s)\Big)ds, \quad a.\ e.\ \ on\ \ \mathbb{R}_+.$$

Therefore it follows from Lemma 3.2.2 that $u_n(t)+v(t) \leq u(t)$. But by our induction assumption, we have

$$\sum_{i=1}^{n-1} u_i(t) \leq v(t) \quad a.\ e.\ \ on\ \mathbb{R}_+.$$

which implies

$$\sum_{i=1}^{n} u_i(t) \leq u(t) \quad a.\ e.\ \ on\ \ \mathbb{R}_+.$$

□

Remark 3.2.1 The literature on the Volterra type inequalities is quite rich, we refer to the books by Lakshmikantham and Leela [338], Martinjuk and Gutowski [387] and Walter [634], and the references therein.

In the sequel, we first introduce some results of comparison theorems for the Volterra integral equations,although comparison theorems for ordinary differential equations (see, e.g., [116, 311]) are widely known and used.

Now we shall give two comparison theorems for Volterra integral equations of the second kind

$$y_i(x) = f_i(x) + \int_a^x K(x,t)y_i(t)dt, \qquad i = 1, 2. \tag{3.2.22}$$

In the real case, these theorems give us sufficient conditions for the validity of the inequality $y_1 \leq y_2$, where y_i is the unique solution of (3.2.22).

These theorems are simple consequences of well-known facts concerning integral equations of the form (3.2.22). These facts are first briefly summarized; we refer to [403] and [219] for the details. As in [219, 403], we shall deal with Lebesgue square integral functions throughout so that all equalities and inequalities are of the almost everywhere kind. We also give an upper bound for $\mid y_i(x) \mid$ in (3.2.22) and deal with integral inequalities related to Volterra integral equations of the second kind. These results are closely related to the positivity of the operator $L(y) = y - Ky$, and include an extension to the L^2 case of the recent theorem of Chu and Metcalfe [135]. As pointed out in [135], this theorem includes the classical Gronwall inequality and some of its linear extensions. We should also note that it was pointed out by Bellman in [47] that such inequalities are closely related to the positivity of operators.

For completeness, and because the following results are essential in the sequel, we state here the principal results for Volterra integral equations.

Let $I = [a, b]$, where $-\infty \leq a < b \leq +\infty$. By a Volterra type kernel on $I \times I$, we mean any complex-valued function $K \in L^2(I \times I)$ such that $K(x, t) = 0$ for $a \leq x < t \leq b$. By the resolvent kernel of K for the (complex) value λ, we mean the function Γ given by the series

$$\Gamma(x, t; \lambda) = \sum_{n=1}^{+\infty} \lambda^{n-1} K^{(n)}(x, t), \tag{3.2.23}$$

where $K^{(1)} = K$, and for all $n \geq 2$,

$$K^{(n)}(x, t) = \int_I K(x, s) K^{(n-1)}(s, t) ds = \int_t^x K(x, s) K^{(n-1)}(s, t) ds. \tag{3.2.24}$$

Each of the iterated kernels $K^{(n)}$ is also a Volterra type kernel on $I \times I$. The above series is, for each complex λ, to mean square convergent on $I \times I$, and Γ is, for almost all $(x, t) \in I \times I$, an entire function of λ. Moreover, Γ is also a Volterra type kernel on $I \times I$.

Without loss of generality, we shall always take $\lambda = 1$, and write $\Gamma(x, t; 1) = \Gamma(x, t)$. For each complex-valued function $f \in L^2(I)$, the Volterra integral equation for all $x \in I$,

$$y(x) = f(x) + \int_a^x K(x, t) y(t) dt, \tag{3.2.25}$$

has a unique L^2-solution y given by the formula for all $x \in I$,

$$y(x) = f(x) + \int_a^x \Gamma(x, t) f(t) dt. \tag{3.2.26}$$

Whenever the function K_1 defined by, for all $x \in I$,

$$K_1(x) = \sup_{a \le t \le x} | K(x,t) |, \tag{3.2.27}$$

is integrable over I, we may apply the method used by Bellman in [61] to obtain a bound for $| y(x) |$, as given in the next theorem.

Theorem 3.2.6 (Beesack [49]) *Let K be a Volterra type kernel on $I \times I$, and let $f \in L^2(I)$. If the function K_1 defined by (3.2.27) is integrable over I, and if y is the unique L^2-solution of (3.2.25), then for all a. e. on I,*

$$\begin{aligned} | y(x) | \le | f(x) | &+ K_1(x) \exp\left(\int_a^x K_1(s)ds\right) \\ &\times \int_a^x | f(t) | \exp\left(-\int_a^t K_1(s)ds\right) dt. \end{aligned} \tag{3.2.28}$$

Proof Noting that from (3.2.25), we derive

$$| y(x) | \le | f(x) | + K_1(x) \int_a^x | y(t) | \, dt.$$

Hence, setting $R(x) = \int_a^x | y(t) | \, dt$, we obtain,

$$R'(x) - K_1(x)R(x) \le | f(x) | \, .$$

The rest of the proof follows more or less as in Theorem 3.2.10 below. □

We note that by using the resolvent equation satisfied by Γ, it follows that

$$\begin{aligned} | \Gamma(x,t) | \le | K(x,t) | &+ K_1(x,t) \exp\left(\int_t^x K_1(s,t)ds\right) \\ &\times \int_t^x | K(u,t) | \exp\left(-\int_t^u K_1(s,t)ds\right) du \end{aligned} \tag{3.2.29}$$

almost everywhere on $I \times I$, provided that for all $(x,t) \in I \times I$,

$$K_1(x,t) = \max_{t \le s \le x} | K(x,s) |,$$

is integrable over I for almost all $t \in I$.

We begin with a simple lemma which, in the case that f and K are non-negative, gives us an obvious lower bound for the unique L^2-solution of Eq. (3.2.26).

Lemma 3.2.3 (Beesack [49]) *Let $f \in L^2(I)$ with $f \geq 0$ on I, and K be a non-negative Volterra type kernel on $I \times I$. If Γ is the* resolvent kernel *of K (for the value $\lambda = 1$), and if y is the unique L^2-solution of problem (3.2.26), then*

$$\Gamma(x,t) \geq K(x,t), \quad a.\ e.\ on\ I \times I, \tag{3.2.30}$$

$$y(x) \geq f(x), \quad a.\ e.\ on\ I. \tag{3.2.31}$$

Proof It follows at once from (3.2.24) that all the iterated kernels of K are non-negative on $I \times I$. Setting $\lambda = 1$ in (3.2.23), we obtain (3.2.30). The inequality (3.2.31) follows from (3.2.30) and the representation (3.2.26) of y since $f \geq 0$. □

We now apply Lemma 3.2.3 to prove our first comparison theorem for Volterra integral equations.

Theorem 3.2.7 (Beesack [49]) *Let $f_i \in L^2(I)$, and let K_i, $i = 1, 2$, be Volterra type kernels on $I \times I$, satisfying*

$$\mid f_1(x) \mid \leq f_2(x) \;\; a.\ e.\ \ on\ I, \quad \mid K_1(x,t) \mid \leq K_2(x,t) \;\; a.\ e.\ \ on\ I \times I. \tag{3.2.32}$$

If y_i is the unique L^2-solution of the integral equation

$$y_i(x) = f_i(x) = \int_a^x K_i(x,t) y_i(t) dt, \tag{3.2.33}$$

then $\mid y_1(x) \mid \leq y_2(x)$ a. e. on I.

In fact,

$$y_2(x) - \mid y_1(x) \mid \geq f_2(x) - \mid f_1(x) \mid \quad a.\ e.\ on\ I. \tag{3.2.34}$$

Proof Note that f_2 and K_2 are real-valued, non-negative functions, whereas f_1, K_1 could be complex-valued. By Lemma 3.2.3, it follows that $y_2(x) \geq f_2(x) \geq 0$. Noting that

$$y_2(x) = f_2(x) + \int_a^x K_2(x,t) y_2(t) dt,$$

$$\begin{aligned} \mid y_1(x) \mid &\leq \mid f_1(x) \mid + \int_a^x \mid K_1(x,t) \mid\mid y_1(t) \mid dt \\ &\leq \mid f_1(x) \mid + \int_a^x K_2(x,t) \mid y_1(t) \mid dt, \end{aligned}$$

it follows that

$$y_2(x) - \mid y_1(x) \mid \geq f_2(x) - \mid f_1(x) \mid + \int_a^x K_2(x,t) \{ y_2(t) - \mid y_1(t) \mid \} dt.$$

Letting g denote the (positive) difference between the left-hand side and the right-hand side of the above inequality, we have $g \in L^2(I)$, and

$$y_2(x)-\mid y_1(x)\mid = g(x) + \Big(f_2(x)-\mid f_1(x)\mid\Big) + \int_a^x K_2(x,t)\Big(y_2(t)-\mid y_1(t)\mid\Big)dt.$$

Since $g + (f_2 - \mid f_1\mid) \in L^2(I)$, it follows from (3.2.32) and Lemma 3.2.3 that

$$y_2(x)-\mid y_1(x)\mid \geq g(x) + \Big(f_2(x)-\mid f_1(x)\mid\Big),$$

which gives us (3.2.34). □

From now on, we shall deal only with real-valued functions. In this case, the following comparison theorem is sometimes applicable when Theorem 3.2.7 is not.

Theorem 3.2.8 (Beesack [49]) *Let $f_i \in L^2(I)$, and $K_i, i = 1, 2$, be Volterra type kernels on $I \times I$, satisfying*

$$0 \leq f_2(x), \quad f_1(x) \leq f_2(x) \quad a.\,e.\ on\ I, \tag{3.2.35}$$

$$0 \leq K_1(x,t) \leq K_2(x,t) \quad a.\,e.\ on\ I \times I. \tag{3.2.36}$$

If y_i is the unique L^2-solution of the integral equation (3.2.33), then

$$y_2(x) - y_1(x) \geq f_2(x) - f_1(x) \quad a.\,e.\ on\ I. \tag{3.2.37}$$

Proof Let Γ_i denote the resolvent kernel of $K_i, i = 1, 2$. As in Lemma 3.2.3, it follows from the third of the inequalities (3.2.35)–(3.2.36) that

$$0 \leq \Gamma_1(x,t) \leq \Gamma_2(x,t) \quad a.\,e.\ on\ I \times I. \tag{3.2.38}$$

Using an obvious operator notation, we have $y_i = f_i + \Gamma_i f_i$, hence

$$\begin{aligned} y_2 - y_1 &= (f_2 - f_1) + \Gamma_2 f_2 - \Gamma_1 f_1 \\ &= (f_2 - f_1) + (\Gamma_2 - \Gamma_1) f_2 + \Gamma_1(f_2 - f_1). \end{aligned}$$

The inequality (3.2.37) now follows from (3.2.8) and the first two of the inequalities (3.2.35)–(3.2.36). □

Corollary 3.2.1 (Beesack [49]) *Under the hypotheses (3.2.35)–(3.2.36), (3.2.37) can be improved to*

$$y_2 - y_1 \geq (f_2 - f_1) + K_1(f_2 - f_1) \quad a.\,e.\ on\ I. \tag{3.2.39}$$

Proof This follows from Theorem 3.2.8 together with (3.2.30) of Lemma 3.2.3 □

In the same way, we observe that the conclusions of Lemma 3.2.1 and Theorem 3.2.7 can be somewhat improved.

Remark 3.2.2 Keeping the hypothesis $K \geq 0$, we see that if $f \leq 0$ on I, then the conclusion (3.2.31) of Lemma 3.2.3 becomes $y \leq f$, or better, $y \leq f + Kf$. Similarly, if the inequalities are reversed, then so also are the inequalities (3.2.37), (3.2.39).

Remark 3.2.3 Since any initial value problem for linear differential equations can be reduced to a Volterra integral equation of the second kind (see, [625]), the above results can be used to obtain comparison theorems for such problems.

Next, we discuss the positivity of Volterra operators. If K is a Volterra type kernel on $I \times I$, we define the linear operator L on $L^2(I)$ by

$$L(u) = u - Ku. \tag{3.2.40}$$

Following Beckenbach and Bellman [47], the Volterra operator L is said to be positive if $L(u) \geq 0$ implies that $u \geq 0$. Lemma 3.2.3 provides an immediate sufficient condition for the positivity of L.

Theorem 3.2.9 (Beesack [49]) *The operator L defined by (3.2.40) is positive if $K(x,t) \geq 0$, a. e. on $I \times I$.*

For, if $L(u) = f \geq 0$, then $f \in L^2(I)$, and $u = f + Ku$. Hence $u \geq f \geq 0$ by Lemma 3.2.3.

Similarly, according to Remark 3.2.2, L is negative if K is non-positive on $I \times I$.

As pointed out in [47], the inequalities of Gronwall-Bellman are closely related to the question of the positivity of operators. The following result which includes these inequalities is a consequence of Theorem 3.2.9, and is a slight extension of Theorem in [135].

Theorem 3.2.10 (Beesack [49]) *Let $f \in L^2(I)$, and let K be a non-negative Volterra kernel on $I \times I$, and*

$$u(x) \leq f(x) + \int_a^x K(x,t)u(t)dt, \quad a.\ e.\ on\ I, \tag{3.2.41}$$

then

$$u(x) \leq y(x)\ a.\ e.\ on\ I, \tag{3.2.42}$$

where y is the unique L^2-solution of the Volterra integral equation $y = f + Ky$ (Similarly, if the inequality is reversed in (3.2.41), then $u(x) \geq y(x)$ a. e. on I).

Proof Set $u - Ku = g$, so $u = g + Ku$ with $g \in L^2(I)$ and $g \leq f$. Then $L(y-u) = L(y) - L(u) = f - g \geq 0$ and the conclusion follows from Theorem 3.2.9. $\square$

We finally conclude remark concerning possible converses of Theorem 3.2.10. We may, equivalently, formulate such questions in terms of the operator L defined

by (3.2.9). In general, we may ask what conditions on K and u will guarantee that $L(u) \geq 0$? If we consider only non-negative Volterra kernels K, it is clear that the condition $u \geq 0$ is not a sufficient condition for $L(u) \geq 0$, although it is necessary by Theorem 3.2.9. However, the simplest sufficient conditions for $L(u) \geq 0$ turns out to be

$$u \geq 0, \quad u\ non-decreasing\ on\ I, \quad K \geq 0, \quad \int_I K(x,t)dt \leq 1, \tag{3.2.43}$$

these conditions are quite restrictive.

3.3 The Singular Generalizations on the Gronwall-Bellman Inequalities-Henry's Type

In the contemporary theory of semilinear parabolic differential equations, PDEs are studied as evolutions in appropriate infinite dimensional Banach spaces. Linear operators defining linear parts of such equations are unbounded linear operators and it is impossible to apply all standard methods generally used in the theory of ODEs with finite dimensional state spaces. Special semilinear PDEs lead to so-called sectorial evolution equation whose linearizations are defined by sectorial operators (see, e.g., Henry [271] and Hale [251]). These equations can be written as Volterra integral equations with weakly singular kernels. Therefore there are problems with many modifications of estimates usually employed in the theory of ODEs on finite dimensional spaces. Among the basic tools of finite dimensional theory are the well-known Gronwall linear inequalities and also the well-known Bihari nonlinear inequality (see, e.g., [394, 395, 449]). However, the infinite dimensional theory requires solving integral inequalities with singular kernels. Henry [271] proposed a method to find solutions of such inequalities and proved some results concerning linear integral inequalities of this type. A modification of Henry's theorem concerning more general linear integral inequalities was recently proved by Sano and Kunimatsu [572]. All these results have been proved by an iteration argument, and resulting estimates are expressed as integrals with singular kernels from functions defined by power series of a very complicated form which may be convenient for applications.

3.3.1 The One-Dimensional Henry Inequalities

In this section, we present a new method to solve nonlinear integral inequalities of Henry type and their Bihari nonlinear version. The present estimates are quite simple and the resulting formulas are similar to those in the classical Gronwall-Bihari inequalities. We also present results on integral inequalities containing multiple

integrals which are modifications of the results recently published in [395] (see, e.g., [394]). Some modifications of a result by Pachpatte [449] concerning the classical type of integral inequalities are also made here.

Theorem 3.3.1 (The Henry Inequality [272]) *Assume $b \geq 0, \beta > 0$ and let $a(t)$ be a non-negative function locally integrable on $0 \leq t < T$ (for some $T \leq +\infty$), and assume that $u(t)$ is non-negative and locally integrable on $0 \leq t < T$ and satisfies*

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} u(s) ds \tag{3.3.1}$$

on this interval, then for all $t \in [0, T)$,

$$u(t) \leq a(t) + \int_0^t \left(\sum_{n=1}^{+\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right) ds. \tag{3.3.2}$$

Proof Let $B\phi(t) = b \int_0^t (t-s)^{\beta-1}\phi(s)ds,\ t \geq 0$, for locally integrable function ϕ. Then $u \leq a + Bu$ implies $u \leq \sum_{k=0}^{n-1} B^k a + B^n u$, and $B^n u(t) = \int_0^t (b\Gamma(\beta))^n (t-s)^{n\beta-1} u(s)ds/\Gamma(n\beta) \to 0$ as $n \to +\infty$ for each $t \in [0, T)$. Then from (3.3.1) it follows

$$u(t) \leq a(t) + \int_0^t \sum_{n=1}^{+\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) ds. \tag{3.3.3}$$

□

Remark 3.3.1 In fact, the original form of *(3.3.2)* in Henry [272] should be the following

$$u(t) \leq a(t) + \theta \int_0^t E'_\beta(\theta(t-s)) a(s) ds \tag{3.3.4}$$

where

$$\theta = [b\Gamma(\beta)]^{1/\beta}, \quad E_\beta(z) = \sum_{n=0}^{+\infty} z^{n\beta}/\Gamma(n\beta+1), \quad E'_\beta(z) = \frac{d}{dz} E_\beta(z),$$

and $E'_\beta(z) \simeq z^{\beta-1}/\Gamma(\beta)$ as $z \to 0^+$, $E'_\beta(z) \simeq e^z/\beta$ as $z \to +\infty$, and $E_\beta(z) \simeq e^z/\beta$ as $z \to +\infty$. If $a(t) \equiv a$, a constant, then

$$u(t) \leq aE_\beta(\theta t). \tag{3.3.5}$$

Theorem 3.3.2 (Henry [272]) *Assume $\beta > 0, \gamma > 0, \beta + \gamma > 1$ and $a \geq 0, b \geq 0$, and let u be non-negative and $t^{\gamma-1}u(t)$ locally integrable on $0 \leq t < T$, and satisfy for a. e. $t \in (0, T)$,*

$$u(t) \leq a + b \int_0^t (t-s)^{\beta-1} s^{\gamma-1} u(s) ds, \tag{3.3.6}$$

then

$$u(t) \leq a E_{\beta,\gamma} \left((b\Gamma(\beta))^{1/\nu} t \right) \tag{3.3.7}$$

where $\nu = \beta + \gamma - 1 > 0, E_{\beta,\gamma}(s) = \sum_{m=0}^{+\infty} C_m s^{m\nu}$ with $C_0 = 1, C_{m+1}/C_m = \Gamma(m\nu + \gamma)/\Gamma(m\nu + \gamma + \beta)$ for $m \geq 0$. As $s \to +\infty$, we have

$$E_{\beta,\gamma}(s) = O\left(s^{1/2(\nu/\beta - \gamma)} \exp\left(\frac{\beta}{\gamma} s^{\nu/\beta} \right) \right). \tag{3.3.8}$$

Proof If

$$B\phi(t) = b \int_0^t (t-s)^{\beta-1} s^{\gamma-1} \phi(s) ds,$$

then an easy induction from (3.3.6) shows

$$u(t) \leq a \sum_{m=0}^{n} C_m [b\Gamma(\beta)]^m t^{m\nu} + B^{n+1} u(t). \tag{3.3.9}$$

Also

$$B^n u(t) = \int_0^t K_n(t,s) s^{\gamma-1} u(s) ds, \tag{3.3.10}$$

where for $\gamma \geq 1$,

$$\begin{cases} K_n(t,s) \leq Q_n t^{(n-1)(\gamma-1)} (t-s)^{n\beta-1}, \\ Q_1 = b, \; Q_{n+1}/Q_n = b\Gamma(\beta)/\Gamma(n\beta)/\Gamma(n\beta+\beta). \end{cases}$$

If $\gamma \in (0, 1)$, then we have

$$\begin{cases} K_n(t,s) \leq Q_n (t-s)^{n\nu-\gamma}, \quad Q_1 = b, \\ Q_{n+1}/Q_n = b\Gamma(\beta)/\Gamma(n\nu)/\Gamma(n\nu+\beta). \end{cases}$$

In either case, $Q_{n+1}/Q_n = O(n^{-\beta})$ as $n \to +\infty$, so $B^n u(t) \to 0$ as $n \to +\infty$, and

$$u(t) \leq a E_{\beta,\gamma} \left([b\Gamma(\beta)]^{1/\nu} t \right). \tag{3.3.11}$$

Now $\Gamma(z+p)/\Gamma(z+q) = z^{p-q}\{1+(p-q)(p+q-1)/2z+O(z^{-2})\}$ as $z \to +\infty$ so if

$$\delta = (\beta\gamma + \nu)/2\nu, \quad \frac{\Gamma((n+1)\beta + \delta)C_{n+1}}{\Gamma(n\beta + \delta)C_n} = (\beta/\nu)^{\beta}[1 + O(n^{-2})]. \tag{3.3.12}$$

Thus $C_n\Gamma(n\beta + \delta)(\beta/\nu)^{-n\beta}$ converges as $n \to +\infty$ and has an upper bound K for all $n \geq 0$. Then for any $s > 0$,

$$E_{\beta,\gamma}(s^{\beta/\nu})s^{\delta-1} \leq K\sum_{n=0}^{+\infty} \frac{(\beta/\nu)^{n\beta}}{\Gamma(n\beta + \delta)} s^{n\beta+\delta-1}. \tag{3.3.13}$$

The Laplace transform of the right-hand side of (3.3.13) is

$$K\lambda^{-\delta}/\Big[1 - (\beta/\nu\lambda)^{\beta}\Big],$$

therefore, the series is $O(\exp(\beta s/\nu))$ as $s \to +\infty$ which proves the result.

The estimate of $E_\beta(z)$ and $E'_\beta(z)$ as $z \to +\infty$ follows from the fact that the Laplace transform

$$\int_0^{+\infty} e^{-\lambda z}E_\beta(z)dz = \lambda^{-1}/(1 - \lambda^{-\beta})$$

has a simple pole at $\lambda = 1$ (see, e.g., Evgrafov [204]). For example, we can choose $\gamma \in (0, 1)$ such that $1 - \lambda^{-\beta} \neq 0$ for $Re\lambda \geq \gamma, \lambda \neq 1$, and then for all $z > 0$,

$$E_\beta(z) = \frac{1}{\beta}e^z + \frac{1}{2\pi i}\lim_{n\to+\infty}\int_{\gamma-iN}^{\gamma+iN} e^{\lambda z}\lambda^{-1}/(1 - \lambda^{-\beta})d\lambda \tag{3.3.14}$$

where the shift in the line of integration is justified by $e^{\lambda z}\lambda^{-1}/(1 - \lambda^{-\beta}) \to 0$ as $Im\lambda \to \pm\infty$ for $Re\lambda$ bounded. Integration by parts in the integral on the right-hand side of (3.3.14) shows as $z \to +\infty$,

$$\mid E_\beta(z) - \frac{1}{\beta}e^z \mid = O(e^z).$$

□

The following result was established by Nagumo in [425], which can be viewed as a generalization of Theorem 3.3.2 for a special case when $a \equiv 0, \beta = 1, \gamma = 0\ (\beta + \gamma = 1)$.

Corollary 3.3.1 (Nagumo [425]) *Given a non-negative function* $v(t) \in C[0, b]$ *such that* $v(0) = 0$ *and* $\lim_{h\to 0^+} v(h)/h = 0$. *If* $v(t)$ *satisfies for all* $t \in (0, b]$,

$$v(t) \le \int_0^t v(s)/s ds, \tag{3.3.15}$$

then $v(t) \equiv 0$, *for all* $t \in [0, b]$.

Proof For all $t > 0$, $\varepsilon > 0$, $t > \varepsilon$, let $F(t) = \int_0^t v(s)/s ds$ for $s \in [\varepsilon, t]$. Then we have $F'(t) = v(t)/t$. If we add the condition $F'(0) = 0$, then since $\lim_{h\to 0^+} v(h)/h = 0$, we deduce that $F'(t) \in C[0, b]$. By (3.3.15), we know that $F'(t) = v(t)/t \le F(t)/t$ for all $t > 0$, that is, for all $t > 0$,

$$\big(\log F(t)\big)' \le 1/t. \tag{3.3.16}$$

Integrating (3.3.16) over $[\varepsilon, t]$ for any $\varepsilon > 0$ $(t \ge \varepsilon)$ implies that

$$F(t) \le F(\varepsilon)t/\varepsilon. \tag{3.3.17}$$

By the L'Hospital Rule and noting that $\lim_{h\to 0^+} v(h)/h = 0$, we deduce for all fixed $t > 0$,

$$F(t) \le t \lim_{\varepsilon\to 0^+} F(\varepsilon)/\varepsilon = t \lim_{\varepsilon\to 0^+} F'(\varepsilon) = 0$$

which implies for all $t \in (0, b]$,

$$v(t) = 0. \tag{3.3.18}$$

Combining $v(0) = 0$ and (3.3.18), we complete the proof. □

In the same manner, we may prove the following result (see, e.g., Henry [272]); the proof is left to the reader.

Theorem 3.3.3 (Henry [272]) *If* α, β, γ *are positive with* $\beta + \gamma - 1 = \nu > 0, \delta = \alpha + \gamma - 1 > 0$, *and for all* $t > 0$,

$$u(t) \le at^{\alpha-1} + b\int_0^t (t-s)^{\beta-1} s^{\gamma-1} u(s) ds, \tag{3.3.19}$$

then

$$u(t) \le at^{\alpha-1} \sum_{m=0}^{+\infty} C'_m (b\Gamma(\beta))^m t^{m\nu} \tag{3.3.20}$$

where $C'_0 = 1, C'_{m+1} = C'_m = \Gamma(m\nu + \delta)/\Gamma(m\nu + \delta + \beta)$.

Corollary 3.3.2 (Henry [272]) *Under the hypotheses of Theorem 3.3.3, let $a(t)$ be a non-decreasing function on $[0, T)$. Then*

$$u(t) \leq a(t)E_\beta \left(g(t)\Gamma(\beta)t^\beta\right), \tag{3.3.21}$$

where E_β is the Mittag-Leffler function defined by $E_\beta(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\beta+1)}$.

In order to formulate the following singular Gronwall-Bellman inequality (see, e.g., Theorem 3.3.4) which can be viewed as a generalization of the above theorem (see, e.g., Amann [28]), we need to introduce some basic concepts.

By a vector space, we always understand a vector space over K where $K = \mathbb{R}$ or $K = \mathbb{C}$. If M is a subset of a vector space, we set

$$\dot{M} := M\backslash\{0\}.$$

If X is a topological space, by $BC(X, E)$ we denote the closed linear subspace of $B(X, E)$ consisting of all bounded and continuous functions.

Let J be a perfect subinterval of $\mathbb{R}$. Then

$$J_\Delta := \left\{(t, s) \in J \times J : \ s \leq t\right\}$$

and we set

$$J_\Delta^* := \left\{(t, s) \in J_\Delta : \ s < t\right\}.$$

Assume that J is a perfect subinterval of $\mathbb{R}_+$ containing 0 and let

$$J_T := J \cap [0, T], \quad T \in \mathbb{R}^+.$$

Given any $\alpha \in \mathbb{R}$, by $\mathcal{K}(E, F, \alpha)$ we denote the Fréchet space of all $k \in C(J_\Delta^*, \mathcal{L}(E, F))$ satisfying

$$||k||_{(\alpha),T} := ||k||_{(\alpha),T,\mathcal{L}(E,F)} := \sup_{0\leq s<t\leq T} (t-s)^\alpha ||k(t, s)||_{\mathcal{L}(E,F)} < +\infty, \quad T \in \dot{J},$$

equipped with the topology induced by the seminorms $\{||\cdot||_{(\alpha),T} : \ T \in \dot{J}\}$. Moreover, $\mathcal{K}(E, \alpha) := \mathcal{K}(E, E, \alpha)$. Note that for all $T \in \dot{J}$,

$$|| \cdot ||_{(\alpha),T} \leq T^{\alpha-\beta} || \cdot ||_{(\beta),T}, \quad \alpha > \beta, \tag{3.3.22}$$

so that

$$\mathcal{K}(E, F, \beta) \hookrightarrow \mathcal{K}(E, F, \alpha), \quad \alpha > \beta. \tag{3.3.23}$$

Let

$$||k||_{(\alpha)} := \sup_{(t,s)\in J_\Delta^*} (t-s)^\alpha ||k(t,s)||_{\mathcal{L}(E,F)}$$

and denote by

$$\mathcal{K}_\infty(E, F, \alpha)$$

the Banach space consisting of all $k \in \mathcal{K}(E, F, \alpha)$ satisfying $||k||_{(\alpha)} < +\infty$, equipped with norm $||\cdot||_{(\alpha)}$. Note that

$$\mathcal{K}_\infty(E, F, \alpha) \hookrightarrow \mathcal{K}(E, F, \alpha) \tag{3.3.24}$$

and

$$\mathcal{K}_\infty(E, F, 0) = BC(J_\Delta^*, \mathcal{L}(E, F)). \tag{3.3.25}$$

If $\alpha < 0$, each $k \in \mathcal{K}(E, F, \alpha)$ can be continuously extended over J_Δ by putting $k(t,t) = 0$ for all $t \in J$ so that

$$\mathcal{K}(E, F, \alpha) \hookrightarrow C(J_\Delta, \mathcal{L}(E, F)), \quad \alpha < 0. \tag{3.3.26}$$

If $E = K$, we canonically identify $\mathcal{L}(K, F)$ with F via

$$\mathcal{L}(K, F) \ni B \leftrightarrow B \cdot 1 \in F.$$

Then $k \in \mathcal{K}(K, F, \alpha)$ if and only if $k \in C(J_\Delta^*, F)$ and $T \in \dot{J}$,

$$\sup_{0\le s<t\le T} (t-s)^\alpha ||k(t,s)||_{\mathcal{L}(E,F)} < +\infty,$$

In particular, we have an embedding

$$BC(\dot{J}, F) \hookrightarrow \mathcal{K}_\infty(K, F, 0) = BC(J_\Delta^*, F) \tag{3.3.27}$$

by the identification

$$C(\dot{J}, F) \ni u \leftrightarrow \Big[(t,s) \mapsto u(t)\Big] \in C(J_\Delta^*, F). \tag{3.3.28}$$

Let G be a Banach space. Assume that $k \in \mathcal{K}(E, F, \alpha)$ and $h \in \mathcal{K}(F, G, \beta)$ with $\alpha, \beta \in (-\infty, 1)$, and set

$$h * k(t,s) := \int_s^t h(t,\tau)k(\tau,s)d\tau, \quad (t,s) \in J_\Delta.$$

It is easy to verify that

$$h * k \in \mathcal{K}(E, G, \alpha + \beta - 1) \tag{3.3.29}$$

and

$$||h * k||_{(\alpha+\beta-1),T} \leq B(1-\alpha, 1-\beta)||h||_{(\alpha),T}||k||_{(\beta),T}, \quad T \in \dot{J}, \tag{3.3.30}$$

where B is Euler's beta function. It is a consequence of Fubini's theorem that the operation $*$ is associative.

Assume that $k \in \mathcal{K}(E, \alpha)$ for some $\alpha \in [0, 1)$. By an easy induction argument, we conclude that

$$||\underbrace{k * k * \cdots * k(t,s)}_{n}||_{\mathcal{L}(E)} \leq \frac{[\Gamma(1-\alpha)||k||_{(\alpha),T}]^n}{\Gamma(n(1-\alpha))}(t-s)^{n(1-\alpha)-1} \tag{3.3.31}$$

for all $n \in \mathbb{N}$ and $0 \leq s < t \leq T$. Set

$$\omega := \sum_{j=1}^{+\infty} \underbrace{k * \cdots * k}_{j}. \tag{3.3.32}$$

In the sequel, we shall prove the following generalized Gronwall-Bellman inequality. First, we give the following two lemmas.

Lemma 3.3.1 (Amann [29]) *The function $w \in \mathcal{K}(E, \alpha)$ satisfies the estimate ,*

$$(t-s)^\alpha ||w(t,s)||_{\mathcal{L}(E)} \leq c(\alpha, \varepsilon) m e^{(1+\varepsilon)m^{1/(1-\alpha)}(t-s)} \tag{3.3.33}$$

for any given $\varepsilon > 0$ and $0 \leq s < t \leq T, T \in \dot{J}$, where

$$m := \Gamma(1-\alpha)||k||_{(\alpha),T}.$$

Proof Let $\beta := 1-\alpha \in (0, 1]$. Thanks to (3.3.31), it suffices to prove, for all $x > 0$,

$$\sum_{j=1}^{+\infty} \frac{x^{j-1}}{\Gamma(\beta j)} \leq c(\beta, \varepsilon) e^{(1+\varepsilon)x^{1/\beta}}. \tag{3.3.34}$$

Stirling's formula implies the existence of $\theta(t) \in (0, 1)$ such that

$$\Gamma(t) = \sqrt{2\pi} t^{t-1/2} e^{-t+\theta(t)/(12t)}, \quad t > 0,$$

from which we deduce for all $j \in \mathbb{N}$ that

$$\frac{\Gamma(j+1)^\beta}{\Gamma(\beta j)} = \frac{[j\Gamma(j)]^\beta}{\Gamma(\beta j)} \le (2\pi)^{\frac{\beta-1}{2}} e^{\frac{\beta}{12}} \beta^{1/2} \frac{j^{(1+\beta)/2}}{\beta^{\beta j}}.$$

Hence, by Hölder's inequality,

$$\begin{aligned}\sum_{j=1}^{+\infty} \frac{x^j}{\Gamma(\beta j)} &= \sum_{j=1}^{+\infty} \frac{x^j}{(j!)^\beta} \frac{\Gamma(j+1)^\beta}{\Gamma(\beta j)} \\ &\le c(\beta)\left[\sum_{j=1}^{+\infty} \frac{(\eta x^{1/\beta})^j}{j!}\right]^\beta \left[\sum_{j=1}^{+\infty} \frac{j^{(1+\beta)/(2(1-\beta))}}{(\eta\beta)^{j\beta/(1-\beta)}}\right]^{1-\beta}\end{aligned}$$

where $\eta > 0$ is arbitrary. Since the last series converges for $\eta > 1/\beta$, it follows that

$$\sum_{j=1}^{+\infty} \frac{x^{j-1}}{\Gamma(\beta j)} \le c(\beta,\eta)\left(\frac{e^{\eta x^{1/\beta}} - 1}{\eta x^{1/\beta}}\right)^\beta \le c(\beta,\eta) e^{\beta\eta x^{1/\beta}}$$

for all $x > 0$ and $\eta > 1/\beta$, which implies (3.3.33) for $\eta := (1+\varepsilon)/\beta$. $\square$

Now we easily prove the following existence and uniqueness theorem for abstract linear Volterra equations.

Lemma 3.3.2 (Amann [29]) *Assume that* $\alpha, \beta \in [0,1)$ *and* $k \in \mathcal{K}(E,\alpha)$. *Then the linear Volterra equations*

$$u = a + u * k, \quad v = b + k * v \tag{3.3.35}$$

possess for each $(a,b) \in \mathcal{K}(E,F,\beta) \times b \in \mathcal{K}(F,E,\beta)$ *a unique solution* (u,v) *such that*

$$u \in \mathcal{K}(E,F,\beta), \qquad v \in \mathcal{K}(F,E,\beta),$$

which are given by

$$u = a + a * \omega, \quad v = b + \omega * b \tag{3.3.36}$$

respectively, where ω*, the resolvent kernel of* (3.3.35)*, belongs to* $\mathcal{K}(E,\alpha)$ *and is given by* (3.3.32).

Proof We consider the first equation in (3.3.35). The second one can be treated in a same manner.

Define ω by (3.3.32) and u by (3.3.36), and observe that $\omega \in \mathcal{K}(E,\alpha)$ and $u \in \mathcal{K}(E,F,\beta)$ by Lemma 3.3.1 and by (3.3.23) and (3.3.29), respectively. It is obvious that u solves (3.3.35).

Let $T \in \dot{J}$ be fixed. By replacing J by J_T, it follows from (3.3.22), (3.3.23), (3.3.29), (3.3.30) and (3.3.31) that $*k \in \mathcal{L}(\mathcal{K}_\infty(E, F, \beta))$ and that the spectral radius of this operator equals zero. Hence (3.3.35) has at most one solution 'on J_T' for each $T \in \dot{J}$. This proves the assertion. □

Remark 3.3.2 In the definition of $\mathcal{K}(E, F, \alpha)$, we can replace the assumption that $k \in C(J_\Delta^*, \mathcal{L}(E, F))$. Then everything remains true if the following hold

(1) $\sup_{0 \le s < t \le T}$ is replaced by $ess - \sup_{0 \le s < t \le T}$ everywhere.
(2) *(3.3.26)* is replaced by

$$\mathcal{K}(E, F, \alpha) \cap C(J_\Delta^*, \mathcal{L}(E, F)) \hookrightarrow C(J_\Delta, \mathcal{L}(E, F)), \quad \alpha < 0.$$

(3) $C(J_\Delta^*, F)$ is replaced by $L_{\infty,loc}(J_\Delta^*)$ in the interpretation of $\mathcal{K}(K, F, \alpha)$.
(4) BC is replaced by L_∞ in *(3.3.25)* and *(3.3.27)*.

Note that with this new definition of $\mathcal{K}(E, F, \alpha)$, and by using obvious notation,

$$\mathcal{K}(F, G, \beta) * \mathcal{K}(E, F, \alpha) \hookrightarrow \mathcal{K}(E, G, \alpha + \beta - 1) \cap C(J_\Delta, \mathcal{L}(E, G))$$

if $\alpha + \beta < 1$.

By Lemma 3.3.2, we prove the following generalized Gronwall-Bellman inequality.

Theorem 3.3.4 (Amann [29]) *Given $\alpha, \beta \in [0, 1)$ and $\varepsilon > 0$, there exists a positive constant $c := c(\alpha, \beta, \varepsilon)$ such that the following is true. If $u : J \to \mathbb{R}$ satisfies*

$$\left[t \mapsto t^\beta u(t) \right] \in L_{\infty,loc}(J, \mathbb{R}) \tag{3.3.37}$$

and for a. e. $t \in \dot{J}$,

$$u(t) \le At^{-\beta} + B \int_0^t (t - \tau)^{-\alpha} u(\tau) d\tau, \tag{3.3.38}$$

where A and B are positive constants, then a. e. $t \in \dot{J}$,

$$u(t) \le At^{-\beta} \left(1 + cBt^{1-\alpha} e^{(1+\varepsilon)\mu(\alpha,B)t} \right), \tag{3.3.39}$$

where $\mu(\alpha, B) := \Big(\Gamma(1 - \alpha) B \Big)^{1/(1-\alpha)}$.

Proof Let $E := F := \mathbb{R}$ and $k(t, s) := B(t - s)^{-\alpha}$ for all $(t, s) \in J_\Delta^*$. Then we see that $k \in \mathcal{K}(E, \alpha)$ and $||k||_{(\alpha),T} = B$ for all $T \in \dot{J}$. Let $a(t) := At^{-\beta}$ and observe that (3.3.28) implies $a \in \mathcal{K}(E, \beta)$ and $||a||_{(\beta),T} = A$ for all $T \in \dot{J}$. Since

$u \in \mathcal{K}(E, \beta)$ by (3.3.37) and Remark 3.3.2, it follows from (3.3.29) and (3.3.23) that

$$b := a + k * u - u \in \mathcal{K}(E, \beta).$$

Hence $u = a - b + k * u$, and Lemma 3.3.2 implies that

$$u = (a - b) + \omega * (a - b).$$

Observe that $b \geq 0$ by (3.3.38) and that $k \geq 0$ implies $\omega \geq 0$. Thus $u \leq a + \omega * a$, that is, for almost all $t \in \dot{J}$,

$$u(t) \leq At^{-\beta} + A \int_0^t \omega(t - \tau)\tau^{-\beta} d\tau.$$

By Lemma 3.3.1, we have

$$\omega(t) \leq c(\alpha, \varepsilon)Bt^{-\alpha} e^{(1+\varepsilon)\mu(\alpha,B)t}, \quad t > 0, \quad \varepsilon > 0.$$

Since for all $t > 0$ and $\nu \geq 0$,

$$\begin{aligned} \int_0^t e^{\nu(t-\tau)}(t - \tau)^{-\alpha}\tau^{-\beta} d\tau &\leq e^{\nu t} \int_0^t (t - \tau)^{-\alpha}\tau^{-\beta} d\tau \\ &= B(1 - \alpha, 1 - \beta)t^{1-\alpha-\beta} e^{\nu t}, \end{aligned}$$

the assertion (3.3.29) follows. □

Corollary 3.3.3 *Assume* (3.3.37) *and* (3.3.38) *hold. Then, for any given* $\varepsilon > 0$, *there exists a constant* $c := c(\varepsilon, \alpha, \beta, B) > 0$ *such that for a. a.* $t \in \dot{J}$,

$$u(t) \leq Act^{-\beta} e^{(1+\varepsilon)\mu(\alpha,B)t}.$$

Remark 3.3.3

(a) It should be noted that, in general, the constant $c(\alpha, \varepsilon)$ in the estimate of Lemma 3.3.1—and, consequently, the constant c in Theorem 3.3.4 and Corollary 3.3.3 as well—tend to infinity if $\varepsilon \to 0$. Note that if $\alpha = 0$, then $\varepsilon = 0$ is possible and $c(0, 0) = 1$. In this case, the constant c of Theorem 3.3.4 equals $1/(1 - \alpha)$ and *(3.3.39)* is then a consequence of the classical Gronwall inequality.

(b) Of course, the factor $e^{(1+\varepsilon)\mu(t-s)}$ in Lemma *3.3.1* and in Theorem *3.3.4* and Corollary *3.3.3*—where $\mu := m^{1/(1-\alpha)}$ in Lemma *3.3.1*—can be replaced by $e^{(\mu+\varepsilon)(t-s)}$.

The following result may be found in Ye and Li [666], it is a corollary of a special case of Theorem 1.4.4 with $\beta = 0$ and $\alpha \in (0, 1)$.

Corollary 3.3.4 (Ye-Li [666]) *Let $v(t) \geq 0$ be continuous on $[t_0, T]$. If there are positive constants a, b and $\alpha < 1$ such that for all $t \in [t_0, T]$,*

$$v(t) \leq a + b \int_{t_0}^{t} (t-s)^{\alpha-1} v(s) ds, \tag{3.3.40}$$

then there is a constant $M > 0$, independent of a, such that for all $t \in [t_0, T]$,

$$v(t) \leq Ma. \tag{3.3.41}$$

Proof By iterating (3.3.40) and exploiting the identity

$$\int_0^t (t-s)^{-\alpha-1}(s-\tau)^{\beta-1} ds = (t-\tau)^{\alpha+\beta-1} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

we obtain

$$\begin{aligned} v(t) &\leq a + b \int_{t_0}^{t} (t-s)^{\alpha-1} \Big\{ a + b \int_{t_0}^{s} (s-\tau)^{\alpha-1} v(\tau) d\tau \Big\} ds \\ &\leq a \Big\{ 1 + b \frac{(T-t_0)^\alpha}{a} \Big\} + b^2 \int_{t_0}^{t} \Big\{ \int_{\tau}^{t} (t-s)^{\alpha-1} (s-\tau)^{\alpha-1} ds \Big\} v(\tau) d\tau \\ &= a(1+b) \Big\{ 1 + b \frac{(T-t_0)^\alpha}{a} \Big\} + b^2 \frac{\Gamma^2(\alpha)}{\Gamma(2\alpha)} \int_{t_0}^{t} (t-\tau)^{2\alpha-1} v(\tau) d\tau \end{aligned}$$

which implies that

$$v(t) \leq a \sum_{j=0}^{n-1} \left[\frac{b(T-t_0)^\alpha}{a} \right]^j + \frac{[b\Gamma(\alpha)]^n}{\Gamma(n\alpha)} \int_{t_0}^{t} (t-\tau)^{n\alpha-1} v(\tau) d\tau. \tag{3.3.42}$$

Choosing n so large that $n\alpha - 1 > 0$, we conclude

$$v(t) \leq C_1 a + C_2 \int_{t_0}^{t} v(\tau) d\tau \tag{3.3.43}$$

where C_1, C_2 are positive constants depending only on $T - t_0$ and b, but not on α and a. Thus (3.3.41) follows from (3.3.43) by the Gronwall-Bellman inequality (see, Theorem 1.1.2). □

Recently, Ye et al. [665] gave the following inequality to prove the continuous dependence on parameters of fractional differential equations, which can be viewed as a general form of the above theorem.

Theorem 3.3.5 (Tokuaka [624]) *Let $\beta > 0$, and assume that $a(t)$ is a non-negative function locally integrable on $0 \leq t < T$ for some $T \leq +\infty$ and let $g(t)$ be a*

non-negative, non-decreasing continuous function defined on $0 \le t < T$, $g(t) \le M$ (*a constant*), *and assume that* $u(t)$ *is non-negative and locally integrable on* $0 \le t < T$ *and satisfies*

$$u(t) \le a(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s)ds \tag{3.3.44}$$

on this interval. Then for all $0 \le t < T$

$$u(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{+\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s) \right] ds. \tag{3.3.45}$$

Proof Let $B\varphi(t) = g(t) \int_0^t (t-s)^{\beta-1} \varphi(s)ds$, for all $t \ge 0$, for locally integrable functions φ. Then

$$u(t) \le a(t) + Bu(t)$$

implies

$$u(t) \le \sum_{k=0}^{n-1} B^k a(t) + B^n u(t).$$

Let us prove that

$$B^n u(t) \le \int_0^t \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} u(s)ds \tag{3.3.46}$$

and $B^n u(t) \to 0$ as $n \to +\infty$ for each $0 \le t < T$.

We know that this relation (3.3.46) is true for $n = 1$. Assume that it is true for some $n = k$. If $n = k+1$, then the induction hypothesis implies

$$B^{k+1}u(t) = B(B^k u(t)) \le g(t) \int_0^t (t-s)^{\beta-1} \left[\int_0^s \frac{(g(t)\Gamma(\beta))^k}{\Gamma(k\beta)} (s-\tau)^{k\beta-1} u(\tau)d\tau \right] ds. \tag{3.3.47}$$

Since $g(t)$ is non-decreasing, it follows that

$$B^{k+1}u(t) \le (g(t))^{k+1} \int_0^t (t-s)^{\beta-1} \left[\int_0^s \frac{(\Gamma(\beta))^k}{\Gamma(k\beta)} (s-\tau)^{k\beta-1} u(\tau)d\tau \right] ds. \tag{3.3.48}$$

By interchanging the order of integration, we obtain

$$B^{k+1}u(t) \le \int_0^t \frac{(g(t)\Gamma(\beta))^{k+1}}{\Gamma((k+1)\beta)} (t-s)^{(k+1)\beta-1} u(s)ds, \tag{3.3.49}$$

where the integral

$$\int_{\tau}^{t}(t-s)^{\beta-1}(s-\tau)^{k\beta-1}ds = (t-\tau)^{k\beta+\beta-1}\int_{0}^{1}(1-z)^{\beta-1}z^{k\beta-1}dz$$
$$= (t-\tau)^{k\beta+\beta-1}B(k\beta,\beta) = \frac{\Gamma(\beta)\Gamma(k\beta)}{\Gamma((k+1)\beta)}(t-\tau)^{k\beta+\beta-1}$$

is evaluated with the help of the substitution $s = \tau + z(t-\tau)$ and the definition of the beta function (cf. [526]). This gives us the relation (3.3.46). Since $B^n u(t) \le \int_0^t \frac{(M\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}u(s)ds \to 0$ as $n \to +\infty$ for any $t \in [0,T)$, the theorem is thus proved. □

For $g(t) \equiv constant = b$ in the theorem, we obtain the following inequality, which can be found in Henry [272].

Corollary 3.3.5 *Assume that $b \ge 0, \beta > 0$, and let $a(t)$ be a non-negative function locally integrable on $0 \le t \le T$ for some $T \le +\infty$, and assume that $u(t)$ is non-negative and locally integrable on $0 \le t < T$ such that for all $t \in [0,T]$,*

$$u(t) \le a(t) + b\int_0^t (t-s)^{\beta-1}u(s)ds \tag{3.3.50}$$

then for all $t \in [0,T]$,

$$u(t) \le a(t) + \int_0^t \left[\sum_{n=1}^{+\infty}\frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)}(t-s)^{n\beta-1}a(s)\right]ds. \tag{3.3.51}$$

Corollary 3.3.6 *Under the hypothesis of Theorem 3.3.5, let $a(t)$ be a non-decreasing function on $[0,T]$. Then for all $t \in [0,T]$,*

$$u(t) \le a(t)E_\beta(g(t)\Gamma(\beta)t^\beta), \tag{3.3.52}$$

where E_β is the Mittag-Leffler function defined by $E_\beta(z) = \sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(k\beta+1)}$.

3.3.2 The Volterra Integral Inequalities for Functions Defined in Partially Ordered Spaces

In this section, we introduce the results due to Ronkov and Bainov [560], on the inequalities of Volterra type for functions defined in partially ordered spaces.

Now we study Volterra type integral operators action on numeric functions defined in partially ordered topological spaces with a measure. Integral equations and inequalities for such operators have been considered. Note that Bainov et al.

[41] is the first contribution to consider linear integral equations and inequalities of Volterra type for functions defined in metric spaces.

We shall consider complex functions defined in the partially ordered set T $(T, <)$, employing the following notations:

$T_x := \{y : y \in T \textit{ and } y < x\}$ for the segment of the element x and

$$\chi(x, y) = \begin{cases} 1, & \text{if} \quad y \leq x \\ 0, & \text{otherwise} \end{cases}$$

For its characteristic function, concerning the set T, we shall assume that the following conditions hold:

(C_1) T is a partially ordered connected topological space with positive measure μ.
(C_2) For every $x \in T$, the function $\chi(x, \cdot)$ is μ-measurable.
(C_3) If x_α is a generalized sequence of elements of T, convergent to x, then $||\chi(x_\alpha, \cdot) - \chi(x, \cdot)||_2$ tends to 0.
(C_4) There exists element $x_0 \in T$ such that

$$||\chi(x_0, \cdot)||_2 = 0,$$

where

$$||\varphi||_2^2 = \int_T |\varphi(x)|^2 d\mu(x).$$

Definition 3.3.1 The operator $\phi : L^2(T) \to L^2(T)$ is called to be characteristic provided for all $x \in T$, $\psi \in L^2(T)$ and $\chi(x, \cdot)\psi(\cdot) = \chi(x, \cdot)\phi\psi(\cdot)$ hold.

Remark 3.3.4 If ϕ is a linear characteristic operator, then for any $x \in T$ and every function $\psi \in L^2(T)$, the equality $\chi(x, \cdot)\phi\psi(x) = \chi(x, \cdot)\phi\psi_x(x)$ holds, where $\psi_x(\cdot) := \chi(x, \cdot)\psi(\cdot)$.

Indeed, since

$$\chi(z, \cdot)(\varphi(\cdot) - \varphi_z(\cdot)) = \varphi_z(\cdot) - \varphi_z(\cdot) = 0,$$

then

$$\chi(z, \cdot)\phi(\varphi(\cdot) - \varphi_z(\cdot)) = 0$$

whence, noting the fact that the operator ϕ is linear, it follows that

$$\chi(z, \cdot)\phi\varphi(\cdot) - \chi(z, \cdot)\phi\varphi_z(\cdot) = 0.$$

Definition 3.3.2 We define the integral operator V in $L^2(T)$ as

$$Vf(x) := \int_T \chi(x,y)K(x,y)\phi f(y)d\mu(y)$$
$$= \int_T \chi(x,y)K(x,y)\phi f(y)d\mu(y) \tag{3.3.53}$$

where

(1) The kernel $K(x,y) \in L^2(T \times T)$.
(2) ϕ is a bounded linear characteristic operator.

Remark 3.3.5 Obviously, V is compact operator acting from $L^2(T)$ onto $L^2(T)$ since it is the composition of the operator ϕ with a Hilbert-Schmidt operator, the latter is a compact operator from $L^2(T)$ onto $L^2(T)$ (see, e.g., [254, Chap. 15, Problem 135]).

To prove the next theorem, we need the following lemma.

Lemma 3.3.3 (Ronkov-Bainov [560]) *The integral equation*

$$\varphi = \lambda V\varphi \tag{3.3.54}$$

possesses the trivial solution (here λ is an arbitrary complex number).

Proof Let the function $\varphi \in L^2(T)$ be a solution of the integral equation (3.3.54). By T_0 we denote the following subset of T:

$$T_0 := \{x : x \in T \ and \ \chi(x,\cdot)\varphi(\cdot) = 0\}.$$

We shall show that $T_0 = T$ whence it follows that $\varphi = 0$. Indeed,

$$||\varphi||_2^2 = \int_T |\varphi|^2 d\mu(x)$$
$$= |\lambda|^2 \int_T |K(x,y)\chi(x,y)\phi\varphi(y)|^2 d\mu(x) = 0$$

because, in view of Remark 3.3.4, $\chi(x,\cdot)\phi\varphi(\cdot) = \chi(x,\cdot)\phi\varphi_x(\cdot)$, where $\varphi_x(\cdot) = \chi(x,\cdot)\phi = 0$.

Since T is a connected topological space, then in order to establish that $T_0 = T$, it is sufficient to show that T_0 is not empty and that it is closed and open at the same time.

In view of condition (C_4), an element $x_0 \in T$ exists such that $||\chi(x_0,\cdot)||_2 = 0$ whence $||\varphi\chi(x_0,\cdot)||_2 = 0$ i.e., $x_0 \in T$ and therefore T_0 is not empty.

We shall show that T_0 is a closed set. Indeed, let $\{x_\alpha\}$ be a generalized sequence of elements of T_0, convergent to x. But then

$$||\varphi(\cdot)\chi(x,\cdot)||_2 = ||\varphi(\cdot)\chi(x,\cdot) - \varphi(\cdot)\chi(x_\alpha,\cdot)||_2 \int_{T_x \Delta T_{x_\alpha}} ||\varphi(y)||^2 d\mu(y)$$

and since by (C_3), $||\chi(x,\cdot)-\chi(x_\alpha,\cdot)||_2$ tends to 0, then $||\varphi(\cdot)\chi(x,\cdot)||_2 = 0$. Hence, $x \in T_0$, which implies that T_0 is closed.

Now we show that T_0 is open, too.

Let $z_0 \in T_0$. According to the condition (C_3), for every $\delta > 0$, a neighborhood $U(z_0,\delta)$ of z_0 exists such that if $z \in U(z_0,\delta)$, then $\mu(T_z/T_{z_0}) \le \delta$. We shall show that if $\delta > 0$ is sufficiently small, $U(z_0,\delta) \subset T_0$.

Let $z \in T$ and consider the function $\varphi_z(y) := \chi(z,y)\varphi(y)$. For the square of the norm of the function φ_z, the following holds

$$\begin{aligned}
||\varphi_z||_2^2 &= \int_T \chi^2(z,x)|\varphi(x)|^2 d\mu(x)\\
&\le |\lambda|^2 \int_T \chi(z,x)\Big(\int_T \chi(x,y)|K(x,y)||\phi\varphi(y)|d\mu(y)\Big)^2 d\mu(x)\\
&= |\lambda|^2 \int_T \Big(\int_T \chi(z,x)\chi(x,y)|K(x,y)\phi\varphi(y)|d\mu(y)\Big)^2 d\mu(x)\\
&\le |\lambda|^2 \int_T \Big(\int_{T_z} |K(x,y)\phi\varphi(y)|d\mu(y)\Big)^2 d\mu(x).
\end{aligned}$$

Since $z_0 \in T_0$, then $||\varphi\chi(z_0,\cdot)||_2 = 0$, and therefore, $||\phi\varphi\chi(z_0,\cdot)||_2 = 0$ and

$$\int_{T_z} |K(x,y)\phi\varphi(y)|d\mu(y) = \int_{T_z/T_{z_0}} |K(x,y)\phi\varphi(y)|d\mu(y).$$

On the other hand,

$$\begin{aligned}
&\int_{T_z/T_{z_0}} |K(x,y)\phi\varphi(y)|d\mu(y)\\
&\quad\le \Big(\int_{T_z/T_{z_0}} |K(x,y)|^2 d\mu(y)\Big)^{1/2}\Big(\int_{T_z} |\phi\varphi(y)|^2 d\mu(y)\Big)^{1/2}.
\end{aligned}$$

However, by Remark 3.3.4, we have

$$\begin{aligned}
\int_{T_z} |\phi\varphi(y)|^2 d\mu(y) &= \int_T |\chi(z,y)\phi\varphi(y)|^2 d\mu(y) = \int_T |\chi(z,y)\phi\varphi_z(y)|^2 d\mu(y)\\
&\le ||\phi\varphi_z||_2^2 \le ||\phi||_2^2 ||\varphi_z||_2^2,
\end{aligned}$$

whence,

$$||\varphi_z||_2^2 \le |\lambda|^2 \int_T \Big(\int_{T_z/T_{z_0}} |K(x,y)|^2 d\mu(y)\Big) d\mu(x) \cdot ||\phi||_2^2 ||\varphi_z||_2^2.$$

By assumption $|K(x,y)|^2 \in L(T \times T)$ and since the set of the functions of the form $f(x,y) = \sum_{s=1}^{n} X_s(x)Y_s(y)$ is dense in $L(T \times T)$, where $X_s(\cdot)$ and $Y_s(\cdot)$ are proportional of characteristic functions of measurable sets with finite measure, then, without loss of generality, we can consider that $|K(x,y)|^2 = \sum_{s=1}^{n} X_s(x)Y_s(y)$.

Thus

$$\begin{aligned}\int_{T_z}\Big(\int_{T_z/T_{z_0}} |K(x,y)|^2 d\mu(y)\Big) d\mu(x) &= \int_{T_z}\Big(\int_{T_z/T_{z_0}} \Sigma_{s=1}^{n} X_s(x)Y_s(y)d\mu(y)\Big)d\mu(x)\\ &\le \Big(\int_{T_z} |\Sigma_{s=1}^{n} X_s(x)|d\mu(x)\Big)\cdot C_1 \cdot \mu(T_z/T_{z_0})\\ &\le C\cdot \mu(T_z/T_{z_0})\end{aligned}$$

where

$$C_1 = \max_{s=1,\ldots,n\ y\in T} |Y_s(x)|;\ C = C_1 \int_T |\sum_{s=1}^{n} X_s(x)|d\mu(x).$$

Let δ be a positive number so small that $\delta \cdot C|\delta|^2||\phi||_2^2 < 1$. Then if $z \in U(z_0,\delta)$, then $U(z_0,\delta) < \delta$ and hence $||\varphi_z||_2^2 \le \delta \cdot c \cdot |\lambda|^2||\phi||_2^2||\varphi_z||_2^2$ which implies that $||\varphi_z||_2 = 0$. Therefore, $U(z_0,\delta) \subseteq T_0$, i.e., T_0 is an open set. This completes the proof. $\square$

Theorem 3.3.6 (Ronkov-Bainov [560]) *Let the function g be from $L^2(T)$ and for the space T, the conditions (C_1)–(C_4) hold. Then the integral equation*

$$\varphi = g + V\varphi \tag{3.3.55}$$

possesses a unique solution $\varphi \in L^2(T)$, and

$$\varphi = \sum_{n=0}^{+\infty} V_g^n.$$

Proof Since V is a compact operator, then Lemma 3.3.3 implies that the spectral radius of the operator V equals 0, i.e., $\lim_{n\to+\infty} ||V^n||^{1/n} = 0$.

However, in this case the Cauchy criterion implies that the series $\sum_{n=0}^{+\infty} V^n$ is convergent with respect to norm, and hence it is convergent. Its sum, as is seen by an immediate verification, is an operator inverse to the operator $I - V$, here I is the identity operator on $L^2(T)$. Therefore, the integral equation (3.3.55), which can be written as

$$(I - V)\varphi = g$$

possesses a unique solution

$$\varphi = \sum_{n=0}^{+\infty} V^n g$$

This completes the proof. □

We shall use the following definitions.

Definition 3.3.3 If f and g are two real functions from $L^2(T)$, then $f \leq g \Leftrightarrow f(x) \leq g(x)$ for almost every $x \in T$.

Definition 3.3.4 The operator $W : L^2(T) \to L^2(T)$ is called monotone if $Wf \leq Wg$ for $f \leq g$.

Definition 3.3.5 The operator $W : L^2(T) \to L^2(T)$ is said to be characteristically monotone if, for every x from T and for any two real functions f and g from $L^2(T)$, for which $(f(\cdot) - g(\cdot))\chi(x, \cdot) \leq 0$, the inequality $Wf \leq Wg$ holds. Obviously, if W is a characteristically monotone operator, then W is monotone.

Theorem 3.3.7 (Ronkov-Bainov [560]) *Assume that following assumptions hold,*

(1) For the space T, conditions (C_4) hold,
(2) The kernel $K(x, y)$ of the integral operator V is non-negative, while the operator ϕ is monotone,
(3) For some $x \in T$ and for two real functions f and g from $L^2(T)$, the following inequality holds

$$\Big(f(\cdot) - g(\cdot) - Vf(\cdot)\Big)\chi(x, \cdot) \leq 0, \quad \Big(\textit{respectively}\ (f(\cdot) - g(\cdot) - Vf(\cdot))\chi(x, \cdot) \geq 0\Big). \tag{3.3.56}$$

Then

$$(f(\cdot) - \varphi(\cdot))\chi(x, \cdot) \leq 0, \quad \Big(\textit{respectively}\ (f(\cdot) - \varphi(\cdot))\chi(x, \cdot) \geq 0\Big)$$

where φ is the solution of the integral equation (3.3.55).

Proof We shall first show that the operator V is characteristically monotone. Indeed, if $h \in L^2(T)$ and $h_z = \chi(z, \cdot)h(\cdot) \leq 0$, then in view of Remark 3.3.4, $\chi(z, \cdot)\phi h(\cdot) = \chi(z, \cdot)\phi h_z(\cdot)$ and since $h_z(\cdot) \leq 0$, and ϕ is monotone, then $\chi(z, \cdot)\phi h(\cdot) \leq 0$. On the other hand, $Vh(z) = \int_T K(z, y)\chi(z, y)\phi h(y) d\mu(y)$ and since $K(x, y) \geq 0$, then $Vh(z) \leq 0$.

Now we shall show that inequality (3.3.56) implies that

$$(Vf(\cdot) - Vg(\cdot) - V^2 f(\cdot))\chi(x, \cdot) \leq 0.$$

Indeed, if $z < x$, then (3.3.56) yields that $(f(\cdot) - g(\cdot) - Vf(\cdot))\chi(x,\cdot) \leq 0$. But then, since V is characteristically monotone, $Vf(\cdot) - Vg(\cdot) - V^2f(\cdot) \leq 0$. Therefore, $(Vf(\cdot) - Vg(\cdot) - V^2f(\cdot))\chi(x,\cdot) \leq 0$. Analogously, by induction, it is easy to check that for every $n = 0, 1, 2, \ldots,$ the inequality $(V^nf(\cdot) - V^ng(\cdot) - V^{n+1}f(\cdot))\chi(x,\cdot) \leq 0$ holds. Summing by $n = 0, 1, 2, \ldots$, and taking into account that $V^nf \to 0$ and that $\varphi = \sum_{n=1}^{+\infty} V^ng$ is a solution of the integral equation (3.3.55), hence it follows that $(f(\cdot) - \varphi(\cdot))\chi(x,\cdot) \leq 0$. The inverse inequality is proved analogously. □

Theorem 3.3.8 (Ronkov-Bainov [560]) *If, under the conditions of Theorem 3.3.7, the inequality* $f(\cdot) - g(\cdot) - Vf(\cdot) \leq 0$ *holds (or* $f(\cdot) - g(\cdot) - Vf(\cdot) \geq 0$*), then*

$$f \leq \varphi = \sum_{n=1}^{+\infty} V^ng \quad (f \geq \varphi,\ \textit{respectively}).$$

Proof The inequality $f(\cdot) - g(\cdot) - Vf(\cdot) \leq 0$ implies that $(Vf(\cdot) - Vg(\cdot) - V^2f(\cdot))\chi(x,\cdot) \leq 0$ for every $x \in T$.

However, by Theorem 3.3.7, $(f(\cdot) - \varphi(\cdot))\chi(x,\cdot) \leq 0$ for any x from T and since the operator V is characteristically monotone, this implies that $Vf \leq Vg$, Hence

$$f \leq g + Vf \leq g + V\varphi = \varphi.$$

The inverse inequality is proved analogously. □

Corollary 3.3.7 (Ronkov-Bainov [560]) *If, in the conditions of Theorem 3.3.7 (Theorem 3.3.8),* $g = 0$*, then it follows that the inequality* $f - Vf \leq 0$ $(\chi(x,\cdot)(f - Vf) \leq 0)$*, respectively) implies the inequality* $f \leq 0 \quad (\chi(x,\cdot)f \leq 0)$*, respectively. Then, since* V *is a linear operator, then the inequality* $f - Vf \leq h - Vh$ *(*h *is a real function from* $L^2(T)$*) implies that* $f \leq h$*.*

We now give an example (see [560]).

Example 3.3.1 Let $T = [t_0, t_1)$ (here t_1 may be $+\infty$ as well). Consider in T the usual topology (with respect to which T is connected) and the Lebesgue measure denoted by μ. Let for any $x \in T$, $T_x := [t_0, t(x))$, where $t(\cdot)$ is such a continuous function defined in T that for every $x \in T$, the inequalities $t(t_0) \leq t(x) \leq x$ hold. By V we denote the operator defined in $L^2(T)$ in the following way

$$Vf(x) \leq \int_{t_0}^{t(x)} K(x,y)f_1(\varphi(y))d\mu(y)$$

where the kernel $K \in L^2(T \times T)$, φ is an invertible real function with continuous derivative for which $\varphi(x) \leq x$, while $f_1 = f$ if $t_0 \leq t$ and $f_1 = 0$ if $t_0 \geq t$.

It is not difficult to verify that all assumptions of Theorem 3.3.6 are satisfied. Hence, the integro-functional equations

$$\begin{aligned} h(x) &= g(x) + \int_{t_0}^{t(x)} K(x,y)h_1(\varphi(y))d\mu(y) \\ &= g(x) + Vh(x) \quad (g + L^2(T)) \end{aligned}$$

possesses a unique solution. By Theorem 3.3.7, if the kernel K is non-negative and under the assumption that $f(y) \le g(y) + Vf(y)$ for almost all $y \in [t_0, t(x))$. In view of Corollary 3.3.7, if for two functions $f_i,\ f_s \in L^2(T)$, the inequality $f_i - Vf_i \le f_s - Vf_s$ holds, then $f_i \le f_s$.

Now consider the case when the operator ϕ is identity. Then

$$Vf(x) = \int_{T_s} K(x,y)f(y)d\mu(y)$$

and, in view of Theorem 3.3.6, the integral equation

$$\varphi(x) = g(x) + \int_{T_s} K(x,y)\varphi(y)d\mu(y) \tag{3.3.57}$$

(here g denotes an arbitrary real function from $L^2(T)$) possesses a unique solution $\varphi \in L^2(T)$. We shall try to obtain some explicit estimates for this solution which, by Corollary 3.3.7, will hold for the solutions f of the corresponding integral inequality

$$f \le g + Vf. \tag{3.3.58}$$

Similar estimates, when functions defined in $\mathbb{R}$ or $\mathbb{R}^n$ are considered, are usually obtained with the help of differentiation. Here this technique seems inapplicable and we shall employ Theorem 3.3.6 which implies that the solution φ of (3.3.57) is actually the sum of the Neumann series $\sum_{n=0}^{+\infty} V^n g$. The idea is to compare the terms of this series with the ones of an exponential series, whence the demanded estimate follows.

Theorem 3.3.9 (Ronkov-Bainov [560]) *Assume that*

(1). For the space T, conditions (C_1)–(C_4) hold, and, for the ordering in T, beside being transitive, it is assumed that it satisfies the requirement if $x < y$ and $y < x$, then $x = y$.

(2). The diagonal $D := \{(x,x) : x \in T\}$ of the space $(T^2, \Sigma^2, \mu^2) := (T, \Sigma, \mu) \times (T, \Sigma, \mu)$ is a μ_2-null set.

(3). The kernel $K(x,y)$ of the integral operator V is a non-negative function from $L^2(T^2)$, which, for a fixed y from T is a non-decreasing function of x.

(4). $g(x)$ is a non-decreasing, non-negative function from $L^2(T)$.
Then for the solution φ of the integral equation (3.3.57), it holds that

$$\varphi(x) \leq g(x) \cdot \exp\left(\int_{T_s} K(x,y)d\mu(y)\right). \tag{3.3.59}$$

Proof By Theorem 3.3.6, the integral equation (3.3.57) possesses a unique solution $\varphi = \sum_{n=0}^{+\infty} V^n g$. On the other hand,

$$\exp\Big(\int_{T_s} K(x,y)d\mu(y)\Big) = \exp(V(1)(x)) = \sum_{n=0}^{+\infty} \frac{(V(1)(x))^n}{n!}.$$

Hence, in order to obtain the estimate (3.3.59), it is sufficient to prove that the inequality

$$n!V^n g \leq g(V(1))^n \tag{3.3.60}$$

holds for every natural $n = 0, 1, \ldots$ (For $n = 0$, we obviously have an equality).

$$\begin{aligned}
V^n g(x) &= \int_{T_s} K(x,y_1)d\mu(y_1)\int_{T_{y_1}} K(y_1,y_2)\cdots\int_{T_{y_{n-1}}} K(y_{n-1},y_n)g(y_n)d\mu(y_n)\\
&= \int_{T^n} \chi(x,y_1)\chi(y_1,y_2)\chi(y_{n-1},y_n)\cdots K(x,y_1)K(y_1,y_2)\\
&\quad \cdots K(y_{n-1},y_n)g(y_n)d\mu_n(y)\\
&= \int_{T^n} \chi(x,y_{\alpha_1})\chi(y_{\alpha_1,\alpha_2})\chi(y_{\alpha_{n-1}},y_{\alpha_n})\cdots K(x,y_{\alpha_1})K(y_{\alpha_1,\alpha_2})\\
&\quad \cdots K(y_{\alpha_{n-1}},y_{\alpha_n})g(y_{\alpha_n})d\mu_n(y),
\end{aligned}$$

where α is an arbitrary element from the aggregate Π of all permutations of $\{1, 2, \cdots, n\}$. Then the monotonicity of g and K implies that

$$V^n g(x) \leq g(x)\int_{T^n} \chi(x,y_{\alpha_1})\chi(y_{\alpha_1,\alpha_2})\chi(y_{\alpha_{n-1}},y_{\alpha_n})\cdots K(x,y_2)\cdots K(x,y_n)d\mu_n(y).$$

Since the number of all permutations of $\{1, 2, \ldots, n\}$ is $n!$, then

$$n!V^n g(x) \leq g(x)\int_T (\sum_{\alpha\in\Pi} \chi(x,y_{\alpha_1})\chi(y_{\alpha_1,\alpha_2})\chi(y_{\alpha_{n-1}},y_{\alpha_n}))\cdots K(x,y_2)\cdots K(x,y_n)d\mu_n(y). \tag{3.3.61}$$

Now we shall show that the inequality

$$\sum_{\alpha\in\Pi}\chi(x,y_{\alpha_1})\chi(y_{\alpha_1,\alpha_2})\cdots\chi(y_{\alpha_{n-1}},y_{\alpha_n})\leq\chi(x,y_1)\chi(x,y_2)\cdots\chi(x,y_n) \tag{3.3.62}$$

holds almost everywhere in T^n.

Indeed, let $x,\ y_1,\ ,\dots,\ y_n$ be elements of T. In view of condition (3.3.55), without loss of generality, we may consider that taken two-by-two they are different. However, if the left-hand sides of (3.3.62) is different from zero, then for some permutation $\alpha\in\Pi$,

$$y_{\alpha_n}<y_{\alpha_{n-1}}<\cdots y_{\alpha_1}<x \tag{3.3.63}$$

will be fulfilled. Since $x,\ y_1,\ ,\dots,\ y_n$ are different from one another, then there will not be another similar permutation and hence

$$\sum_{\alpha\in\Pi}\chi(x,y_{\alpha_1})\chi(y_{\alpha_1,\alpha_2})\cdots\chi(y_{\alpha_{n-1}},y_{\alpha_n})=1.$$

Moreover, (3.3.63) yields that in this case the right-hand side of (3.3.62) also assumes the value one.

This, in fact, proves that inequality (3.3.62) holds almost everywhere in T^n. Then inequalities (3.3.61) and (3.3.62) it follows that

$$\begin{aligned}n!V^n g(x)&\leq g(x)\int_{T^n}\chi(x,y_1)\chi(x,y_2)K(x,y_2))\cdots\chi(x,y_n)K(x,y_n)d\mu_n(y)\\&=g(x)(\int_{T_s}K(x,z)d\mu(z))^n=g(x)(V(1)(x))^n,\end{aligned}$$

i.e., inequality (3.3.60) holds for every natural number n. This completes the proof. □

Corollary 3.3.8 (Ronkov-Bainov [560]) *If under the conditions of Theorem 3.3.9 for some function $f\in L^2(T)$, inequality (3.3.58) holds, then it holds that*

$$f(x)\leq g(x)\exp\left(\int_{T_s}K(x,y)d\mu(y)\right)$$

since, in view of Theorem 3.3.8, $f\leq\varphi$.

Theorem 3.3.9 was proved under the assumption that the measure of the diagonal of the space $T\times T$ is zero. A sufficient condition for this assumption to be satisfied is provided by the following lemma.

Lemma 3.3.4 (Ronkov-Bainov [560]) *Let $T=(T,\Sigma,\mu)$ be a space of non-negative measure. If for any positive ϵ, a sequence $\{U_{\epsilon,n}\}$ exists, consisting of sets*

that are measurable with respect to μ and such that

$$T = \cup_{n=1}^{\infty} U_{\epsilon,n}, \qquad \mu(U_{\epsilon,n}) < \epsilon$$

for every $n = 1, 2, \cdots$, then the diagonal $D = \{(x, x) : x \in T\}$ of the space $T \times T = (T, \Sigma, \mu) \times (T, \Sigma, \mu)$ is of zero measure.

Proof Since T can be represented as a denumerable sum of sets having a finite measure, the T is a space with σ-finite measure. First consider the case when $\mu(T) \leq +\infty$. Let ϵ be an arbitrary positive number and $T = \cup_{n=1}^{+\infty} U_{\epsilon,n}$ and $U_{\epsilon,n} \in \Sigma$ and $\mu(U_{\epsilon,n}) < \epsilon$ for every natural number n. Without loss of generality, we can consider that $U_{\epsilon,n} \cap U_{\epsilon,m} = \emptyset$ when $n \neq m$ because, otherwise, we could have set

$$W_{\epsilon,1} := U_{\epsilon,1};\ W_{\epsilon,2} := U_{\epsilon,2}/U_{\epsilon,1};\ W_{\epsilon,3} := U_{\epsilon,2}/(U_{\epsilon,1} \cup U_{\epsilon,2}).$$

Consider the set $E_\epsilon := \cup_{n=1}^{\infty}(U_{\epsilon,n} \times U_{\epsilon,n})$. Obviously, E_ϵ is a measurable set with respect to the measure of the product $\mu_2 = \mu \times \mu$ and $D \subset E_\epsilon$. By E_y we denote the following subset of T : $E_y := \{X : (x, y) \in E_\epsilon\}$. As is known (see, e.g., [199], III.II.7), $\mu_2(E_\epsilon) = \int_T \mu(E_y) d\mu(y) \leq \epsilon\mu(T)$ since $\mu(E_y) < \epsilon$ for every $y \in T$ because $U_{\epsilon,n} \cap U_{\epsilon,m} = \emptyset$ when $n \neq m$ and $\mu(U_{\epsilon,n}) < \epsilon$ for any n. Therefore, taking into account that $\mu(T) < +\infty$ and ϵ is arbitrary, it follows that $\mu(D) = 0$.

Now consider the case when the measure of T is not finite. Since $D = \cup_{n=1}^{\infty} D_{\epsilon,n}$ where $D_{\epsilon,n} := \{(x, x) : \ x \in U_{\epsilon,n}\}$, and besides $\mu(U_{\epsilon,n}) < \epsilon$ for any n, then $\mu(D_{\epsilon,n}) = 0$, whence it also follows that $\mu(D) = 0$.

This completes the proof of Lemma 3.3.4. □

In Theorem 3.3.9, the function g from the integral equation (3.3.57) was assumed to be non-negative and non-decreasing, while the next theorem gives us an estimate for the solution of (3.3.57) without these assumptions.

Theorem 3.3.10 (Ronkov-Bainov [560]) *Under assumptions (1), (2), and (3) of Theorem 3.3.9, if g is an arbitrary function from $L^2(T)$, then for the solution φ of the integral equation (3.3.57), the following estimate holds,*

$$|\varphi(x)| \leq |g(x)| + \int_{T_x} |g(y)| K(x, y) \exp\Big(\int_{T_x/T_y} K(x, z) d\mu(z)\Big) d\mu(y), \tag{3.3.64}$$

which implies immediately the weaker but simpler estimate

$$|\varphi(x)| \leq |g(x)| + \int_{T_x} |g(y)| K(x, y) d\mu(y) \cdot \exp\Big(\int_{T_x/T_y} K(x, z) d\mu(z)\Big). \tag{3.3.65}$$

Proof For the solution φ of the integral equation (3.3.57), we get

$$|\varphi(x)| = |g(x) + \int_{T_x} g(y) K(x, y) d\mu(y)|, \tag{3.3.66}$$

whence

$$|\varphi(x)| \le |g(x)| + \int_{T_x} |\varphi(y)| K(x,y) d\mu(y). \tag{3.3.67}$$

Noting that $K(x,y) \ge 0$, we get

$$\begin{aligned}\int_{T_x} |\varphi(y)| K(x,y) d\mu(y) \le \int_{T_x} &|g(y)| K(x,y) d\mu(y) \\ &+ \int_{T_x} K(x,y) \Big(\int_{T_x} |\varphi(y)| K(x,z) d\mu(z) \Big) d\mu(y).\end{aligned} \tag{3.3.68}$$

Obviously, $g_1(x) = \int_{T_x} |g(y)| K(x,y) d\mu(y)$ is a non-negative and non-decreasing function. Then the estimate (3.3.65) follows from Corollary 3.3.7 and inequalities (3.3.68) and (3.3.67).

Estimate (3.3.64) can be obtained means of calculations analogous to those done in the proof of Theorem 3.3.9. Thus, if by φ_1 we denote the solution of the integral equation

$$\varphi_1 = g_1 + V\varphi_1,$$

then Theorem 3.3.6 implies that $\varphi_1 = \sum_{n=0}^{+\infty} V^n g_1$, and it follows from (3.3.68) and Theorem 3.3.7 that

$$\int_{T_x} |\varphi(y)| K(x,y) d\mu(y) \le \varphi_1 = \sum_{n=0}^{+\infty} V^n g_1. \tag{3.3.69}$$

However,

$$\begin{aligned}V^n g_1 &= \int_{T_s} K(x,y_1) d\mu(y_1) \int_{T_{y1}} K(y_1,y_2) d\mu(y_2) \cdots \int_{T_{y_{n-1}}} K(y_{n-1},y_2) d\mu(y_n) \\ &\quad \times \int_{T_{yn}} K(y_n,y_0) |g(y_0)| d\mu(y_0) \\ &= \int_{T^{n+1}} K(x,y_1) K(y_1,y_2) \cdots K(y_{n-1},y_0) |g(y_0)| \\ &\quad \times \chi(x,y_1) \chi(y_1,y_2) \cdots \chi(y_{n-1},y_n) \chi(y_{n-1},y_n) \chi(y_n,y_0) d\mu_{n+1}(y) \\ &= \int_{T^{n+1}} K(x,y_{\alpha_1}) K(y_{\alpha_1},y_{\alpha_2}) \cdots K(y_{\alpha_{n-1}},y_{\alpha_n}) K(y_{\alpha_n},y_0) |g(y_0)| \\ &\quad \times \chi(x,y_{\alpha_1}) \chi(y_{\alpha_1},y_{\alpha_2}) \cdots \chi(y_{\alpha_{n-1}},y_{\alpha_n}) \chi(y_{\alpha_n},y_0) d\mu_{n+1}(y)\end{aligned}$$

where α denotes any element of the aggregate Π of all permutation of $\{1, 2, \ldots, n\}$. Since for a fixed y, $K(x, y)$ is a non-decreasing function of x, then

$$n!V^n g_1 \leq \int_{T^{n+1}} K(x, y_0)K(x, y_1) \cdots K(x, y_n)|g(y_0)| \times\Big(\sum_{\alpha\in\Pi} \chi(x, y_{\alpha_1})\chi(y_{\alpha_1}, y_{\alpha_2}) \cdots \chi(y_{\alpha_{n-1}}, y_{\alpha_n})\chi(y_{\alpha_n}, y_0)\Big)d\mu_{n+1}(y).$$

On the other hand, since for almost everywhere in T^n, the following inequality holds,

$$\sum_{\alpha\in\Pi} \chi(x, y_{\alpha_1})\chi(y_{\alpha_1}, y_{\alpha_2}) \cdots \chi(y_{\alpha_{n-1}}, y_{\alpha_n})\chi(y_{\alpha_n}, y_0) \leq \chi(x, y_0)\Big(\chi(x, y_{\alpha_1}) - \chi(y_0, y_{\alpha_1})\Big)\Big(\chi(x, y_{\alpha_2}) - \chi(y_0, y_{\alpha_2})\Big) \cdots \Big(\chi(x, y_{\alpha_n}) - \chi(y_0, y_{\alpha_n})\Big), \tag{3.3.70}$$

which can be proved quite analogously to inequality (3.3.62) of Theorem 3.3.9, then

$$\begin{aligned} n!V^n g_1 &\leq \int_{T_x} |g(y_0)|K(x, y_0)\Big[\int_{T^n} K(x, y_1)K(x, y_2) \cdots K(x, y_n)(\chi(x, y_1) - \chi(y_0, y_1)) \\ &\quad \times\Big(\chi(x_1, y_2) - \chi(y_0, y_2)\Big) \cdots \Big(\chi(x, y_n) - \chi(y_0, y_n)\Big)d\mu_n(y)\Big]d\mu(y_0) \\ &= \int_{T_x} |g(y_0)|K(x, y_0)\Big(\int_{T_x/T_{y_0}} K(x, z)d\mu(z)\Big)^n d\mu(y_0) \end{aligned}$$

whence

$$\varphi_1(x) = \sum_{n=0}^{+\infty} V^n g_1 \leq \int_{T_x} |g(y)|K(x, y) \cdot \exp\Big(\int_{T_x/T_y} K(x, z)d\mu(z)\Big)^n d\mu(y).$$

Therefore (3.3.64) follows from the above expression and inequalities (3.3.69) and (3.3.67). This completes the proof. □

Corollary 3.3.9 (Ronkov-Bainov [560]) *If, under the conditions of Theorem 3.3.10, for some real function f from $L^2(T)$, inequality (3.3.58) holds, then there holds that*

$$f(x) \leq |g(x)| + \int_{T_x} |g(y)|K(x, y) \cdot \exp\Big(\int_{T_x/T_y} K(x, z)d\mu(z)\Big)^n d\mu(y).$$

Remark 3.3.6 If the ordering in T is linear, then there will be an equality in (3.3.70). Then, if by T we denote the real interval $[a, B)$ (here B may be $+\infty$ as well) having the usual ordering and topology, and if μ denotes the Lebesgue measure, then

assumptions (1) and (2) of Theorem 3.3.10 are obviously fulfilled. Let $K(x, y) = K(y)$ be a non-negative function from $L^2(T)$. In this case, for every solution f of the integral inequality (3.3.58), it is possible to give a more precise estimate

$$f(x) \leq g(x) + \int_a^x g(y)K(y) \exp\Big(\int_x^y K(x,z)d\mu(z)\Big)^n d\mu(y) \tag{3.3.71}$$

which, in this case, follows from the fact that, (3.3.70) is an equality and hence

$$\int_a^x g(y)K(y) \cdot \exp\Big(\int_{T_x/T_y} K(x,z)d\mu(z)\Big)^n d\mu(y)$$
$$= \sum_{n=0}^{+\infty} V^n\Big(\int_a^x g(y)K(y)d\mu(y)\Big) = \varphi(x),$$

which, by Theorem 3.3.6, is a solution of the equation

$$\varphi(x) = \int_a^x g(y)K(y)d\mu(y) + \int_a^x \varphi(y)K(y)d\mu(y).$$

However, inequality (3.3.58), since $K(y) \geq 0$, implies that

$$\int_a^x f(y)K(y)d\mu(y) \leq \int_a^x g(y)K(y)d\mu(y) + \int_a^x K(y)(\int_a^y f(z)K(z)d\mu(z))d\mu(y)$$

whence, by Theorem 3.3.8,

$$\int_a^x f(y)K(y)d\mu(y) \leq \varphi(x).$$

The above expression and inequality (3.3.58) imply the estimate (3.3.71), which in fact is the well-known Gronwall-Bellman inequality (see, e.g., [338]).

Chapter 4
Applications of Linear One-Dimensional Inequalities

4.1 Applications of Theorems 1.1.2, 2.1.13–2.1.14 to Sublinear Perturbations of the Differential Equation $y^{(n)} = 0$ and of the Analogous Difference Equation

In this section, we give some applications of Theorems 1.1.2, 2.1.13–2.1.14 to sublinear perturbations of the differential equation $y^{(n)} = 0$ and of the analogous difference equation, which is due to Mate and Nevai [392]. We recall that according to a result of Ghizzetti [225], for any solution $y(t)$ of the differential equation

$$y^{(r)} + \sum_{i=0}^{n-1} g_i(t)y^{(i)}(t) = 0, \quad \textit{for all } t \geq 1, \tag{4.1.1}$$

where

$$\int_1^{+\infty} |\, g_i(x)\,| \, x^{n-i-1} dx < +\infty, \quad 0 \leq i \leq n-1, \tag{4.1.2}$$

either $y(t) = 0$ for all $t \geq 1$ or there is an integer r with $0 \leq r \leq n-1$ such that $\lim_{t\to+\infty} y(t)/t^r$ exists and $\neq 0$. Related results are obtained for difference and differential inequalities. A special case of the former has interesting applications in the study of orthogonal polynomials.

The study of asymptotic behavior of difference equations was initiated by the following result of Poincaré [525] (see also, [408, 433]).

Theorem 4.1.1 (Mate-Nevai, Poincaré [392, 525]) *Let $n \geq 1$ and let f be a function defined for integers $k \geq 1$ such that $f(k) \neq 0$ infinitely often. Suppose*

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that f satisfies the recurrence equation

$$f(k+n) + \sum_{j=0}^{n-1} a_j(k) f(k+j) = 0, \quad k \geq 1, \tag{4.1.3}$$

where $\lim_{k\to+\infty} a_j(k) = a_j$ $(0 \leq j \leq n-1)$*, and the zeros* $\zeta_1, \cdots, \zeta_n$ *of the equation (called characteristic equation)*

$$z^n + \sum_{j=0}^{n-1} a_j z^j = 0$$

all have distinct absolute values. Then for some l with $1 \leq l \leq n$,

$$\lim_{k\to+\infty} f(k+1)/f(k) = \zeta_l. \tag{4.1.4}$$

This section indicates, in particular, that a solution f of the above equation does not oscillate provided all the ζ_l' s are different from 0. Unfortunately, the assumption on the absolute values of the zeros of the characteristic equation is often too restrictive. In particular, of special interest are perturbations of the equation $\triangle^2 f = 0$, where $\triangle$ is the forward difference operator, that is,

$$\triangle f(x) = f(x+1) - f(x),$$

$\triangle^0 f = f$, and $\triangle^{j+1} f = \triangle(\triangle^j f)\,(j \geq 0)$. The following result was shown by Chihara and Nevai [133].

Theorem 4.1.2 (Mate-Nevai, Chihara-Nevai [133, 392]) *Suppose that f satisfies the difference equation*

$$\Delta^2 f(k) = a_{k+1} f(k+2) + b_k f(k+1) + a_k f(k), \quad k \geq 1, \tag{4.1.5}$$

where

$$\sum_{k=1}^{+\infty} k(|a_k| + |b_k|) < +\infty. \tag{4.1.6}$$

Then either

$$f(k) = 0, \tag{4.1.7}$$

for every large enough k, or

$$\lim_{k\to+\infty} f(k)/k^r \tag{4.1.8}$$

exists and $\neq 0$ *for* $r = 0$ *or* 1.

Note that they used this result to give an elementary proof of a theorem of Geronimo and Case [224] on characterizing the support of measures associated with certain orthogonal polynomials (cf., Theorem 1 on p. 371 of their paper [133]), but the result analogous to Theorem 4.1.2 for differential equations is due to Bellman [64], Theorem 5 on p. 14 as follows.

Theorem 4.1.3 (Bellman, Mate-Nevai [64, 392]) *Suppose that f satisfies the differential equation*

$$f''(t) + g(t)f(t) = 0, \quad \text{for all } t \geq 1, \tag{4.1.9}$$

where

$$\int_1^{+\infty} t|g(t)|dt < +\infty. \tag{4.1.10}$$

Then either

$$f(t) = 0 \tag{4.1.11}$$

for all $t \geq 1$ *or*

$$\lim_{t\to+\infty} f(t)/t^r \tag{4.1.12}$$

exists and $\neq 0$ *for* $r = 0$ *or* 1.

In fact, later on, Atkinson [33] slightly generalized the above theorem by formulating a result about an integral equation involving Stieltjes integrals that implies Theorem 4.1.3 as well as the analogous result for the difference equation

$$\triangle^2 f(k) + a_k f(k) = 0 \tag{4.1.13}$$

under the assumption

$$\sum_{k=1}^{+\infty} k|a_k| < +\infty. \tag{4.1.14}$$

However, Atkinson's result does not imply Theorem 4.1.2 above, and it seems to have some difficulties in modifying Atkinson's formulation in a natural way so

as to obtain a result that implies both Theorems 4.1.2 and 4.1.3. Bellman's proof of Theorem 4.1.3 does not lead itself to generalizations in that it does not appear possible to apply his method to prove the result above ever in the case $n = 2$ because of the presence of the term $g_1(t)y'(t)$.

The result of Bellman grew out of investigations by Haupt, who studied the first equation given at the beginning of Sect. 4.1, and proved that the assumptions there imply that $\lim_{t\to+\infty} y^{(n-1)}(t)$ exists, but may be equal to 0; see [261]. Haupt's result actually goes somewhat further the first equation at the beginning of Sect. 4.1 by replacing the 0 on the right-hand side by a function whose integral in the interval $(1, t)$ has a limit as $t \to +\infty$. Lemma 4.1.1 below is essentially the same as Haupt's result for the homogeneous equation. After some slight modifications, the proof for Lemma 4.1.1 can be used to prove Haupt's result for the inhomogeneous equation as well.

Note that Haupt's result was also proved independently but slightly later by Wilkins [645]. Another proof can be found in Belmann [62]. But approximately at the same time that Haupt proved his result, Boas, and Levinson [86] established this for the particular case $n = 2$. Nonoscillation results for the equation in Theorem 4.1.3 were also studied by Hille [280] and other.

Under much stronger assumptions than those in Theorem 4.1.3, we can get results, due to Mate and Nevai [392], about the stability at infinity of solutions of systems of differential equations. We shall only state a very special case, related to Theorems 4.1.2 and 4.1.3, of a result of Yakubovic [653], see also Nemytskii and Stepanov [427].

Theorem 4.1.4 (Mate-Nevai, Yakubovic [392, 653]) *Suppose that f satisfies the differential equation*

$$f^{(n)}(t) + g(t)f(t) = 0, \quad \text{for all } t > 1, \tag{4.1.15}$$

where $n \geq 1$ and

$$\int_1^{+\infty} x^{2n-2}|g(t)|dt < +\infty. \tag{4.1.16}$$

Then there is a polynomial $P(t)$ of degree $< n$ such that

$$\lim_{t\to+\infty} |f(t) - P(t)| = 0. \tag{4.1.17}$$

In the sequel, we shall introduce the results of Mate and Nevai [392], which have generalized Theorems 4.1.2 and 4.1.3 for linear difference and differential inequalities of arbitrary order n, whereby we also strengthen the assertion for the case $n = 2$. The following results, due to Mate and Nevai [392], Theorems 4.1.5 and 4.1.7, have applications for orthogonal polynomials similarly as Theorem 4.1.2 does. Orthogonal polynomials satisfy a second-order recurrence equations, and it may occasionally be advantageous to rewrite this equation as a more manageable

higher-order recurrence equation. This was done in [429], where the original second-order recurrence equation was rewritten as a fourth-order equation.

We first state the result generalizing Theorem 4.1.3. All integrals below are Lebesgue integrals, and all functions whose integrals are taken are assumed to be measurable (but not necessarily integrable); this assumption of measurability only applies to the functions themselves, and not to their derivatives, ever if those are integrated. For example, the measurability of $f^{(n)}$ in Theorem 4.1.5 and Lemma 4.1.1, and of $f^{(k)}$ in Lemma 4.1.2, below, is a consequence of a well-known theorem of Lebesgue in integration theory, and is not covered by the implicit measurability assumption made above.

Theorem 4.1.5 (Mate-Nevai [392]) *Let* $n \geq 1$. *Suppose that the function* f *is differentiable* $n-1$ *times in* $[1, +\infty)$ *and* $f^{(n-1)}$ *is absolutely continuous. Suppose further that* f *satisfies the differential inequality*

$$|f^{(n)}(t)| \leq \sum_{i=0}^{n-1} g_i(t)\,|f^{(i)}(t)|, \quad \text{for a. e. in } [1, +\infty), \tag{4.1.18}$$

where $g_i \geq 0$ *and*

$$\int_1^{+\infty} g_i(x) x^{n-i-1} dx < +\infty, \quad 0 \leq i \leq n-1. \tag{4.1.19}$$

Then either

$$f(t) = 0 \textit{ for all } t \geq 1 \tag{4.1.20}$$

or there is an integer r *with* $0 \leq r \leq n-1$ *such that*

$$\lim_{t\to+\infty} f(t)/t^r \quad \text{exists and} \neq 0. \tag{4.1.21}$$

Theorem 4.1.5 is the immediate consequence of the following result.

Theorem 4.1.6 (Mate-Nevai [392]) *Assume that all the conditions of Theorem 4.1.5 hold. Then either*

$$f = 0 \tag{4.1.22}$$

vanishes for all $t \geq 1$ *or there exists an integer* r *with* $0 \leq r \leq n-1$ *such that*

$$\lim_{t\to+\infty} f^{(j)}(t) = f^{(j)}(+\infty) \quad \text{exists for } r \leq j \leq n-1, \tag{4.1.23}$$

and

$$f^{(r)}(+\infty) \neq 0 \quad \text{and} \quad f^{(j)}(+\infty) = 0 \quad \text{for} \quad r < j \leq n-1. \tag{4.1.24}$$

Moreover, there exists a constant $M > 0$ such that

$$|f^{(j)}(t) - f^{(j)}(+\infty)| \le M \sum_{i=0}^{j} \int_{t}^{+\infty} g_i(x) x^{n-i-1} dx \qquad (4.1.25)$$

holds for all $t \ge 1$ and $r \le j \le n-1$.

On the basis of the proof of this theorem below, it is possible to give an estimate for the size of the constant M in (4.1.25). In fact, writing

$$I(t) = \sum_{i=0}^{n-1} \int_{t}^{+\infty} g_i(x) x^{n-i-1} dx,$$

and choosing a $\tau_0 \ge 1$ such that

$$I(\tau_0) e^{I(\tau_0)} \le 1/2,$$

we can show that (4.1.25) holds for all $t \ge \tau_0$ with

$$M = 11 \max\{ |f^{(i)}(\tau_0)| : 0 \le i \le n-1 \}. \qquad (4.1.26)$$

This estimate for M may be useful when making calculations with the differential equation (4.1.1) for given initial values $f^{(i)}(\tau_0), 0 \le i \le n-1$. In fact, supposing that the value of r is known, we may calculate a $t_r \ge \tau_0$ such that

$$M \sum_{i=0}^{r} \int_{t_r}^{+\infty} g_i(x) x^{n-i-1} dx < \frac{1}{2} |f(t_r)|,$$

where M is given by (4.1.26). Then (4.1.25) with $j = r$ implies that

$$|f^{(r)}(t) - f^{(r)}(+\infty)| < \frac{1}{2} |f^{(r)}(t_r)| \qquad (4.1.27)$$

holds for all $t \ge t_r$. From this inequality it follows that $f^{(r)}(t) \ne 0$ for all $t \ge t_r$. Then we may be able to calculate numbers $t_r \le t_{r-1} \le t_{r-2} \le \cdots \le t_0$ such that $f^{(i)}(t) \ne 0$ for all $t \ge t_i$ $(0 \le i \le r-1)$ simply by noting that $f^{(i)}$ is monotonic for all $t \ge t_{i+1}$ and tends to $\pm\infty$. This way may be feasible to calculate all zeros of $f(t)$.

In practice, however, we may not know the value of r. Even in this case, (4.1.26) can be used to disprove the hypothesis $r = l$ for some l with $0 \le l \le n-1$. To this end, it is enough to find a $t_l > \tau_0$ such that

$$M \sum_{i=0}^{l} \int_{t_l}^{+\infty} g_i(x) x^{n-i-1} dx < \frac{1}{2} |f^{(l)}(t_l)|$$

where M is given by (4.1.26), and then to find a $t \geq t_l$ such that

$$|f^{(l)}(t) - f^{(l)}(t_l)| \geq |f^{(l)}(t_l)|.$$

If now $l = r$, then (4.1.17) should be valid, in contradiction to the last inequality. This approach to disprove $l = r$ is expected to work in case $l < r$, since $\lim_{t\to+\infty} f^{(l)}(t) = \pm\infty$ then, but it will clearly not work in case $l > r$, since then (4.1.26) it satisfied with $l = j$.

To prove Theorem 4.1.7, we need several lemmas.

Lemma 4.1.1 (Mate-Nevai [392]) *If the assumptions of Theorem 4.1.5 hold, then* $\lim_{t\to+\infty} f^{(n-1)}(t)$ *exists.*

Proof We first show that $f^{(n-1)}(t)$ is bounded. To this end, observe that (4.1.18) implies for $1 \leq t_0 \leq t$,

$$|f^{(n-1)}(t) - f^{(n-1)}(t_0)| \leq \int_{t_0}^{t} \sum_{i=0}^{n-1} g_i(x)\,|f^{(i)}(x)|\,dx. \tag{4.1.28}$$

By Taylor's formula with the integral remainder term, for all $x \geq 1$ and $0 \leq i \leq n-2$, we have

$$\begin{aligned} f^{(i)}(x) &= \sum_{j=0}^{n-i-2} f^{(i+j)}(1)(x-1)^j/j! \\ &\quad + \frac{1}{(n-i-2)!}\int_1^x (x-s)^{n-i-2} f^{n-1}(s)ds \\ &= P_{n-i-2}(x) + \frac{1}{(n-i-2)!}\int_1^x (x-s)^{n-i-2} f^{n-1}(s)ds, \end{aligned}$$

where $P_{n-i-2}(x)$ is a polynomial of degree $\leq n-i-2$ in x whose coefficients depend only on $f^{(k)}(1), k = 0, \ldots, n-2$. Substituting this into (4.1.28) with $t_0 = 1$, we obtain

$$\begin{aligned} |f^{(n-1)}(t)| &\leq |f^{(n-1)}(1)| + \int_1^{+\infty} \sum_{i=0}^{n-2} g_i(x)\,|P_{n-i-2}(x)|\,dx \\ &\quad + \int_1^t \sum_{i=0}^{n-2} g_i(x) \int_1^x x^{n-i-2}\,|f^{(n-1)}(s)|\,dsdx + \int_1^t g_{n-1}(x)\,|f^{(n-1)}(x)|\,dx, \end{aligned}$$

where we write $+\infty$ for upper limit of the first integral, replaced $x - s$ by x in the third term on the right-hand side, and omitted the factors $1/(n-i-2)!$ from there. It thus follows from (4.1.19) that the first integral on the right-hand side is convergent. In fact, by looking back on the definitions of the polynomials P_{n-i-2}, it follows that

this integral is

$$\leq \sum_{i=0}^{n-2} c_i \, |f^{(i)}(1)\,| \,,$$

where the positive numbers c_i depend only on the integrals in (4.1.6). Interchanging the integrals in the third term on the right-hand side and then extending the domain of integrating to $+\infty$ in the inner integral, and writing $c_{n-1} = 1$, we obtain for any $t \geq 1$,

$$|f^{(n-1)}(t)\,| \leq \sum_{i=0}^{n-1} c_i \, |f^{(i)}(1)\,| + \int_1^t |f^{(n-1)}(s)\,| \Big(\sum_{i=0}^{n-2} \int_s^{+\infty} x^{n-i-2} g_i(x) dx + g_{n-1}(s) \Big) ds.$$

Note that the integrals on the right-hand side are convergent in view of (4.1.19), since $f^{(n-1)}$ was assumed to be continuous. Hence we derive from Theorem 1.1.2 that

$$|f^{(n-1)}(t)\,| \leq \sum_{i=0}^{n-1} c_i \, |f^{(i)}(1)\,| \exp \int_1^t \Big(\sum_{i=0}^{n-2} \int_s^{+\infty} x^{n-i-2} g_i(x) dx + g_{n-1}(s) \Big) ds$$

holds for every $t \geq 1$. Changing the limit of the outer integral to $+\infty$ on the right-hand side and then interchanging the order of integration, we obtain by (4.1.19) that for any $t \geq 1$,

$$|f^{(n-1)}(t)\,| \leq c$$

where c depends only on $f^i(1)$ for $i = 0, \cdots, n-1$ and on the integrals in (4.1.19). Thus the boundedness of $f^{(n-1)}(t)$ is established.

Integrating this inequality $n - i - 1$ times for $i = 0, 1, \cdots, n-1$, we obtain that there is a positive constant c' depending only on $f^{(j)}(1)$ for $j = 0, \cdots, n-1$ such that for all $t \geq 1$,

$$|f^{(i)}(t)\,| \leq ct^{n-i-1} + c' t^{n-i-2}, \quad 0 \leq i \leq n-1. \tag{4.1.29}$$

Substituting this into (4.1.28), it follows from (4.1.19) that

$$\lim_{t_0 \to +\infty, t \to +\infty} |f^{(n-1)}(t) - f^{(n-1)}(t_0)\,| = 0,$$

i.e., that $\lim_{t\to+\infty} f^{(n-1)}(t) = f^{(n-1)}(+\infty)$ exists. This completes the proof. □

We can now easily derive (4.1.25) with $j = n-1$; thus if $f^{(n-1)}(+\infty) \neq 0$, then Theorem 4.1.6 follows with $r = n-1$. Indeed, making $t \to +\infty$ in (4.1.28) and

then replacing t_0 by t, we obtain that for all $t \geq 1$,

$$|f^{(n-1)}(t) - f^{(n-1)}(+\infty)| \leq \int_t^{+\infty} \sum_{i=0}^{n-1} g_i(x)\,|f^{(i)}(x)|\,dx. \tag{4.1.30}$$

Estimate (4.1.25) with $j = n-1$ now follows with the aid of (4.1.29). Thus Theorem 4.1.6 is established in case $f^{(n-1)}(+\infty) \neq 0$. If, on the other hand, $f^{(n-1)}(+\infty) = 0$, then we obtain from (4.1.30) that

$$|f^{(n-1)}(t)| \leq \int_t^{+\infty} \sum_{i=0}^{n-1} g_i(x)\,|f^{(i)}(x)|\,dx.$$

This motivates the following lemma, which is to support the induction in the proof of Theorem 4.1.5, this lemma will be applied with $k = n-1, n-2, \cdots, 1$.

Lemma 4.1.2 (Mate-Nevai [392]) *Let $k \geq 1$. Suppose $f^{(k-1)}$ is absolutely continuous and f^k is essentially bounded on $[1, +\infty)$, and for a. e. in $[1, +\infty)$,*

$$|f^{(k)}(t)| \leq \int_t^{+\infty} \sum_{i=0}^{k} G_i(x)\,|f^{(i)}(x)|\,dx, \tag{4.1.31}$$

where $G_i \geq 0$ and

$$\int_1^{+\infty} G_i(x) x^{k-i} dx < +\infty, \quad 0 \leq i \leq k. \tag{4.1.32}$$

Then $\lim_{t\to+\infty} f^{(k-1)}(t) = f^{(k-1)}(+\infty)$ exists and for all $t \geq 1$,

$$|f^{(k-1)}(t) - f^{(k-1)}(+\infty)| \leq c \int_t^{+\infty} \sum_{i=0}^{k-1} x G_i(x)\,|f^{(i)}(x)|\,dx, \tag{4.1.33}$$

where

$$c = \exp\left(\int_1^{+\infty} G_k(x)dx\right). \tag{4.1.34}$$

Proof It follows from (4.1.31) that for all $s \geq s_0 \geq 1$, we have for all $s \bar{\in} E$,

$$|f^{(k)}(s)| \leq \int_{s_0}^{+\infty} \sum_{i=0}^{k-1} G_i(x)\,|f^{(i)}(x)|\,dx + \int_s^{+\infty} G_k(x)\,|f^{(k)}(x)|\,dx,$$

where $E \subset [1, +\infty)$ is the set of measure zero where (4.1.31) fails. The integrals on the right-hand side here are convergent in view of (4.1.32), since $f^{(k)}$ was assumed to

be essentially bounded, and so $f^{(i)}(x) = O(x^{k-i})$. Hence we obtain by Theorem 1.1.2 that for all $s \bar{\in} E$

$$|f^{(k)}(s)| \le \int_{s_0}^{+\infty} \sum_{i=0}^{k-1} G_i(x)\,|f^{(i)}(x)|\,dx \cdot \exp \int_s^{+\infty} G_k(x)dx$$
$$\le c \int_{s_0}^{+\infty} \sum_{i=0}^{k-1} G_i(x)\,|f^{(i)}(x)|\,dx,$$

where c was defined in (4.1.34). Writing $s = s_0$ and integrating, we obtain for all $t \ge t_0 \ge 1$,

$$|f^{(k-1)}(t) - f^{(k-1)}(t_0)| \le c \int_{t_0}^{t} \int_s^{+\infty} \sum_{i=0}^{k-1} G_i(x)\,|f^{(i)}(x)|\,dsdx. \tag{4.1.35}$$

If we assume that $\lim_{t\to+\infty} f^{(k-1)}(t)$ exists, then (4.1.33) follows from (4.1.35) via (4.1.32) by making $t \to +\infty$ and then interchanging to the order of integration.

It still remains to show that $\lim_{t\to+\infty} f^{(k-1)}(t)$ exists. First we shall show that $f^{(k-1)}$ is bounded. To this end, note that (4.1.35) implies for all $t \ge t_0 \ge 1$,

$$|f^{(k-1)}(t)| \le |f^{(k-1)}(t_0)| + c \int_{t_0}^{t} \int_s^{+\infty} \sum_{i=0}^{k-1} G_i(x)\,|f^{(i)}(x)|\,dsdx. \tag{4.1.36}$$

As $f^{(k)}(t)$ is essentially bounded, we have, for all $t \ge t_0$,

$$|f^{(k-1)}(t)| \le A_0 t + A_1, \tag{4.1.37}$$

with some constant A_0 and A_1. Integrating this $0, \ldots, k-1$ times, we obtain that for all $t \ge t_0$,

$$|f^{(i)}(t)| \le A_0 t^{k-i} + A_1 t^{k-i-1} + A_2 t^{k-i-2}, 0 \le i \le k-1,$$

where A_2 depends only on $f^{(j)}(t_0)$ for $j = 0, \ldots, k-2$ (note that $t_0 \ge 1$). Substituting this into (4.1.36) and making extensions of the domain of integration, we obtain

$$|f^{(k-1)}(t)| \le |f^{(k-1)}(t_0)| + c \int_1^{t} \int_{t_0}^{+\infty} \sum_{i=0}^{k-1} A_0 G_i(x) dsdx$$
$$+c \int_{t_0}^{+\infty} \int_s^{+\infty} \sum_{i=0}^{k-1} (A_1 t^{k-i-1} + A_2 t^{k-i-2}) G_i(x)\,|f^{(i)}(x)|\,dsdx$$
$$\le |f^{(k-1)}(t_0)| + A_0 c J(t_0) t + (A_1 + A_2) c J(t_0), \tag{4.1.38}$$

where we have interchanged the order of integration in the second integral and then introduced the abbreviation

$$J(t_0) = \int_{t_0}^{+\infty} \sum_{i=0}^{k-1} x^{k-i} G_i(x) dx;$$

this integral is finite according to (4.1.32). Thus inequality (4.1.37) leads to inequality (4.1.38), where, as we noted above, A_2 depends only on $f^{(j)}(t_0)$ for $j = 0, \ldots, k-2$; in particular, it is independent of A_0 and A_1. Estimate (4.1.38) is the same type of inequality as (4.1.37), with $A_0 cJ(t_0)$ replacing A_0 and $|f^{(k-1)}(t_0)| + (A_1 + A_2)cJ(t_0)$ replacing A_1. Thus, in the same way as we obtained (4.1.38) from (4.1.37), we can use (4.1.38) to obtain a new inequality of the same kind. If t_0 is large enough so that $cJ(t_0) < 1$, then we can iterate this procedure infinitely many times, we obtain for every $t \geq t_0$,

$$\begin{aligned} |f^{(k-1)}(t)| &\leq (|f^{(k-1)}(t_0)| + A_2 cJ(t_0)) \sum_{j=0}^{\infty} \\ &= (|f^{(k-1)}(t_0)| + A_2 cJ(t_0))/(1 - cJ(t_0)). \end{aligned}$$

This shows that $f^{(k-1)}(t)$ is bounded.

It is easy to show that $\lim_{t\to+\infty} f^{(k-1)}(t)$ exists. Indeed, as we have just shown, for all $x \geq 1$,

$$|f^{(k-1)}(x)| \leq C.$$

Integrating this $0, \cdots, k-1$ times, we obtain that for all $x \geq 1$,

$$|f^{(i)}(x)| \leq Cx^{k-i-1} + C' x^{k-i-2}, \quad 0 \leq i \leq k-1. \tag{4.1.39}$$

Substituting this into (4.1.35), then extending the upper limit of the outer integral on the right-hand side to $+\infty$, and then interchanging the order of integration, it follows from (4.1.32) that

$$\lim_{t\to+\infty, t_0\to+\infty} |f^{(k-1)}(t) - f^{(k-1)}(t_0)| = 0,$$

i.e., that $\lim_{t\to+\infty} f^{(k-1)}(t)$ exists. This completes the proof. □

Note that the integral in (4.1.34) is convergent in view of (4.1.32), and it is an easy consequence of the assumptions that $\operatorname{esslim}_{t\to+\infty} f^{(k)}(t) = 0$, but we shall not use this fact directly.

Proof of Theorem 4.1.6 Applying Lemma 4.1.1 and then repeatedly applying Lemma 4.1.2, if possible, with $k = n-1, n-2, \cdots, 1$ to the f in Theorem 4.1.6, we can conclude that either (i) $\lim_{t\to+\infty} f^{(n-1)}(t) \neq 0$ or (ii) there is an l with

$1 \le l \le n-1$ such that $\lim_{t\to+\infty} f^{(l)}(t) = 0$ and $\lim_{t\to+\infty} f^{(l-1)}(t)$ exists and $\neq 0$, or (iii) $\lim_{t\to+\infty} f(t) = 0$ and for all $t \ge 1$,

$$|f(t)| \le K \int_t^{+\infty} x^{n-1} g_0(x)\, |f(x)|\, dx, \tag{4.1.40}$$

with some constant $K > 0$; this inequality follows from (4.1.33) in the last application (with $k = l = 1$) of Lemma 4.1.2.

In cases (i) and (ii), (4.1.23) and (4.1.24) follow with $r = n-1$ or $r = l-1$, respectively. As for (4.1.25), for $j = n-1$ this follows from (4.1.29) and (4.1.30), as was pointed out right after (4.1.30), and for $r \le j < n-1$ (which can only occur in case (ii)), it follows from (4.1.33) and (4.1.39) with $k = j+1$. In case (iii), $f(t) = 0$ for all $t \ge 1$ follows from (4.1.40) via Theorem 1.1.5 with $n(t) = 0$ (note that the integral on the right-hand side of (4.1.40) is convergent in view of (4.1.19), as f is bounded in the case considered). This completes the proof of Theorem 4.1.6. □

In the sequel, we shall prove the analogues of Theorems 4.1.5 and 4.1.6 for difference inequalities. Although the proofs of Theorems 4.1.7 and 4.1.8 closely parallel that of Theorems 4.1.5 and 4.1.6, we shall give the proof in almost complete detail as there are several subtle differences that are difficult to appreciate without actually following the proof closely. It was only the final argument in the proof of Lemma 4.1.3 where we thought the methods of the proof of Lemma 4.1.2 can be carried over without using special technical tricks peculiar to finite differences; accordingly, we did not give this part of the proof of Lemma 4.1.3 in detail. While the proof of Theorem 4.1.8 is self-contained aside from the above-mentioned exception, it is still easier to follow it if we read the proof of Theorem 4.1.5 first, which contains fewer technical complications.

In order to emphasize the parallel between the proofs of Theorems 4.1.6 and 4.1.8, we tried to retain the notation above as much as possible. In particular, we shall continue to use s, t, and x for denoting independent variables, but now it is always assumed that these variables range over integers. The functions f, g, F, and G will now be defined only for integers. $\triangle$ will denote the forward difference operator, that is,

$$\triangle f(x) = f(x+1) - f(x),$$

$\triangle^0 f = f$, and $\triangle^{j+1} f = \triangle(\triangle^j f)$ $(j \ge 0)$.

Theorem 4.1.7 (Mate-Nevai [392]) *Let f be a function defined for integers such that $f(t) = 0$ for any integer $t < t'$, where $t' \le 0$. Let $n > 0$ be an integer and suppose that f satisfies the difference inequality*

$$|\triangle^n f(t)| \le \sum_{i=0}^{n-1} g_i(t) \sum_{l=0}^{n-i} |\triangle^i f(t+l)| \tag{4.1.41}$$

for every integer $t \geq 1$*, where* $g_i \geq 0$ *and*

$$\sum_{x=1}^{+\infty} g_i(x)x^{n-i-1} < +\infty, \quad 0 \leq i \leq n-1. \tag{4.1.42}$$

Then either $f(t) = 0$ *for every large enough* t *or there is an integer* r *with* $0 \leq r \leq n-1$ *such that*

$$\lim_{t \to +\infty} f(t)/t^r \quad \text{exists and} \quad \neq 0 \tag{4.1.43}$$

This theorem is an easy consequence of the following result (to derive (4.1.43) from (4.1.44) and (4.1.45), see, e.g., (4.1.53) below).

Theorem 4.1.8 (Mate-Nevai [392]) *Assume that all the conditions of Theorem 4.1.7 hold. Then either* f *vanishes for all large enough* t *or there exists an integer* r *with* $0 \leq r \leq n-1$ *such that*

$$\lim_{t \to +\infty} \triangle^j f(t) = \triangle^j f(+\infty) \quad \text{exists for } r \leq j \leq n-1, \tag{4.1.44}$$

and

$$\triangle^r f(+\infty) \neq 0 \quad \text{and} \quad r < j \leq n-1. \tag{4.1.45}$$

Moreover, there exists a constant $M > 0$ *such that*

$$|\, \triangle^j f(t) - \triangle^j f(+\infty)\,| \leq M \sum_{i=0}^{j} \sum_{x=t-n+i}^{+\infty} g_i(x)x^{n-i-1} \tag{4.1.46}$$

holds for $r \leq j \leq n-1$ *and for every large enough* t*.*

Similarly as remarked after Theorem 4.1.6, on the basis of the proof of this theorem, it is possible to give an estimate for the size of the constant M in (4.1.46). In fact, writing

$$S(t) = \sum_{i=0}^{n-1} \sum_{x=t-n+i}^{+\infty} g_i(x)x^{n-i-1},$$

and choosing a $\tau_0 > n^2$ such that

$$2neS(\tau_0)e^{2neS(\tau_0)} \leq \frac{1}{2},$$

we can show that (4.1.46) holds with

$$M = 22n \max\{\, |\triangle^i f(\tau_0)|\, : 0 \le i \le n-1\}. \tag{4.1.47}$$

This estimate for M may be useful when making calculations with difference equations analogous to the first differential equation in this section. Similarly as (4.1.26) was useful in finding a zero-free interval $[t_0, +\infty)$ of a solution of that differential equation, (4.1.47) may be used to find an interval $[t_0, +\infty)$ in which a certain solution of the analogous difference equation has constant sign. Also, (4.1.46) can be used to disprove the hypothesis $r = l$ in (4.1.46) for some l with $0 \le l \le n-1$, similarly as it was possible to use (4.1.26) to disprove the hypothesis $r = l$ in (4.1.25).

These applications of (4.1.47) may be of significance, for example, in describing the support of the measure associated with a system of orthogonal polynomials. If, for example, $p_k(x)$ is a set of polynomial defined by the recurrence relation

$$p_{k+2}(x) - 2xp_{k+1}(x) + p_k(x) = a_{k+1}p_{k+2}(x) + b_k p_{k+1}(x) + a_k p_k(x) \tag{4.1.48}$$

for $k \ge -1$, where

$$p_{-1}(x) = 0 \quad \text{and} \quad p_0(x) = 1, \tag{4.1.49}$$

then it is easy to check that for real a_k, b_k the assumption $a_k < 1$ for $k \ge -1$ ensures that these polynomials form a system of orthogonal polynomials with respect to a certain measure $\mathrm{d}\alpha$ on the real line. The uniqueness of this measure is guaranteed by certain conditions on the coefficients a_k, b_k; for example, relation (4.1.6) guarantees uniqueness (because it guarantees that the support of $\mathrm{d}\alpha$ is bounded, which is a condition sufficient for uniqueness). We often consider the matrix

$$\begin{pmatrix} \beta_{-1} & \alpha_0 & 0 & 0 & \cdots \\ \alpha_0 & \beta_0 & \alpha_1 & 0 & \cdots \\ 0 & \alpha_1 & \beta_1 & \alpha_2 & \cdots \\ 0 & 0 & \alpha_2 & \beta_2 & \cdots \\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \end{pmatrix}$$

a so-called Jacobi matrix, where $\alpha_k = (1 - a_k)/2$ and $\beta_k = -b_k/2$; these substitutions convert (4.1.48) into the formula

$$xp_{k+1}(x) = \alpha_{k+1}p_{k+2}(x) + \beta_k p_{k+1}(x) + \alpha_k p_k(x).$$

For example, condition (4.1.6) guarantees that this matrix is a bounded self-adjoint operator on the space l^2 and its spectrum equals the support of the measure $d\alpha$. In fact, if $\sum_{k=0}^{+\infty}\lambda_k p_k(x)$ is the orthogonal expansion of function $f(x)$ on the real line, then the coefficients of the expansion of the function $xf(x)$ is given by $A\lambda$, where λ is the column vector whose kth component is λ_k. Determination of the spectrum of A is important in various applications, we may refer to, e.g., Dunford and Schwartz [199], Shohat and Tamarkin [590], and Stone [609] on Jacobi matrices and the related Hamburger moment problem.

Condition (4.1.6) in Theorem 4.1.2 ensures that the spectrum of A consists of subset of the interval $[-1, 1]$ plus finitely many points. It may be possible to give an estimate for the number of these points by using (4.1.47). What we need to do is to consider a solution of Eq. (4.1.5) for $k \geq -1$ with the initial conditions $f(-1) = 0$ and $f(0) = 1$; then we have $f(k) = p_k(1)$ according to (4.1.48) and (4.1.49). If we can find an integer k_0 such that $p_k(l) = f(k)$ has constant sign for $k \geq k_0$, then we can conclude that, for any $k \geq k_0$, the number of zeros of $p_k(x)$ and of $p_{k_0}(x)$ in the interval $[1, +\infty)$ is the same. This follows from a simple property of the zeros of orthogonal polynomials (see, e.g., Szegö [616]). Thus $p_k(x)$ has at most k_0 zeros in the interval $(1, +\infty)$ (see, Szegö [616]). In a similar way, we may also obtain a bound for the number of points in the interval $(-\infty, -1)$ that belong to the spectrum of A. Some of the details of the above argument can be found in Chihara and Nevai [133].

To prove Theorem 4.1.8, we need Theorems 2.1.14–2.1.15, which parallel Theorems 1.1.2 and 1.1.4 respectively.

Lemma 4.1.3 (Mate-Nevai [392]) *If the assumptions of Theorem 4.1.7 hold, then* $\lim_{t\to+\infty} \triangle^{n-1}f(t)$ *exists.*

Proof We first show that $\triangle^{n-1}f(t)$ is bounded. To this end, observe that (4.1.41) implies, for $1 \leq t_0 \leq t$,

$$\begin{aligned} |\,\triangle^{n-1}f(t) - \triangle^{n-1}f(t_0)\,| &= |\sum_{x=t_0}^{t-1}\triangle^n f(x)\,| \\ &\leq \sum_{x=t_0}^{t-1}\sum_{i=0}^{n-1} g_i(x)\sum_{l=0}^{n-i}|\,\triangle^i f(x+l)\,|\,. \end{aligned} \tag{4.1.50}$$

It is easy to show that if h is a function defined on integers such that $h(x) = 0$ for any $x < x_0$, then

$$h(x) = \sum_{s=-\infty}^{x-k}\binom{x-s-1}{k-1}\triangle^k h(s)$$

holds for each positive integer k and every integer x. Thus, for $0 \leq i \leq n-2$, we have

$$\triangle^i f(x) = \sum_{s=-\infty}^{x-n+i+1} \binom{x-s-1}{n-i-2} \triangle^{n-1} f(s). \tag{4.1.51}$$

Hence, for $0 \leq i \leq n-2$ and $0 \leq l \leq n-i$,

$$\begin{aligned} |\triangle^i f(x+l)| &= \sum_{s=-\infty}^{x+1} (x+n-s)^{n-i-2} |\triangle^{n-1} f(s)| \\ &\leq |P_{n-i-2}(x,t_0)| + \sum_{s=t_0}^{x+1} (x+n)^{n-i-2} |\triangle^{n-1} f(s)|, \end{aligned}$$

where P_{n-i-2} is a polynomial of degree $\leq n-i-2$ in x whose coefficients depend only on $\triangle^{n-1} f(s)$ for $s < t_0$ (all but finitely many of these are zero). Substituting this into (4.1.50) we obtain

$$\begin{aligned} |\triangle^{n-1} f(t)| &\leq |\triangle^{n-1} f(t_0)| + \sum_{i=0}^{n-1}\sum_{x=t_0}^{+\infty} n g_i(x) |P_{n-i-2}(x,t_0)| \\ &\quad + \sum_{i=0}^{n-2}\sum_{x=t_0}^{t-1} n g_i(x) \sum_{s=t_0}^{x+1} (x+n)^{n-i-2} |\triangle^{n-1} f(s)| \\ &\quad + \sum_{x=t_0}^{t-1} g_{n-1}(x)(|\triangle^{n-1} f(x)| + |\triangle^{n-1} f(x+1)|), \end{aligned} \tag{4.1.52}$$

where we changed the upper limit if the summation on x to $+\infty$ in the second term. It follows from (4.1.52) and the definition of the polynomials P_{n-i-2} that the sum of the first and second terms on the right-hand side here can be dominated b a constant $K(t_0) \geq 0$ that depends only on the sums in (4.1.42) and on $\triangle^{n-1} f(s)$ for $s \leq t_0$. After interchanging the two inner sums in the third term and then extending the summation with respect to x to infinity, we can see that this term is dominated by

$$\sum_{i=0}^{n-2}\sum_{x=t_0}^{t} |\triangle^{n-1} f(s)| \sum_{x=s-1}^{+\infty} n g_i(x)(x+n)^{n-i-2}.$$

Finally, changing the variable x to s in the last term and making a simple rearrangement, we can see that this term is dominated by

$$\sum_{s=t_0}^{t} |\triangle^{n-1} f(s)| (g_{n-1}(s-1) + g_{n-1}(s)).$$

Thus, (4.1.52) becomes

$$| \triangle^{n-1} f(t) | \le K(t_0) + \sum_{s=t_0}^{t} | \triangle^{n-1} f(s) | \times \left(\sum_{i=0}^{n-2} \sum_{x=s-1}^{+\infty} n g_i(x)(x+n)^{n-i-2} + g_{n-1}(s-1) + g_{n-1}(s) \right),$$

where $1 \le t_0 \le t$. It follows from (4.1.42) that the coefficient of $\triangle^{n-1} f(s)$ in the right-hand side tends to 0 as $s \to +\infty$. Choose $t_1 \ge 1$ so large that this coefficient is $\le \frac{1}{2}$ for $s \ge t_1$. Then it follows from the above inequality with t_1 replacing t_0 by virtue of Theorem 2.1.14 (i) that

$$| \triangle^{n-1} f(t) | \le 2K(t_1) \exp 2 \sum_{s=t_1}^{t-1} \Big(\sum_{i=0}^{n-2} \sum_{x=s-1}^{+\infty} n g_i(x)(x+n)^{n-i-2} + g_{n-1}(s-1) + g_{n-1}(s) \Big)$$

holds for $s \ge t_1$. Changing the upper limit of the outer summation to $+\infty$ on the right-hand side and then interchanging the summations with respect to x and s, we obtain by (4.1.42) that $| \triangle^{n-1} f(t) |$ is bounded, i.e., there is a number $c \ge 0$ such that

$$| \triangle^{n-1} f(t) | \le c$$

holds for any integer t. Summing this $n-i-1$ times for $i = 0, 1, \ldots, n-1$, we obtain that there is a c' depending only on $\triangle^{n-1} f(s)$ for $s \le 0$ such that for all $t \ge 1$,

$$| \triangle^i f(t) | \le c t^{n-i-1} + c' t^{n-i-2}, \quad 0 \le i \le n-1. \tag{4.1.53}$$

Substituting this into (4.1.50) it follows from (4.1.42) that

$$\lim_{t_0 \to +\infty, t \to +\infty} | \triangle^{n-1} f(t) - \triangle^{n-1} f(t_0) | = 0,$$

i.e., that $\lim_{t \to +\infty} \triangle^{n-1} f(t) = \triangle^{n-1} f(+\infty)$ exists. This completes the proof of the lemma. □

If $\triangle^{n-1} f(+\infty) \ne 0$, then Theorem 4.1.7 easily follows with $r = n-1$. To see this, we have to establish (4.1.46). Taking $t \to +\infty$ in (4.1.50) and then replacing t_0 by t, we obtain that for all $t \ge 1$,

$$| \triangle^{n-1} f(t) - \triangle^{n-1} f(+\infty) | \le \sum_{x=t}^{+\infty} \sum_{i=0}^{n-1} g_i(x) \sum_{l=0}^{n-i} | \triangle^i f(x+l) | . \tag{4.1.54}$$

Equation (4.1.46) now follows with the aid of (4.1.53). If, on the other hand, $\triangle^{n-1}f(+\infty) = 0$, then we derive from (4.1.54) that for all $t \geq 1$,

$$
\begin{aligned}
|\triangle^{n-1}f(t)| &\leq \sum_{x=t}^{+\infty}\sum_{i=0}^{n-1} g_i(x) \sum_{l=0}^{n-i} |\triangle^i f(x+l)| \\
&\leq \sum_{x=t}^{+\infty}\sum_{i=0}^{n-1} G_i^*(x) |\triangle^i f(x)| \qquad (4.1.55)
\end{aligned}
$$

holds, where

$$
G_i^*(x) = \sum_{l=0}^{n-i} g_i(x-l). \qquad (4.1.56)
$$

This motivates the following lemma, which is to support the induction in the proof of Theorem 4.1.8; this lemma will be applied with $k = n-1, n-2, \cdots, 1$.

Lemma 4.1.4 (Mate-Nevai [392]) *Let $k \geq 1$, and suppose that $f(t) = 0$ for all $t < t'$, where $t' \leq 0$. Suppose, further, that $\triangle^k f$ is bounded and for every $t \geq t_1$,*

$$
|\triangle^k f(t)| \leq \sum_{x=t}^{+\infty}\sum_{i=0}^{k} G_i(x) |\triangle^i f(x)| \qquad (4.1.57)
$$

holds, where $t_1 > 0, G_i \geq 0$, and

$$
\sum_{x=1}^{+\infty} G_i(x) x^{k-i} < +\infty, \quad 0 \leq i \leq k. \qquad (4.1.58)
$$

Then $\lim_{t\to+\infty} \triangle^{k-1}f(t) = \triangle^{k-1}f(+\infty)$ exists, and there is an integer $t_2 > 0$ such that for every $t \geq t_2$,

$$
|\triangle^{k-1}f(t) - \triangle^{k-1}f(+\infty)| \leq c \sum_{x=t}^{+\infty}\sum_{i=0}^{k-1} x G_i(x) |\triangle^i f(x)| \qquad (4.1.59)
$$

holds, where

$$
c = 2\exp\Big(2 \sum_{x=t_2+1}^{+\infty} G_k(x)\Big). \qquad (4.1.60)
$$

Note that the sum in (4.1.60) is convergent in view of (4.1.58).

Proof It follows from (4.1.57) that for any s and s_0 with $s \geq s_0 \geq t_1$, we have

$$|\triangle^k f(s)| \leq \sum_{x=s_0}^{+\infty}\sum_{i=0}^{k-1} G_i(x)\,|\triangle^i f(x)| + \sum_{x=s}^{+\infty} G_k(x)\,|\triangle^k f(x)|.$$

The infinite sums on the right-hand side are convergent in view of (4.1.58), since $\triangle^k f$ was assumed to be bounded, and so $\triangle^i f(x) = O(x^{k-i})$. Choose $t_2 \geq t_1$ so that $G_k(x) \leq 1/2$ for all $x \geq t_2$. Then we obtain from Theorem 2.1.15 (ii) that for every s (and s_0) with $s \geq t_2$ (and $s \geq s_0 \geq t_1$),

$$\begin{aligned}|\triangle^k f(s)| &\leq 2\sum_{x=s_0}^{+\infty}\sum_{i=0}^{k-1} G_i(x)\,|\triangle^i f(x)|\,\exp\left(2\sum_{x=s+1}^{+\infty} G_k(x)\right)\\ &\leq c\sum_{x=s_0}^{+\infty}\sum_{i=0}^{k-1} G_i(x)\,|\triangle^i f(x)|\end{aligned} \tag{4.1.61}$$

holds, where c is as given in (4.1.60). Writing $s = s_0$ in (4.1.61) and then summing for s, we obtain that for any t and t_0 with $t \geq t_0 \geq t_2$,

$$|\triangle^{k-1} f(t) - \triangle^{k-1} f(t_0)| \leq c\sum_{s=t_0}^{t-1}\sum_{x=s}^{+\infty}\sum_{i=0}^{k-1} G_i(x)\,|\triangle^i f(x)| \tag{4.1.62}$$

holds. If we assume $\lim_{t\to+\infty} \triangle^{k-1} f(t)$ exists, then (4.1.59) follows from here by taking $t \to +\infty$ and then interchanging the summations with respect to s and x.

It will remains to show that $\lim_{t\to+\infty} \triangle^{k-1} f(t)$ exists. This can be done by using arguments analogous to the ones described in the proof of Lemma 4.1.2. Instead of formula (4.1.35), we have to use formula (4.1.62), and the details can be easily worked out. The proof of the lemma is complete. □

Proof of Theorem 4.1.8 Applying Lemma 4.1.3 and then repeatedly applying Lemma 4.1.4, if possible, with $k = n-1, n-2, \ldots, 1$, to the f in Theorem 4.1.5, we can conclude that either (i) $\lim_{t\to+\infty} \triangle^{n-1} f(t) \neq 0$ (this limit exists according to Lemma 4.1.3; it is not possible to apply Lemma 4.1.4 at all in this case), or (ii) there is an l with $1 \leq l \leq n-1$ such that $\lim_{t\to+\infty} \triangle^l f(t) = 0$ and $\lim_{t\to+\infty} \triangle^{l-1} f(t)$ exists and $\neq 0$ (this is obtained through $n-l$ applications of Lemma 4.1.4, with $k = l$ in the last application, and with no more applications possible), or (iii) $\lim_{t\to+\infty} f(t) = 0$ and for every $t \geq t^*$,

$$|f(t)| \leq K\sum_{x=t}^{+\infty} x^{n-1} G_0^*(x)\,|f(x)| \tag{4.1.63}$$

where G_0^* was defined in (4.1.56), $K > 0$ is a constant, and $t^* > 0$ is an integer; this inequality follows from (4.1.59) in the last application (with $k = l = 1$) of Lemma 4.1.4.

In cases (i) and (ii), (4.1.44) and (4.1.45) follow with $r = n - 1$ or $r = l - 1$, respectively. As for (4.1.46), for $j = n-1$ this follows from (4.1.53) and (4.1.54), as was pointed out right after (4.1.54) (as we indicated there, the second summation on the right-hand side of (4.1.46) can start with $x = t$ in this case). To establish (4.1.46) for $r \le j < n - 1$ combining (4.1.59) for $k = j + 1$ with the analogue of (4.1.39) (which is not given above, but would occur in a detailed presentation of the end of the proof of Lemma 4.1.4), it follows that for every large enough t,

$$| \triangle^j f(t) - \triangle^j f(+\infty) | \le M' \sum_{x=t}^{+\infty} \sum_{i=0}^{j} G_i^*(x) x^{n-i-1}$$

holds, where G_i^* was defined in (4.1.56) and $M' > 0$ is a constant. This inequality gives (4.1.46) for $r \le j < n - 1$ as well. In case (iii), let $t_0 \ge t^*$ be such that $x^{n-1} G_0^*(x) \le 1/2$ for all $x \ge t_0$; then $f(t) = 0$ for $t \ge t_0$ follows from (4.1.63) via Theorem 2.1.15 (ii) with $C = 0$ (note that the sum on the right-hand side of (4.1.63) is convergent in view of (4.1.42), as f is bounded in the case considered). This completes the proof of Theorem 4.1.8. $\square$

4.2 Applications of Theorem 1.1.5 to Terminal Value Problem of Differential Equations

In this section, we present some applications of Theorem 1.1.5 to study certain properties of solutions of the following terminal value problem for differential equation

$$\begin{cases} u'(t) = f(t, u(t)) + p(t), & (4.2.1) \\ u(+\infty) = u_\infty, & (4.2.2) \end{cases}$$

where $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $p : \mathbb{R}_+ \to \mathbb{R}$ are continuous functions and $u_{+\infty} \in \mathbb{R}$.

Following theorem provides estimates for a solution of problem (4.2.1)–(4.2.2).

Theorem 4.2.1 (Pachpatte-Pachpatte [510]) *Suppose that*

$$|f(t, u)| \le b(t) |u|, \tag{4.2.3}$$

$$|u_\infty - Q(t)| \le a(t), \tag{4.2.4}$$

where $a(t)$ *is decreasing continuous function defined on* $\mathbb{R}_+$ *and* $b(t) \geq 0$ *is a continuous function defined on* $\mathbb{R}_+$ *satisfying* $\int_0^{+\infty} b(t)dt < +\infty$ *and* $Q(t) = \int_t^{+\infty} p(s)ds$.

If $u(t)$ *is a solution of problem (4.2.1)–(4.2.2), then for all* $t \in \mathbb{R}_+$,

$$|\, u(t)\,| \leq a(t)\exp\Big(\int_t^{+\infty} b(s)ds\Big). \tag{4.2.5}$$

Proof If $u(t)$ is solution of problem (4.2.1)–(4.2.2), then it can be written as (see [42], p. 80) for all $t \in \mathbb{R}_+$,

$$u(t) = u_\infty - \int_t^{+\infty} [f(s, u(s)) + p(s)]ds. \tag{4.2.6}$$

From (4.2.6) and (4.2.3)–(4.2.4), we derive

$$|\, u(t)\,| \leq a(t) + \int_t^{+\infty} b(s)\,|\, u(s)\,|\, ds. \tag{4.2.7}$$

Now applying Theorem 1.1.5 to (4.2.7) yields the required estimate (4.2.5). □

Next we shall prove the uniqueness of the solutions of problem (4.2.1)–(4.2.2).

Theorem 4.2.2 (Pachpatte-Pachpatte [510]) *Suppose that the function* f *in (4.2.1) satisfies the condition*

$$|f(t,u) - f(t,v)\,| \leq b(t)\,|\, u - v\,|, \tag{4.2.8}$$

where $b(t)$ *is defined as in Theorem 4.2.1. Then the problem (4.2.1)–(4.2.2) has at most one solution on* $\mathbb{R}_+$.

Proof The problem (4.2.1)–(4.2.2) is equivalent to the integral equation (4.2.6). Let $u(t)$ and $v(t)$ be two solutions of problem (4.2.1)–(4.2.2) on $\mathbb{R}_+$. Thus from (4.2.6) and (4.2.8) it follows

$$|\, u(t) - v(t)\,| \leq \int_t^{+\infty} b(s)\,|\, u(s) - v(s)\,|\, ds. \tag{4.2.9}$$

Now applying Theorem 1.1.5 to (4.2.9) yields $u(t) = v(t)$, i.e., there is at most one solution of the problem (4.2.1)–(4.2.2). □

The next result shows the continuous dependence of solutions to problem (4.2.1)–(4.2.2) on terminal values.

Theorem 4.2.3 (Pachpatte-Pachpatte [510]) *Let $u_1(t)$ and $u_2(t)$ be two solutions of Eq. (4.2.1) with the given terminal conditions*

$$u_1(+\infty) = u_{1\infty} \tag{4.2.10}$$

and

$$u_2(+\infty) = u_{2\infty} \tag{4.2.11}$$

respectively, where $u_{1\infty}, u_{1\infty} \in \mathbb{R}$. Suppose that the function f in (4.2.1) satisfies the condition (4.2.8) in Theorem 4.2.2. Then for all $t \in \mathbb{R}_+$,

$$|\, u_1(t) - u_2(t) \,| \;\leq\; |\, u_{1\infty} - u_{2\infty} \,| \exp\Big(\int_t^{+\infty} b(s)ds \Big). \tag{4.2.12}$$

Proof By using the facts that $u_1(t)$ and $u_2(t)$ are the solutions of problem (4.2.1), (4.2.10) and problem (4.2.1), (4.2.11) respectively, we have

$$u_1(t) - u_2(t) = u_{1\infty} - u_{2\infty} - \int_t^{+\infty} [f(s, u_1(s)) - f(s, u_2(s))]ds. \tag{4.2.13}$$

We deduce from (4.2.13) and (4.2.8),

$$|\, u_1(t) - u_2(t) \,| \;\leq\; |\, u_{1\infty} - u_{2\infty} \,| + \int_t^{+\infty} b(s) \,|\, u_1(s) - u_2(s) \,|\, ds. \tag{4.2.14}$$

Now applying Theorem 1.1.5 to (4.2.14) yields the required estimate in (4.2.12). □

We next consider the following differential equations

$$u'(t) = f(t, u(t), \mu), \tag{4.2.15}$$

$$u'(t) = f(t, u(t), \mu_0), \tag{4.2.16}$$

with the given terminal value condition (4.2.2), where $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and μ, μ_0 are real parameters.

The following theorem shows the continuous dependence of solutions to problem (4.2.15), (4.2.2) and problem (4.2.16), (4.2.2) on pure parameters.

Theorem 4.2.4 (Pachpatte-Pachpatte [510]) *Suppose that the function satisfies the conditions*

$$|f(t, u, \mu) - f(t, v, \mu)\,| \;\leq b(t) \,|\, u - v \,|, \tag{4.2.17}$$

$$|f(t, u, \mu) - f(t, v, \mu)\,| \;\leq c(t) \,|\, \mu - \mu_0 \,|, \tag{4.2.18}$$

where $b(t), c(t) : \mathbb{R}_+ \to \mathbb{R}_+$ *are continuous functions. If* $u_1(t)$ *and* $u_2(t)$ *are the solutions of problem (4.2.15), (4.2.2) and problem (4.2.16), (4.2.2) for all* $t \in \mathbb{R}_+$ *respectively, then for all* $t \in \mathbb{R}_+$,

$$|\, u_1(t) - u_2(t)\,| \le B(t)\exp\Big(\int_t^{+\infty} b(s)ds\Big), \tag{4.2.19}$$

where for all $t \in \mathbb{R}_+$,

$$B(t) = |\, \mu - \mu_0\,| \int_t^{+\infty} c(s)ds. \tag{4.2.20}$$

Proof Let $z(t) = u_1(t) - u_2(t)$, for all $t \in \mathbb{R}_+$. As in the proof of Theorem 4.2.3, from the hypotheses, we derive

$$\begin{aligned} |z(t)| &= \left|\int_t^{+\infty} [f(s, u_1(s), \mu) - f(s, u_2(s), \mu) + f(s, u_2(s), \mu) - f(s, u_2(s), \mu_0)]ds\right| \\ &\le B(t) + \int_t^{+\infty} b(s)\,|\,z(s)\,|\,ds \end{aligned} \tag{4.2.21}$$

where $B(t)$ is defined by (4.2.20). Clearly $B(t)$ is non-negative, continuous and non-increasing for all $t \in \mathbb{R}_+$. Therefore applying Theorem 1.1.5 to (4.2.21) yields the required estimate (4.2.19). □

4.3 Applications of Theorem 1.2.12 to Perturbations of Nonlinear Systems of Differential Equations

In this section, we shall use Theorem 1.2.12 and the nonlinear variation of constants formula of Alekseev [25] to study the behavior of solutions of a nonlinear differential system of the form

$$y'(t) = f(t, y(t)) + g(t, y(t), Ty(t)), \tag{4.3.1}$$

which will be studied as a perturbation of the nonlinear differential system

$$x'(t) = f(t, x(t)). \tag{4.3.2}$$

Here x, y, f and g are the elements of $\mathbb{R}^n$, an n-dimensional Euclidean space. Primes will always denote differentiation with respect to t. Let I be the interval$0 \le t < +\infty$ and $C(X, Y)$ denote the space of continuous functions from X to Y where X and Y are any convenient spaces. We shall assume that $f \in C(I \times \mathbb{R}^n, \mathbb{R}^n)$, and $f_x(t, x)$ exists and is continuous on $I \times \mathbb{R}^n$ into $\mathbb{R}^n$, that $g \in C(I \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$,

and that T is a continuous operator which maps $\mathbb{R}^n$ into $\mathbb{R}^n$. We denote by $y(t, t_0, y_0)$ the solution of Eq. (4.3.1) through the initial point (t_0, x_0) for $t_0 > 0$. The symbol $|\cdot|$ will denote some convenient norm on $\mathbb{R}^n$ as well as a corresponding consistent matrix norm. This result is chosen from [460].

It is well-known [150] that the derivative matrix

$$\frac{\partial}{\partial x_0}[x(t, t_0, x_0)] = \Phi(t, t_0, x_0)$$

exists, and satisfies the variational equation

$$z'(t) = f_x(t, x(t, t_0, x_0))z(t) \tag{4.3.3}$$

such that

$$\Phi(t_0, t_0, x_0) = E \text{ (identity matrix)}.$$

Pachpatte [460] defined a new kind of stability as follows in terms of the behavior of solutions of (4.3.2) and the variational equation (4.3.3).

Definition 4.3.1 The solution $x = 0$ of Eq. (4.3.2) is said to be exponentially asymptotically stable in variation if there exist constants $M > 0$, $c > 0$ such that for all $t \geq t_0 \geq 0$ and $|x_0|$ sufficiently small,

$$\begin{cases} |x(t, t_0, x_0)| \leq M|x_0|e^{-c(t-t_0)}, \\ |\Phi(t, t_0, x_0)| \leq Me^{-c(t-t_0)}. \end{cases}$$

Definition 4.3.2 The solution $x = 0$ of Eq. (4.3.2) is said to be uniformly slowly growing in variation if and only if for every $\epsilon > 0$, there exists a constant $M > 0$, possible depending on ϵ, such that for all $t \geq t_0 \geq 0$ and $|x_0| < +\infty$,

$$\begin{cases} |x(t, t_0, x_0)| \leq M|x_0|e^{\epsilon(t-t_0)}, \\ |\Phi(t, t_0, x_0)| \leq Me^{\epsilon(t-t_0)}. \end{cases}$$

We note from [113] that a function $z(t)$ is slowly growing if and only if for every $\epsilon > 0$, there exists a constant M, which may depend on ϵ, such that for all $t \geq t_0 \geq 0$,

$$|z(t)| \leq Me^{\epsilon t}.$$

Theorem 4.3.1 (Pachpatte [460]) *Suppose that for all $t, s \in I$,*

$$|\phi(t, s, y)g[s, y, z]| \leq p(s)(|y| + |z|), \tag{4.3.4}$$

where $p(s) \in C(I, \mathbb{R}_+)$ and $\int_{t_0}^{+\infty} p(s)ds < +\infty$. Furthermore, suppose that the operator T satisfies the inequality

$$|Ty(t)| \le \int_{t_0}^{t} q(s)|y(s)|ds, \tag{4.3.5}$$

where $q(s) \in C(I, \mathbb{R}_+)$ and $\int_{t_0}^{+\infty} q(s)ds < +\infty$. Then for every bounded solution $x(t) = x(t, t_0, x_0)$ of Eq. (4.3.2) on I, the corresponding solution $y(t) = y(t, t_0, y_0)$ of Eq. (4.3.1) is bounded on I.

Proof Using the nonlinear variation of constants formula of Alekseev [25], the solutions of Eqs. (4.3.1) and (4.3.2) with the same initial values are related by

$$y(t) = x(t) + \int_{t_0}^{t} \Phi(t, s, y(s))g(s, y(s), Ty(s))ds. \tag{4.3.6}$$

From (4.3.4)–(4.3.6) it follows

$$|y(t)| \le c + \int_{t_0}^{t} p(s)|y(s)|ds + \int_{t_0}^{t} p(s)[\int_{t_0}^{s} q(\tau)|y(\tau)|d\tau]ds,$$

where $c > 0$ is the upper bound for $|x(t)|$. Thus applying Theorem 1.2.11 yields

$$|y(t)| \le c\left[1 + \int_{t_0}^{t} p(s)\exp\left(\int_{t_0}^{s}[p(\tau) + q(\tau)]d\tau\right)ds\right],$$

which implies the boundedness of $y(t)$ on I, and hence the theorem is proved. □

It is important to note that Theorem 4.3.1 implies not only the boundedness, but also the stability of $y(t)$, if $c > 0$ is small enough. However, the above estimate does not prove the asymptotic stability.

Recall that Brauer [111] proved that if the trivial solution of Eq. (4.3.2) is exponentially asymptotically stable, and if $g(t, y, Ty) = g(t, y) = o(|y|)$ as $|y| \to 0$ uniformly in t, then the trivial solution of (4.3.1) also exponentially asymptotically stable. The next theorem is closely related to Theorem 4.3.2 given in [111] under more general conditions on the perturbation term g and on the operator T.

Theorem 4.3.2 (Brauer [111]) *Let the solution $x = 0$ of Eq. (4.3.2) be exponentially asymptotically stable in variation. Suppose that the perturbation $g(t, y, z)$ satisfies for all $t \in I$,*

$$|g(t, y, z)| \le p(t)(|y| + |z|), \tag{4.3.7}$$

where $p(t) \in C(I, \mathbb{R}_+)$ *and* $\int_{t_0}^{+\infty} p(s)ds < +\infty$. *Furthermore, suppose that the operator* T *satisfies the inequality*

$$|Ty(t)| \leq e^{-ct} \int_{t_0}^{t} q(s)|y(s)|ds, \tag{4.3.8}$$

where $c > 0$, $q(s) \in C(I, \mathbb{R}_+)$ *and* $\int_{t_0}^{+\infty} q(s)ds < +\infty$. *Then all solutions of Eq. (4.3.1) approach zero as* $t \to +\infty$.

Proof It is known that the solutions of (4.3.1) and (4.3.2) with the same initial values are related to the integral equation (4.3.6). Using (4.3.6), (4.3.7) and (4.3.8) together with the exponential asymptotic stability in variation of Eq. (4.3.2), we obtain

$$|y(t)| \leq M|y_0|e^{-c(t-t_0)} + \int_{t_0}^{t} Me^{-c(t-s)}p(s)[y(s) + e^{-cs}\int_{t_0}^{s} q(\tau)|y(\tau)|d\tau]ds.$$

Multiplying both sides of the above inequality by e^{ct}, applying Theorem 1.2.12 with $u(t) = |y(t)|e^{ct}$, then multiplying by e^{-ct}, we obtain

$$|y(t)| \leq M|y_0|e^{-c(t-t_0)}\left[1 + \int_{t_0}^{t} Mp(s)\exp\left(\int_{t_0}^{s} (Mp(\tau) + q(\tau)e^{-c\tau})\,d\tau\right)ds\right].$$

The above estimate yields the desired result if we choose M and $|y_0|$ small enough, and the proof of the theorem thus is complete. □

Brauer and Strauss [113] studied the perturbations of a class of unstable systems, namely those whose solutions grow more slowly then any positive exponential. Theorem 4.3.3 below demonstrates that the solution of Eq. (4.3.1) grows more slowly than any positive exponential.

Theorem 4.3.3 (Brauer-Strauss [113]) *Let the solution* $x = 0$ *of Eq. (4.3.2) be uniformly slowly growing in variation and let the perturbation* $g(t, y, z)$ *satisfy for all* $t \in I$,

$$|g(t, y, z)| \leq p(t)(|y| + |z|),$$

where $p(t) \in C(I, \mathbb{R}_+)$ *and* $\int_{t_0}^{+\infty} p(s)ds < +\infty$. *Suppose that the operator* T *satisfies the inequality*

$$|Ty(t)| \leq e^{\epsilon t} \int_{t_0}^{t} q(s)|y(s)|ds,$$

where $c > 0$, $q(s) \in C(I, \mathbb{R}_+)$ *and* $\int_{t_0}^{+\infty} q(s)ds < +\infty$. *Furthermore, suppose that there exist constants* M *and* k *such that*

$$\int_{t_0}^{+\infty} Mp(s) \exp\left(\int_{t_0}^{s} [Mp(\tau) + q(\tau)e^{\epsilon\tau}]d\tau\right) ds \le k,$$

then all solutions of Eq. (4.3.2) are slowly growing.

Proof The proof of this theorem follows by the similar argument as in the proofs of the above theorems, and hence we omit the details. □

To close this section, we give a simple example to illustrate Theorem 4.3.2. Consider the differential equations

$$x' = e^{-2t}x^3 \tag{4.3.9}$$

and

$$y' = e^{-2t}y^3 + g[t, y, Ty]. \tag{4.3.10}$$

Suppose that the perturbation term g and the operator T in (4.3.10) satisfy the hypotheses (4.3.7) and (4.3.8) of Theorem 4.3.2 with $c = \frac{1}{2}$. The solution of (4.3.9) is given by

$$x(t, t_0, x_0) = y_0 e^{-\frac{1}{2}(t-t_0)}[y_0^2(e^{-(3t-t_0)} - e^{-(t+t_0)}) + e^{-(t-t_0)}]^{-\frac{1}{2}}, \quad t \ge t_0 \ge 0, \tag{4.3.11}$$

where y_0 is a constant. Therefore from (4.3.11), it follows

$$|x(t, t_0, y_0)| \le M|y_0|e^{-\frac{1}{2}(t-t_0)}, \tag{4.3.12}$$

if for a constant $M > 0$,

$$|[y_0^2(e^{-(3t-t_0)} - e^{-(t+t_0)}) + e^{-(t-t_0)}]^{-\frac{1}{2}}| \le M.$$

Here for all $t \ge t_0 \ge 0$,

$$\Phi(t, t_0, y_0) = e^{-\frac{1}{2}(t-t_0)}[y_0^2(e^{-\frac{1}{3}(7t-t_0)} - e^{-\frac{1}{3}(t+5t_0)}) + e^{-\frac{1}{3}(t-t_0)}]^{-\frac{3}{2}}, \tag{4.3.13}$$

and

$$|\Phi(t, t_0, y_0)| \le Me^{-\frac{1}{2}(t-t_0)}, \tag{4.3.14}$$

since

$$\left|[y_0^2(e^{-\frac{1}{3}(7t-t_0)} - e^{-\frac{1}{3}(t+5t_0)}) + e^{-\frac{1}{3}(t-t_0)}]^{-\frac{3}{2}}\right| \leq M.$$

From (4.3.12) and (4.3.14) it follows that the solution $x = 0$ of Eq. (4.3.9) is exponentially asymptotically stable in variation. The solution $y(t) = y(t, t_0, y_0)$ of Eq. (4.3.10) is given by

$$y(t) = x(t) + \int_{t_0}^{t} \Phi(t, s, y(s))g(s, y(s), Ty(s))ds, \tag{4.3.15}$$

where $x(t) = x(t, t_0, y_0)$ and Φ are as given in (4.3.11) and (4.3.14). Now from (4.3.15), (4.3.12) and (4.3.14), in view of the conditions (4.3.7) and (4.3.8) on g and on the operator T, we obtain

$$|y(t)| \leq M|y_0|e^{-\frac{1}{2}(t-t_0)} + \int_{t_0}^{t} Me^{-\frac{1}{2}(t-s)}p(s)\left(y(s) + e^{-\frac{1}{2}s}\int_{t_0}^{s} q(\tau)|y(\tau)|d\tau\right)ds.$$

Now, by following the similar argument as in the proof of Theorem 4.3.2, we have $|y(t)| \to 0$ as $t \to +\infty$, and the conclusion of Theorem 4.3.2 follows readily.

4.4 An Application of Theorem 1.2.13 to Volterra Integral Equations

In this section, we shall use Theorem 1.2.13 to Volterra integral equations.

We first give an estimate of solutions of Volterra integral equation.

Theorem 4.4.1 (Kong and Zhang [317]) *Consider equation*

$$y(x) = f(x) + \int_0^x k(x, s)y(s)ds, \quad x \in \mathbb{R}_+. \tag{4.4.1}$$

Suppose that

1) $k(x, s)$ $(x \geq s)$ are non-negative and continuous on $\mathbb{R}_+ \times \mathbb{R}_+$, and

$$\begin{cases} \dfrac{\partial k(x, s)}{\partial x} & \leq & \displaystyle\sum_{i=1}^{n} q_i(x)h_i(s), \\ k(x, x) & \leq & m(x); \end{cases}$$

2) $f(x), q_i(x), h_i(x)$ $(i = 1, 2, \ldots, n)$ are defined as in Theorem 1.2.13;

3) $m(x) \geq 0$ *is continuous on* $\mathbb{R}_+$.
Then for all $x \in \mathbb{R}_+$,

$$|y(x)| \leq A_n(p),$$

where

$$\begin{cases} p(x) = f(x) + \int_0^x f(s)m(s) \exp\left(\int_s^x m(t)ds\right) ds, \\ g_i(x) = \int_0^x q_i(x) \exp\left(\int_s^x m(t)dt\right) ds, i = 1, 2, \ldots, n, \end{cases}$$

and $A_n(u)$ *is defined as in Theorem 1.2.13.*

Proof From (4.4.1) it follows

$$|y(x)| \leq f(x) + \int_0^x k(x, s)|y(s)|ds.$$

By a similar argument to Theorem 1.2.45, we obtain

$$|y(x)| \leq p(x) + \sum_{i=1}^{n} g_i(x) \int_0^x h_i(s)|y(s)|ds.$$

Thus applying Theorem 1.2.13 to the above inequality yields the desired estimate of the theorem. □

4.5 An Application of Corollary 1.2.15 to Nonlinear Integral Equations

In this section, we shall consider the nonlinear integral differential equation

$$x'(t) = f(t, u(t)) + \int_{t_0}^t g(t, s, x(s))ds, \tag{4.5.1}$$

and the corresponding perturbed equation

$$u'(t) = f(t, u(t)) + \int_{t_0}^t g(t, s, u(s))ds + h(t, u(t), \int_{t_0}^t k(t, s, u(s))ds) \tag{4.5.2}$$

for all $t_0, t \in \mathbb{R}_+$ and $x, u, f, g, h \in \mathbb{R}^n$. If we let $x(t) = x(t; t_0, x_0)$ and $u(t) = u(t; t_0, x_0)$ be the solutions of (4.5.1) and (4.5.2) respectively with $x(t_0) = u(t_0) = x_0$ and $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $f_x : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ $g, k : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, $g_x : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $h : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions in their respective domains. Then we know by [112] that $\partial x/\partial x_0(t, t_0, x_0) = \Phi(t, t_0, x_0)$ exists and satisfies the variational equation

$$x'(t) = f_x(t; x(t; t_0, x_0))z(t) + \int_{t_0}^{t} g_x(t, s, x(s; t_0, x_0))z(s)ds, \quad z(t_0) = I \tag{4.5.3}$$

and

$$\frac{\partial x}{\partial x_0}(t, t_0, x_0) + \Phi(t, t_0, x_0)f(t_0, x_0) \int_{t_0}^{t} \Phi(t, s, x_0)g(s, t_0, x_0)ds = 0. \tag{4.5.4}$$

Thus the solutions $x(t)$ and $u(t)$ are related by

$$u(t) = x(t) \int_{t_0}^{t} \Phi(t, s, u(s))h\left(s, u(s), \int_{t_0}^{t} k(s, t, u(t))dt\right) ds. \tag{4.5.5}$$

Theorem 4.5.1 (Ferreira-Torres [211]) *Let f, f_x, g, g_x, k, h, as above defined, be non-negative continuous functions. Suppose that the following inequalities hold:*

$$|\Phi(t, s, u)| \le Me^{-\alpha(t-s)}, \tag{4.5.6}$$

$$|\Phi(t, s, u)h(s, u, z)| \le p(s)(|u| + |z|), \tag{4.5.7}$$

$$|k(t, s, u)| \le q(s, s)|y|, \tag{4.5.8}$$

for $0 \le s \le t$, $u, z \in \mathbb{R}^n$, $M \ge 1$ and $\alpha > 0$ are constants. If $p(t)$ and $q(t, t)$ are continuous and non-negative and

$$\int_{t_0}^{+\infty} p(s)ds < +\infty, \int_{t_0}^{+\infty} q(s, s)ds < +\infty. \tag{4.5.9}$$

Then for any bounded solution $x(t; t_0, x_0)$ of Eq. (4.5.1) in $\mathbb{R}_+$, then the corresponding solutions of Eq. (4.5.2) is bounded in $\mathbb{R}_+$.

Proof We have from (4.5.6)–(4.5.8) that Eq. (4.5.2) gives

$$|u(t)| \le M|x_0| + \int_{t_0}^{t} p(s)|u(s)|ds + \int_{t_0}^{t} p(s)(\int_{t_0}^{t} q(t, t)|u(t)|dt)ds. \tag{4.5.10}$$

Hence by Theorem 4.5.1, we derive from (4.5.10)

$$|u(t)| \leq M|x_0|\left[1+\int_{t_0}^{t} p(s)\exp\left(\left(\int_{s_0}^{s} p(t)+q(t,t)\right)dt\right)ds\right]. \tag{4.5.11}$$

Hence by (4.5.9), we easily see that $|u(t)|$ is bounded and the proof is now complete. $\square$

4.6 Applications of Theorems 1.2.36 and 4.6.1 to a Retarded Equation

In this section, we apply Theorem 1.2.36 to study the following retarded equation

$$u(t) = k + \int_0^{\alpha(t)} F\left(s, u(s), \int_0^s K(\tau, u(\tau))d\tau\right) ds, \quad t \in [a,b], \tag{4.6.1}$$

where $k \geq 0,\ b > 0,\ \alpha(\cdot) \in C^1([a,b],\mathbb{R})$ is a non-decreasing function with $0 \leq \alpha(t) \leq t,\ u(\cdot) \in C([0,b],\mathbb{R}),\ F(\cdot) \in C([0,b]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $K(\cdot) \in C([0,b]\times\mathbb{R},\mathbb{R})$. The following theorem gives us a bound on the solution of Eq. (4.6.1).

We are now in a position to prove the following result.

Theorem 4.6.1 (Ferreira-Torres [210]) *Suppose that $\alpha(\cdot),\ \beta(\cdot) \in C^1([a,b],\mathbb{R})$ are non-decreasing functions with $\alpha(t),\ \beta(t) \in [a,t]$ for all $t \in [a,b]$. Assume that $u(\cdot), a(\cdot), b(\cdot) \in C([a,b],\mathbb{R}_0)$, $\mathbb{R}_0 = (0,+\infty)$, $(t,s) \to f(t,s) \in C([a,b] \times [a,\alpha(b)],\mathbb{R}_0)$ is non-decreasing in t for every s fixed, $g(\cdot,\cdot) \in C([a,b]\times[a,\beta(b)],\mathbb{R}_0)$, and $(s,\tau) \mapsto k(s,\tau) \in C([a,\beta(b)]\times[a,\beta(b)],\mathbb{R}_0)$ is non-decreasing in s for every τ fixed. Let $W(\cdot),\ \Phi(\cdot) \in C(\mathbb{R}_0,\mathbb{R}_0)$ be non-decreasing functions $\Phi(\cdot)$ sub-multiplicative with $\Phi(x) > 0$ for all $x \geq 1$. Define*

$$G(x) = \int_0^x \frac{ds}{\Phi(1+W(s))}, \quad x \geq 0,$$

$$\eta(\tau) = \max\left\{a(\tau), \int_a^{\beta(\tau)} g(\tau,\theta)d\theta\right\}, \quad \tau \in [a, \max\alpha(b), \beta(b)],$$

and

$$p(s) = \int_a^s k(s,\tau)\Phi\left(\eta(\tau) + b(\tau)\int_a^{\alpha(\tau)} \exp\left(\int_\xi^{\alpha(\tau)} b(\theta)f(\tau,\theta)d\theta\right) f(\tau,\xi)\eta(\xi)d\xi\right) d\tau.$$

If for all $t \in [a, b]$,

$$u(t) \leq a(t) + b(t) \int_a^{\alpha(t)} f(t, s)u(s)ds + \int_a^{\beta(t)} g(t, s)W\left(\int_a^s k(s, \tau)\Phi(u(\tau))d\tau\right) ds, \tag{4.6.2}$$

then there exists a $t_* \in (a, \beta(b)]$ such that $p(t) \in Dom(G^{-1})$ for all $t \in [a, t_*]$, $G^{-1}(\cdot)$ the inverse function of $G(\cdot)$, and

$$u(t) \leq q(t) + b(t) \int_a^{\alpha(t)} \exp\left(\int_s^{\alpha(t)} b(\tau)f(t, \tau)d\tau\right) f(t, s)q(s)ds,$$

where

$$q(t) = a(t) + \int_a^{\beta(t)} g(t, s)W(G^{-1}(p(s)))ds.$$

Proof Let for all $t \in [a, b]$,

$$z(t) = a(t) + \int_a^{\beta(t)} g(t, s)W\left(\int_a^s k(s, \tau)\Phi(u(\tau))d\tau\right) ds.$$

Then (4.6.2) can be restated as

$$u(t) \leq z(t) + b(t) \int_a^{\alpha(t)} f(t, s)u(s)ds. \tag{4.6.3}$$

Applying Theorem 1.2.35 to (4.6.3), we obtain

$$u(t) \leq z(t) + b(t) \int_a^{\alpha(t)} \exp\left(\int_s^{\alpha(t)} b(\tau)f(t, \tau)d\tau\right) f(t, s)z(s)ds. \tag{4.6.4}$$

In order to estimate $z(t)$, we define the function $v(\cdot)$ by

$$v(s) = \int_a^s k(s, \tau)\Phi(u(\tau))d\tau.$$

We have that $z(x) = a(x) + \int_a^{\beta(x)} g(x,\theta)W(v(\theta))d\theta$ and

$$v(s) \le \int_a^s k(s,\tau)\Phi\left[z(\tau) + b(\tau)\int_a^{\alpha(\tau)} \exp\left(\int_\xi^{\alpha(\tau)} b(\theta)f(\tau,\theta)d\theta\right)f(\tau,\xi)z(\xi)d\xi\right]d\tau$$

$$\le \int_a^s k(s,\tau)\Phi\Bigg[\eta(\tau)(1 + W(v(\tau)))$$

$$+b(\tau)\int_a^{\alpha(\tau)} \exp\left(\int_\xi^{\alpha(\tau)} b(\theta)f(\tau,\theta)d\theta\right)f(\tau,\xi)\eta(\xi)d\xi(1 + W(v(\tau)))\Bigg]d\tau$$

$$\le \int_a^s k(s,\tau)\Phi\left[\eta(\tau) + b(\tau)\int_a^{\alpha(\tau)} \exp\left(\int_\xi^{\alpha(\tau)} b(\theta)f(\tau,\theta)d\theta\right)f(\tau,\xi)\eta(\xi)d\xi\right]$$

$$\times\Phi(1 + W(v(\tau)))d\tau.$$

Let $a < t_* \le \beta(b)$ be a number such that $p(t) \in Dom(G^{-1})$ for all $t \in [a, t_*]$. Define $r(\cdot)$ on $[a, s_0]$, where $a < s_0 \le t_*$ is an arbitrary fixed number, by

$$r(s) = \int_a^s k(s_0,\tau)\Phi\left[\eta(\tau) + b(\tau)\int_a^{\alpha(\tau)} \exp\left(\int_\xi^{\alpha(\tau)} b(\theta)f(\tau,\theta)d\theta\right)f(\tau,\xi)\eta(\xi)d\xi)\right]$$

$$\times\Phi(1 + W(v(\tau))d\tau.$$

Then,

$$r'(s) = k(s_0,s)\Phi\left[\eta(s) + b(s)\int_a^{\alpha(s)} \exp\left(\int_\xi^{\alpha(s)} b(\theta)f(s,\theta)d\theta\right)f(s,\xi)\eta(\xi)d\xi\right]$$

$$\times\Phi(1 + W(v(s))$$

$$\le k(s_0,s)\Phi\left[\eta(s) + b(s)\int_a^{\alpha(s)} \exp\left(\int_\xi^{\alpha(s)} b(\theta)f(s,\theta)d\theta\right)f(s,\xi)\eta(\xi)d\xi)\right]$$

$$\times\Phi(1 + W(r(s)),$$

that is,

$$\frac{r'(s)}{\Phi(1 + W(r(s))} \le (s_0,s)\Phi\left[\eta(s) + b(s)\int_a^{\alpha(s)} \exp\left(\int_\xi^{\alpha(s)} b(\theta)f(s,\theta)d\theta\right)f(s,\xi)\eta(\xi)d\xi)\right].$$

Integrating both sides of the last inequality from a to s, and having in mind that $G(r(a)) = 0$, we get

$$G(r(s)) \le \int_a^s k(s_0,\tau)\Phi\left[\eta(\tau)+b(\tau)\int_a^{\alpha(\tau)} \exp\left(\int_\xi^{\alpha(\tau)} b(\theta)f(\tau,\theta)d\theta\right)f(\tau,\xi)\eta(\xi)d\xi\right]d\tau.$$

The choice of t_* permits us to write $r(s_0) \le G^{-1}(p(s_0))$. Since s_0 is arbitrary, we conclude that (the case $s = a$ is trivial), for all $s \in [a, t_*]$,

$$r(s) \le G^{-1}(p(s))). \tag{4.6.5}$$

To complete the proof, we observe that for $a \le s \le t_*$ the inequality $\beta(\alpha(s)) \le t_*$ holds. Hence, we can insert inequality (4.6.5) into inequality (4.6.4). □

Remark 4.6.1 Theorem 4.6.1 is new even in the particular setting studied in [177] with $\alpha(t) = \beta(t) = t$, $b(t) = 1$, and $f(t,s) = g(t,s) = f(s)$. Indeed, we may choose in Theorem 4.6.1 a sub-multiplicative function $\Phi(\cdot)$ that is not sub-additive, e.g., $\Phi(x) = x^2$ for all $x \ge 0$. This choice of $\Phi(\cdot)$ is not a possibility in Theorem 2.1 of [177].

Theorem 4.6.2 (Ferreira-Torres [210]) *Assume that functions $F(\cdot,\cdot,\cdot)$ and $K(\cdot,\cdot)$ in Eq. (4.6.1) satisfy*

$$\begin{cases} |K(t,u)| \le k(t)\Phi(|u|), & (4.6.6)\\ |F(t,u,v)| \le t|u| + |v|, & (4.6.7)\end{cases}$$

with $k(\cdot)$ and $\Phi(\cdot)$ defined as in Theorem 4.6.1. If $u(\cdot)$ is a solution of Eq. (4.6.1), then for all $t \in [a, t_*]$,

$$|u(t)| \le q(t) + t\int_0^{\alpha(t)} \exp\left(t\alpha(t) - s)\right) q(s)ds,$$

for some $t_* \in (a, \alpha(b)]$ such that for all $t \in [a, t_*]$,

$$p(t) \in Dom\ \ (G^{-1}).$$

Here,

$$\begin{cases} q(t) = k + \displaystyle\int_0^{\alpha(t)} G^{-1}(p(s))ds, \quad G(x) = \int_0^x \frac{ds}{\Phi(1+s)}, \qquad x \ge 0,\\ p(s) = \displaystyle\int_a^s k(\tau)\Phi\left[\eta(\tau) + \tau\int_0^{\alpha(\tau)} \exp\left(\tau(\alpha(\tau) - \xi)\right)\eta(\xi)d\xi\right]d\tau,\\ \eta(\tau) = max\{k, \alpha(\tau)\}, \qquad \tau \in [0, \alpha(b)],\end{cases}$$

with $G^{-1}(\cdot)$ representing the inverse function of $G(\cdot)$.

Proof Let $u(\cdot)$ be a solution of Eq. (4.6.1). In view of (4.6.6) and (4.6.7), we get

$$|u(t)| \leq k + \int_0^{\alpha(t)} \left(t\,|u(s)| + \int_0^s k(\tau)\Phi(|u(\tau)|)d\tau \right) ds.$$

Applying Theorem 4.6.1 with $a(t) = k,\ \alpha(t) = \beta(t),\ f(t,s) = t,\ b(t) = g(t,s) = 1$, and $W(u) = U$, we conclude the desired estimate. □

4.7 An Application of Theorem 1.2.37 to an Integrodifferential Equation

In this section, we present some applications of the inequality (a_1) in Theorem 1.2.37 to study certain properties of solutions of the integrodifferential equation

$$x'(t) = F\left(t, x(t-h(t)), \int_{t_0}^t f(t,\sigma,x(\sigma - h(\sigma)))d\sigma\right), \tag{4.7.1}$$

with the given initial condition

$$x(t_0) = x_0, \tag{4.7.2}$$

where $f \in C(I^2 \times \mathbb{R}, \mathbb{R}), F \in C(I \times \mathbb{R}^2, \mathbb{R}), x_0$ is a real constant and $h \in C^1(I,I)$ is non-decreasing with $t - h(t) \geq 0, h'(t) < 1, h(t_0) = 0$.

The following theorem deals with the estimate on the solution of problem (4.7.1)–(4.7.2).

Theorem 4.7.1 (Pachpatte [504]) *Suppose that*

$$|f(t,s,x)| \leq b(t,s)|x|, \quad |F(t,x,w)| \leq a(t)|x| + |w|, \tag{4.7.3}$$

where $a(t), b(t,s)$ are as defined in Theorem 1.2.37 and let

$$M = \max_{t\in I} \frac{1}{1-h'(t)}. \tag{4.7.4}$$

If $x(t)$ is any solution of problem (4.7.1)–(4.7.2), then for all t, η, τ in I,

$$|x(t)| \leq |x_0| \exp\left(\int_{t_0}^{t-h(t)} \left[Ma(s+h(\eta)) + \int_{t_0}^s M^2 b(s+h(\eta), \sigma + h(\tau))d\sigma \right] ds \right). \tag{4.7.5}$$

Proof The solution $x(t)$ of problem (4.7.1)–(4.7.2) can be written as

$$x(t) = x_0 + \int_{t_0}^{t} F\left(s, x(s-h(s)), \int_{t_0}^{s} f(s,\sigma,x(\sigma-h(\sigma)))d\sigma\right) ds. \quad (4.7.6)$$

Using (4.7.3)–(4.7.4) in (4.7.6) and making the change of variables, we have for all t, η, τ in I,

$$|x(t)| \le |x_0| + \int_{t_0}^{t-h(t)} \left[Ma(s+h(\eta))|x(s)| + \int_{t_0}^{s} M^2 b(s+h(\eta), \sigma+h(\tau))|x(\sigma)|d\sigma\right]ds. \quad (4.7.7)$$

Now applying Theorem 1.2.37 yields the required estimate in (4.7.5). □

Next, we shall prove the uniqueness of the solutions of problem (4.7.1)–(4.7.2).

Theorem 4.7.2 (Pachpatte [504]) *Suppose that the functions f, F in problem (4.7.1)–(4.7.2) satisfying*

$$|f(t,s,x) - f(t,s,y)| \le b(t,s)|x-y|, \quad (4.7.8)$$

$$|F(t,x,\bar{x}) - F(t,y,\bar{y})| \le a(t)|x-y| + |\bar{x}-\bar{y}|, \quad (4.7.9)$$

where $a(t), b(t,s)$ are as defined in Theorem 1.2.37 and let M be as in (4.7.4). Then the problem (4.7.1)–(4.7.2) has at most one solution on I.

Proof Let $x(t)$ and $\bar{x}(t)$ be two solutions of problem (4.7.1)–(4.7.2) on I, then we have

$$x(t) - x(\bar{t}) = \int_{t_0}^{t} \Big\{F(s, x(s-h(s)), \int_{t_0}^{s} f(s,\sigma,x(\sigma-h(\sigma)))d\sigma)$$
$$-F(s, \bar{x}(s-h(s)), \int_{t_0}^{s} f(s,\sigma,\bar{x}(\sigma-h(\sigma)))d\sigma)\Big\}ds. \quad (4.7.10)$$

Using (4.7.8), (4.7.9) in (4.7.10) and making the change of variables, we have for all t, η, τ in I,

$$|x(t) - \bar{x}(t)| \le \int_{t_0}^{t-h(t)} \Big[Ma(s+h(\eta))|x(s)-\bar{x}(s)|$$
$$+ \int_{t_0}^{s} M^2 b(s+h(\eta), \sigma+h(\tau))|x(\sigma)-\bar{x}(\sigma)|d\sigma\Big]ds. \quad (4.7.11)$$

Applying Theorem 1.2.37 yields $|x(t) - \bar{x}(t)| \le 0$. Therefore $x(t) = \bar{x}(t)$, i. e., there is at most one solution of problem (4.7.1)–(4.7.2). □

The next result shows the dependency of solutions of problem (4.7.1)–(4.7.2) on initial data.

Theorem 4.7.3 (Pachpatte [504]) *Let $x_1(t)$ and $x_2(t)$ be the solutions of Eq. (4.7.1) with the given initial conditions*

$$x_1(t_0) = x_1, \tag{4.7.12}$$

and

$$x_2(t_0) = x_2, \tag{4.7.13}$$

respectively, where x_1, x_2 are real constants. Suppose that the functions f and F in (4.7.1) satisfy the conditions (4.7.8) and (4.7.9) in Theorem 4.7.2 and let M be as in (4.7.4). Then for all t, η, τ in I,

$$|x_1(t) - x_2(t)| \leq |x_1 - x_2| \\ \times \exp\left(\int_{t_0}^{t-h(t)} \left[Ma(s + h(\eta)) + \int_{t_0}^{s} M^2 b(s + h(\eta), \sigma + h(\tau))d\sigma\right] ds\right). \tag{4.7.14}$$

Proof By using the facts that $x_1(t)$ and $x_2(t)$ are the solutions of problem (4.7.1),(4.7.2) and problem (4.7.12),(4.7.13) respectively, we have

$$x_1(t) - x_2(t) = x_1 - x_2 + \int_{t_0}^{t} \Big\{F(s, x_1(s - h(s)), \int_{t_0}^{s} f(s, \sigma, x_1(\sigma - h(\sigma)))d\sigma) \\ -F(s, x_2(s - h(s)), \int_{t_0}^{s} f(s, \sigma, x_2(\sigma - h(\sigma))))\Big\}. \tag{4.7.15}$$

Using (4.7.8), (4.7.9) in (4.7.15) and making the change of variables, we have for all t, η, τ in I,

$$|x_1(t) - x_2(t)| \leq |x_1 - x_2| + \int_{t_0}^{t-h(t)} \Big[Ma(s + h(\eta))|x_1(s) - x_2(s)| \\ + \int_{t_0}^{s} M^2 b(s + h(\eta), \sigma + h(\tau))|x_1(\sigma) - x_2(\sigma)|d\sigma\Big] ds. \tag{4.7.16}$$

Now applying Theorem 1.2.37 to (4.7.16) yields the required estimate (4.7.14). □

In the same manner, we can also use Theorem 1.2.37 to study the similar properties as in Theorems 4.7.1–4.7.3 for the hyperbolic partial differential equation

$$D_1 D_2 z(x, y) = F(x, y, z(x - h_1(x), y - h_2(y)), Tz(x, y)), \tag{4.7.17}$$

with the given initial boundary conditions

$$z(x, y_0) = a_1(x), \quad z(x_0, y) = a_2(y), \quad a_1(x_0) = a_2(y_0), \tag{4.7.18}$$

where

$$Tz(x, y) = \int_{x_0}^{x} \int_{y_0}^{y} K(x, y, s, t, z(s - h_1(s), t - h_2(t)))dtds, \tag{4.7.19}$$

under some suitable conditions on the functions involved in (4.7.17)–(4.7.19).

4.8 An Application of Theorem 1.2.41 to Continuum Thermodynamics

In this section, we shall apply Theorem 1.2.41 to study continuum thermodynamics. This section is chosen from Morro [419].

First, we introduce some notations.

Henceforth $\mathbf{Y}$, $\mathbf{Z}$, $\mathbf{A}$, and $\mathbf{\Phi}$ denote finite dimensional real normed vector spaces, subject to the requirement dim$\mathbf{A}$ $\leq$dim$\mathbf{Y}$+dim$\mathbf{Z}$, while $\mathbf{L}(\cdot, \cdot)$ stands for the normed vector space of all linear maps from a vector space of all linear maps from a vector space into another. The symbol $|\cdot|$ is adopted to denote the usual norm—$|\mathbf{p}| = (\mathbf{p}, \mathbf{p})^{\frac{1}{2}}$—both in $\mathbf{Y} \times \mathbf{Z}$, $\mathbf{A}$ and in $\mathbf{L}$. A superposed dot designates (material) time differentiation.

A material with hidden variables $\{\mathbf{y_0}, \mathbf{z_0}, \mathbf{a_0}\, \mathrm{U}, \mathbf{V}, \boldsymbol{\varphi}, \mathbf{f}\}$ on $\mathbf{Y} \times \mathbf{Z} \times \mathbf{A}$, the vector $\mathbf{a}$ representing the set of hidden variables, together with an open connected neighborhood $\mathbf{U} \times \mathbf{V}$ of $(\mathbf{y_0}, \mathbf{z_0})$ and the maps

$$\boldsymbol{\varphi} \in \mathbf{C^2}(\mathbf{U} \times \mathbf{A}, \mathbf{\Phi}), \quad \mathbf{f} \in \mathbf{C^2}(\mathbf{U} \times \mathbf{V} \times \mathbf{A}, A).$$

A path is a bounded and piecewise continuously differentiable map $\boldsymbol{\pi} : \mathbb{R} \to \mathbf{U} \times \mathbf{V}$; to save writing the symbol $\boldsymbol{\pi}$ will be used even in connection with the values of the path, i.e., the values of the physical variables $(\mathbf{y}, \mathbf{z}) \in \mathbf{U} \times \mathbf{V}$. The hidden variables are functions on the time $t \in \mathbb{R}$; their growth is determined by the path $\boldsymbol{\pi}$ through the evolution function $\mathbf{f}$, by,

$$\dot{\mathbf{a}}(t) = \mathbf{f}(\boldsymbol{\pi}(t), \mathbf{a}(t)), \quad t \geq t_0, \quad \mathbf{a}(t_0) = \mathbf{a}^*. \tag{4.8.1}$$

For the purpose we have in mind, we may disregard the behavior of the response function $\boldsymbol{\varphi}$. As to the evolution function $\mathbf{f}$, instead, two assumptions are introduced, namely uniform Lipschitz conditions both in the physical variables $\boldsymbol{\pi}$ and in the hidden variables $\mathbf{a}$. These assumptions are made precise as follows.

I) These is a map $\boldsymbol{\Lambda} \in \mathbf{L}(\mathbf{A}, \mathbf{A})$ and a positive constant δ such that

$$|\mathbf{f}(\boldsymbol{\pi}, \mathbf{a}+\mathbf{b}) - \mathbf{f}(\boldsymbol{\pi}, \mathbf{a}) - \boldsymbol{\Lambda}\mathbf{b}| \leq \delta|\mathbf{b}|, \quad \boldsymbol{\pi} \in \mathbf{U} \times \mathbf{V},\ \mathbf{a},\ \mathbf{a}+\mathbf{b} \in \mathbf{A}, \tag{4.8.2}$$

and each eigenvalue of $\boldsymbol{\Lambda} + \delta \mathbf{I}_{\mathbf{A}}$ has a negative real part.

II) There is a positive constant ϵ such that

$$|\mathbf{f}(\boldsymbol{\pi}+\boldsymbol{\omega}, \mathbf{a}) - \mathbf{f}(\boldsymbol{\pi}, \mathbf{a})| \leq \epsilon|\boldsymbol{\omega}|, \quad \boldsymbol{\pi}, \boldsymbol{\pi}+\boldsymbol{\omega} \in \mathbf{U} \times \mathbf{V},\ \mathbf{a} \in \mathbf{A}. \tag{4.8.3}$$

Thus it follows from **I**) and **II**) that

$$\begin{cases} \mathbf{f}(\boldsymbol{\pi}, \cdot) \in \mathrm{Lip}\,(|\boldsymbol{\Lambda}| + \delta), & \boldsymbol{\pi} \in \mathbf{U} \times \mathbf{V}, \\ \mathbf{f}(\cdot, \mathbf{a}) \in \mathrm{Lip}\,\epsilon, & \mathbf{a} \in \mathbf{A}. \end{cases} \tag{4.8.4}$$

As usual, an *equilibrium hidden variable* $\mathbf{E}(\boldsymbol{\pi}) \in \mathbf{A}$ is characterized by $\mathbf{f}(\boldsymbol{\pi}, \mathbf{E}(\boldsymbol{\pi})) = \mathbf{0}$. Letting $\boldsymbol{\pi}_{\mathbf{0}} = (\mathbf{y}_{\mathbf{0}}, \mathbf{z}_{\mathbf{0}})$, we assume that $\mathbf{a}_{\mathbf{0}} = \mathbf{E}(\boldsymbol{\pi}_{\mathbf{0}})$.

We note that starting from a set of assumptions similar to **I**), Day [169] proved that the solution to the evolution equation (4.8.1) is asymptotically stable. Then we might ask if the assumption **I**) leads to the same conclusions. We shall show that this is so and, in addition, that $\mathbf{a}(t)$ is independent of $\boldsymbol{\pi}(t)$. To this end, consider the hidden variables $\mathbf{a}$, $\mathbf{a}+\mathbf{b} \in \mathbf{A}$ corresponding to the paths $\boldsymbol{\pi}, \boldsymbol{\pi}+\omega$, namely,

$$\dot{\mathbf{a}}(t) = \mathbf{f}(\boldsymbol{\pi}(t), \mathbf{a}(t)), \quad t \geq t_0;\ \mathbf{a}(t_0) = \mathbf{a}^*,$$
$$\dot{\mathbf{a}}(t) + \dot{\mathbf{b}}(t) = \mathbf{f}(\boldsymbol{\pi}(t) + \boldsymbol{\omega}(t), \mathbf{a}(t) + \mathbf{b}(t)), \quad t \geq t_0; \quad \mathbf{a}(t_0) + \mathbf{b}(t_0) = \mathbf{a}^* + \mathbf{b}^*.$$

Letting

$$\boldsymbol{\gamma} = \mathbf{f}(\boldsymbol{\pi}+\boldsymbol{\omega}, \mathbf{a}+\mathbf{b}) - \mathbf{f}(\boldsymbol{\pi}, \mathbf{a}+\mathbf{b}),$$
$$\mathbf{r} = \mathbf{f}(\boldsymbol{\pi}, \mathbf{a}+\mathbf{b}) - \mathbf{f}(\boldsymbol{\pi}, \mathbf{a}) - \boldsymbol{\Lambda}\mathbf{b},$$

subtraction of the above two evolution equations allows us to write the one for the difference $\mathbf{b}$ as

$$\dot{\mathbf{b}} = \boldsymbol{\gamma} + \mathbf{r} + \Lambda\mathbf{b}.$$

Accordingly, we find that

$$\frac{d}{dt}[\exp(-t\boldsymbol{\Lambda})\mathbf{b}(t)] = \exp(-t\Lambda)[\boldsymbol{\gamma}(t) + \mathbf{r}(t)].$$

Hence an integration yields

$$\mathbf{b}(t) = \exp((t - t_0)\Lambda)\mathbf{b}(t_0) + \int_{t_0}^{t} \exp((t - s)\Lambda)[\boldsymbol{\gamma}(s) + \mathbf{r}(s)]ds,$$

whence, using **I**), **II**) and denoting by $-m < -\delta$ the real part of the eigenvalue of Λ with the greatest real part, it follows that

$$\begin{aligned} b(t) \leq \exp(-m(t - t_0))b(t_0) + \epsilon \int_{t_0}^{t} \exp(-m(t - s))\omega(s)ds \\ +\delta \int_{t_0}^{t} \exp(-m(t - s))b(s)ds \end{aligned}$$

where $b = |\mathbf{b}|$, $\omega = |\boldsymbol{\omega}|$. Thus, by the identifications

$$\begin{cases} v(t) = b(t), & (4.8.5) \\ g(t) = \exp(-m(t - t_0))b(t_0) + \epsilon \int_{t_0}^{t} \exp(-m(t - s))\omega(s)ds, & (4.8.6) \\ k(t) = \delta \exp(-mt), & (4.8.7) \\ h(t) = \exp(mt), & (4.8.8) \end{cases}$$

we may apply Theorem 1.2.41 to obtain

$$\begin{aligned} b(t) \leq \exp(-(m - \delta)(t - t_0))b(t_0) + \epsilon \int_{t_0}^{t} [\exp(-m(t - s))\omega(s) + \\ +\delta \exp(-(m - \delta)(t - s)) \int_{t_0}^{s} \exp(-m(s - u))\omega(u)du]ds. \end{aligned} \quad (4.8.9)$$

Owing to the cumbersome structure of (4.8.9), a less accurate estimate is desired. In this regard, a satisfactory result is obtained by considering 1 as an upper bound for the expressions $\exp(-m(t - s))$ and $\exp(-m(s - u))$ in (4.8.9); so we find the new estimate

$$\begin{aligned} b(t) \leq \exp(-(m - \delta)(t - t_0))b(t_0) \\ + \frac{\epsilon}{m - \delta}[m - \delta \exp(-(m - \delta)(t - t_0))] \int_{t_0}^{t} \omega(u)du. \end{aligned} \quad (4.8.10)$$

It is worth noting that, starting from Willett's estimate (1.2.304) and following along the same procedure, the counterparts of (4.8.9), (4.8.10) become eventually

$$
\begin{aligned}
b(t) \leq &\exp(-m(t-t_0))\,[1+\delta(t-t_0)\exp(\delta(t-t_0))]\,b(t_0)+\\
&+\epsilon\int_{t_0}^{t}\exp(-m(t-s))\\
&\times\left[\omega(s)+\delta\exp(-\delta(t-t_0))\int_{t_0}^{s}\exp(-m(s-u))\omega(u)du\right]ds
\end{aligned}
\tag{4.8.11}
$$

and

$$
\begin{aligned}
b(t) \leq &\exp(-m(t-t_0))[1+\delta(t-t_0)\exp(\delta(t-t_0))]b(t_0)+\\
&+\epsilon[1-\frac{\delta}{m}\exp(-(m-\delta)(t-t_0))+\frac{\delta}{m}\exp(\delta(t-t_0))]\int_{t_0}^{t}\omega(u)du,
\end{aligned}
\tag{4.8.12}
$$

respectively.

4.9 An Application of Theorem 1.2.61 to Nonlinear Volterra-Fredholm Integral Equations

The this section, we present some applications of Theorem 1.2.61 to study certain properties of the solutions of the nonlinear Volterra-Fredholm integral equation of the form

$$
x(t) = f(t) + \int_{\alpha}^{t} g(t,s,x(s))ds + \int_{\alpha}^{\beta} h(t,s,x(s))ds \tag{4.9.1}
$$

for all $t \in I$, where $x(t)$ is an unknown function, $f \in C(I,\mathbb{R}^n)$, g, $h \in C(\Delta\times\Delta\times\mathbb{R}^n,\mathbb{R}^n)$.

For the study of Volterra-Fredholm integral equations of the type (4.9.1), we refer to [32, 406, 485] and the references cited therein.

The following theorem deals with the estimate on the solution of Eq. (4.9.1).

Theorem 4.9.1 (Pachpatte [500]) *Suppose that the functions f, g, h in Eq. (4.9.1) satisfy the conditions*

$$
\begin{cases}
|f(t)| \leq c, & (4.9.2)\\
|g(t,s,x)| \leq a(t,s)|x|, & (4.9.3)\\
|h(t,s,x)| \leq b(t,s)|x| & (4.9.4)
\end{cases}
$$

where $a(t,s)$, $b(t,s)$ and c are defined as in Theorem 1.2.61. Let $p(t)$ be as in (1.2.443). If $x(t)$ is a solution of Eq. (4.9.1) on I, then for all $t \in I$,

$$x(t) \le \frac{c}{1-p(t)} \exp\left(\int_\alpha^t a(t,s)ds\right). \tag{4.9.5}$$

Proof Since $x(t)$ is a solution of Eq. (4.9.1), from (4.9.1)–(4.9.4) it follows

$$|x(t)| \le c + \int_\alpha^t a(t,s)|x(s)|ds + \int_\alpha^\beta b(t,s)|x(s)|ds. \tag{4.9.6}$$

Now applying Theorem 1.2.61 to (4.9.6) yields the required estimate in (4.9.5). □

The next result deals with the uniqueness of solutions of Eq. (4.9.1).

Theorem 4.9.2 (Pachpatte [500]) *Suppose that the functions g, h in Eq. (4.9.1) satisfy the following conditions*

$$\begin{cases} |g(t,s,x) - g(t,s,y)| \le a(t,s)|x-y|, & (4.9.7) \\ |h(t,s,x) - h(t,s,y)| \le b(t,s)|x-y|, & (4.9.8) \end{cases}$$

where $a(t,s)$, $b(t,s)$ are defined as in Theorem 1.2.61. Let $p(t)$ be as in Theorem 1.2.61. Then Eq. (4.9.1) has at most one solution on I.

Proof Let $u(t)$ and $v(t)$ be two solutions of Eq. (4.9.1) on I. From (4.9.1), (4.9.7) and (4.9.8), we derive

$$|u(t) - v(t)| \le \int_\alpha^t a(t,s)|u(s) - v(s)|ds + \int_\alpha^\beta b(t,s)|u(s) - v(s)|ds. \tag{4.9.9}$$

Now applying Theorem 1.2.61 to (4.9.9) yields $u(t) = v(t)$, i.e., there is at most one solution of Eq. (4.9.1). □

4.10 Applications of Theorems 1.4.8 to the Volterra Integral Equation and Semilinear Evolution Equations

In this section, we shall first use Theorem 1.4.8 to investigate the following Volterra integral equation

$$\phi(t) = h(t) + \int_0^t (t-s)^{\beta-1} P(s, \phi(s))ds, \tag{4.10.1}$$

where $h \in C(\mathbb{R}_+, \mathbb{R}^n), P \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n), 0 < \beta < 1$.

By a solution to Eq. (4.10.1), we understand a continuous mapping $\phi : [0, b) \to \mathbb{R}^n$ satisfying (4.10.1) for all $t \in [0, b)$. If $b = +\infty$, then we say that this solution is global.

Theorem 4.10.1 (Medveď [400]) *Let $h \in C(\mathbb{R}_+, \mathbb{R}^n)$, $P \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, and for all $(t, v) \in \mathbb{R}_+ \times \mathbb{R}^n$,*

$$\|P(t, v)\| \le F(t)\omega(v), \tag{4.10.2}$$

where $f \in C(\mathbb{R}_+, \mathbb{R}^n)$ and ω be as in Theorem 1.4.8. Assume that there exists an $\varepsilon > 0$ such that

$$\int_0^{+\infty} \frac{\tau^{1/\beta-1+\varepsilon}}{\omega(\tau)^{1/\beta+\varepsilon}} d\tau = +\infty.$$

Then any solution to Eq. (4.10.1) is global.

Proof Let $\beta = 1/(1+z), z > 0, \alpha = 1-\beta, q = 1/\beta + \varepsilon = 1 + z + \varepsilon, p = (1+z+\varepsilon)/(z+\varepsilon), \alpha = 1-\beta = z/(1+z)$. Assume that $\phi : [0, T) \to \mathbb{R}^n$ is the solution of Eq. (4.10.1) with $\lim_{t\to T^-} \|\phi(t)\| = +\infty$. By the assertion of Theorem 1.4.8, we have for all $t \in [0, T)$,

$$\Lambda_q(\|\phi(t)\|^q) \le \Lambda_q\left(2^{q-1} \max_{0\le t\le T} \|h(t)\|^q\right) + K_q \int_0^t e^{-qs} F^q(s) ds \tag{4.10.3}$$

where $\Lambda_q = \Lambda_{qr}$ with $r = 1$ (see, Theorem 1.4.8). The right-hand side of (4.10.3) is finite,

$$\begin{aligned}\lim_{t\to T^-} \Lambda_q(\|\phi(t)\|^q) &= \lim_{t\to T^-} \int_0^{\|\phi(t)\|^q} \frac{d\sigma}{[\omega(\sigma^{1/q})]^q} d\sigma \\ &= q\int_0^{+\infty} \frac{\tau^{q-1}}{\omega(\tau)^q} d\tau = q\int_0^{+\infty} \frac{\tau^{1/\beta-1+\varepsilon}}{\omega(\tau)^{1/\beta+\varepsilon}} d\tau = +\infty\end{aligned}$$

which is the desired conclusion. □

Next, we shall study the evolution equation

$$u'(t) + Au = H(t, u), \quad u(0) = u_0 \tag{4.10.4}$$

where $-A$ is the infinitesimal generator of the analytic semigroup $\{S(t)\}$ on a Banach space V, $S(t) \in L(V, E), t > 0, E$ is a Banach space densely and continuously embedded into V, and the nonlinear map $H : \mathbb{R}_+ \times E \to V$ is continuous, satisfying the condition: for all $(t, v) \in \mathbb{R}_+ \times E$,

$$\|H(t, v)\| \le F(t)\omega(v), \tag{4.10.5}$$

and ω is as in Theorem 1.4.8. We shall study the mild solution to (4.10.4), i.e., solutions of the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-s)H(s, u(s))ds, \quad 0 \le t < +\infty. \tag{4.10.6}$$

We have the following result due to [400].

Theorem 4.10.2 (Medveď [400]) *Let $0 < T < +\infty, \beta \in (0,1)$, and $H : \mathbb{R}_+ \times E \to V$ be continuous satisfying the condition (4.10.5) with ω as in Theorem 1.4.8, $F : \mathbb{R}_+ \to \mathbb{R}$ continuous. Assume that there exists an $\varepsilon > 0$ such that*

$$\int_0^{+\infty} \frac{\tau^{1/\beta-1+\varepsilon}}{\omega(\tau)^{1/\beta+\varepsilon}} d\tau = +\infty.$$

Then $\sup_{t\in(0,T)} \|u(t)\|_E < +\infty$ for any $u \in C([0,T), E)$ satisfying (4.10.6), i.e., any solution to Eq. (4.10.6) is global.

Proof Using Theorem 1.4.8, we obtain for $\|u(t)\|_E$ the same inequality as we have obtained for $\|\phi(t)\|$ in the proof of Theorem 4.10.1. If we assume $\sup_{t\in[0,T)} \|u(t)\|_E = +\infty$, this inequality leads to the contradiction. □

4.11 An Application of Theorem 2.1.15 to Quasilinear Differential Equations

In this section, we employ Theorem 2.1.15, due to Lees [358], to investigate properties of certain finite difference approximations to mixed initial-boundary value for second order quasi-linear hyperbolic equations of the form

$$\frac{\partial^2 u}{\partial t^2} - a(x,t)\frac{\partial^2 u}{\partial x^2} = F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial x}\right). \tag{4.11.1}$$

We shall consider finite difference approximation to Eq. (4.11.1) using as principal part the implicit finite difference operator L defined by

$$L\phi(x,t) = \phi_{t\bar{t}}(x,t) = a(x,t)\phi_{x\bar{x}}(x,t) \tag{4.11.2}$$

where the barred subscripts denote backward difference quotients and the unbarred denote forward difference quotients.

The fundamental problem concerning such finite difference approximations to (4.11.1) is to show that their solutions tend with diminishing mesh size to the solution (4.11.1). Actually, it is sufficient to prove that the difference equations are stable since the convergence of a finite difference scheme can be derived from its

stability in a way which has become standard (e.g., see Douglas [190], John [304] and Lax and Richtmyer [355]).

The stability of the difference schemes considered are deduced from an energy inequality satisfied by the difference operator L. This energy inequality is a discrete analogue of the well-known energy inequality of Friedrichs and Lewy [220] for second order hyperbolic equations. The energy inequality for L states that any function ϕ together with the first order difference quotients can be estimated in the mean square along time lines in terms of $L\phi$. It is the fact that the first difference quotients ϕ can be estimated in terms of $L\phi$ that enables us to treat difference approximations to differential equations which have nonconstant coefficients.

Douglas [190] and Lax and Richtmyer [355] have established conditions for the stability of a wide class of difference approximations to hyperbolic equations amenable to Fourier Analysis or separation of variables technique. Their results do not apply to equations of the form (4.11.1).

In this section, we shall consider semi-discrete approximations to Eq. (4.11.1), i.e., difference approximations in which only the derivatives with respect to x are replaced by difference quotients. In addition to establishing the stability and convergence of such semi-discrete approximation schemes, we show that in many interesting cases it is possible to derive an explicit error estimate. Semi-discrete approximations of this type have been investigated by Douglas [191] for parabolic equations.

Let Ω denote the rectangular region $0 < x < 1, 0 < t \leq t_0$ and $\overline{\Omega}$ denote its closure. The set $B = \overline{\Omega} - \Omega$ is called the boundary of Ω. We decompose B into the three segments $B^0(0 \leq x \leq 1, t = 0)$, $B^1(x = 0, 0 < t \leq t_0)$ and $B^2(x = 1, 0 < t \leq t_0)$. Note that B is not the set-theoretical boundary of Ω.

Let M and N be positive integers, and denote by $\overline{D}$ a lattice with mesh (h, k) fitted over $\overline{\Omega}$, i.e., $\overline{D}$ consists of the points of intersection of the coordinate lines

$$\begin{cases} x = nk, & n = 0, 1, \cdots, N, \\ t = mk, & m = 0, 1, \cdots, M \end{cases}$$

where $h = N^{-1}$ and $k = t_0 M^{-1}$. The quantity $\lambda = kh^{-1}$ is called the mesh ratio of $\overline{D}$. The mth row of the lattice $\overline{D}$ is defined to the set

$$\overline{R}(mk) = \{(x, t) | (x, t) \in \overline{D} \quad \text{and} \quad t = mk\}.$$

Put

$$R_0(mk) = \bar{R}(mk) - \{(x, t) | (x, t) \in \overline{D}, x \neq 0 \quad \text{and} \quad x \neq 1\}.$$

The interior of $\overline{D}$ is the set D defined as follows:

$$D = \bigcup_{m=2}^{M} R_0(mk).$$

Let $\partial D = \overline{D} - D$ and $\partial^i D = B^i \bigcap \partial D$, $(i = 0, 1, 2)$.

For function $\phi(x,t)$ defined on the lattice, we employ the following notation for their forward and backward difference quotients

$$\begin{cases} \phi_x(x,t) = h^{-1}\left[\phi(x+h,t) - \phi(x,t)\right], \\ \phi_{\bar{x}}(x,t) = h^{-1}\left[\phi(x,t) - \phi(x-h,t)\right] = \phi_x(x-h,t), \\ \phi_t(x,t) = k^{-1}\left[\phi(x,t+k) - \phi(x,t)\right], \\ \phi_{\bar{t}}(x,t) = k^{-1}\left[\phi(x,t) - \phi(x,t-k)\right] = \phi_t(x,t-k). \end{cases}$$

Difference quotients of the order higher than the first are formed by repeated application of the above formulas, for example,

$$\begin{cases} \phi_{x\bar{x}}(x,t) = [\phi_x(x,t)]_{\bar{x}} = [\phi_{\bar{x}}]_x \\ \qquad\qquad = h^{-2}[\phi(x+h,t) - 2\phi(x,t) + \phi(x-h,t)], \\ \phi_{\bar{t}\bar{t}}(x,t) = [\phi_{\bar{t}}(x,t)]_{\bar{t}} \\ \qquad\qquad = k^{-2}[\phi(x,t) - 2\phi(x,t-k) + \phi(x,t-2k)]. \end{cases}$$

We shall not use subscripts to denote partial derivatives so that no confusion between partial derivatives and partial difference quotients can arise.

We introduce a hyperbolic differential operator M defined as follows

$$Mu \equiv \frac{\partial^2 u}{\partial t^2} - a(x,t)\frac{\partial^2 u}{\partial x^2}, \tag{4.11.3}$$

where $a(x,t)$ satisfies the following two conditions. There exist constants $\mu_i > 0$ $(i = 0, 1, 2)$ such that

$$\begin{cases} 0 < \mu_1 \le a(x,t) \le \mu_2, \quad \textit{for all} \ \ (x,t) \in \overline{\Omega}, & (4.11.4) \\ |a(x,t) - a(x',t')| \le \mu[|x-x'| + |t-t'|], \quad \textit{for all} \ \ (x,t),(x',t') \in \overline{\Omega}. & \\ & (4.11.5) \end{cases}$$

As an approximation to M, we take the implicit finite difference operator L defined by

$$L\phi(x,t) \equiv \phi_{\bar{t}\bar{t}}(x,t) - a(x,t)\phi_{x\bar{x}}(x,t), \tag{4.11.6}$$

which is called an implicit finite difference operator for the following reason: in the equation $L\phi = 0$ in D the values of ϕ on $R_0(mk)$ are defined implicitly in terms of its valued on $R_0[(m-1)k]$, $R_0[(m-2)k]$ and $\bar{R}(mk) \bigcap \partial D$. Thus, if ϕ is prescribed on ∂D, the solution of the equation $L\phi = 0$ in D requires the inversion of $M-2$ systems of $N-1$ unknowns. It is readily verified that the matrix of each of these systems is of the tri-diagonal type; a matrix $A = (a_{ij})$ is tri-diagonal if $a_{ij} = 0$ for $|i-j| > 1$. It follows from (4.11.4) that the matrices associated with the operator L have dominant main diagonal, and according to a theorem of Taussky [617] are nonsingular. Hence, for arbitrary ϕ on ∂D, the equation $L\phi = \psi$ has a unique solution ϕ on D for any ψ defined on $\overline{D}$.

It follows from Taylor's theorem that the difference operator L is consistent [304, 355] with the differential operator M, i.e., for any twice continuously differentiable function u on $\overline{\Omega}$, we have as $h, k \to 0$,

$$|Lu - Mu| \to 0, \tag{4.11.7}$$

at each point of Ω.

If u is any twice continuously differentiable function defined on $\overline{\Omega}$ which vanishes along $x = 0$ and $x = 1$, then the energy inequality of Friedrichs and Lewy [220] states that there exists a constant $C > 0$ depending only on μ_i, $(i = 0, 1, 2)$ and T such that

$$\begin{aligned} &\int_0^1 |u(x,t)|^2 dx + \int_0^1 |\frac{\partial u}{\partial x}(x,t)|^2 dx + \int_0^1 |\frac{\partial u}{\partial t}(x,t)|^2 dx \\ &\leq C\left[\int_0^1 |\frac{\partial u}{\partial x}(x,0)|^2 dx + \int_0^1 |\frac{\partial u}{\partial t}(x,0)|^2 dx + \int_0^t \int_0^1 |Mu|^2 dxdt\right]. \end{aligned} \tag{4.11.8}$$

We shall prove that the difference operator L satisfies an analogous inequality. To this end, we need the following lemma which gives us a finite difference analogue of two differential identities used in the proof of the energy inequality (4.11.8).

Lemma 4.11.1 (Lees [358]) *Let $\phi(x,t)$ and $a(x,t)$ be functions defined on $\overline{D}$. Then at each point $(x,t) \in D$, we have*

$$\begin{cases} \phi\phi_{\bar{t}} = \dfrac{1}{2}[\phi^2]_{\bar{t}} + \dfrac{k}{2}\phi_{\bar{t}}^2, & (4.11.9) \\ \phi_{\hat{t}} a\phi_{x\bar{x}} = [\phi_{\hat{t}} a\phi_x]_{\bar{x}} - a_{\bar{x}}\phi_{\bar{x}}\phi_{\hat{t}} - \dfrac{\bar{a}}{2}[\phi_{\bar{x}}^2]_{\hat{t}} - \dfrac{\bar{a}k}{2}\phi_{\bar{x}\hat{t}}^2, & (4.11.10) \end{cases}$$

where $\bar{a} = a(x-h,t)$.

Proof In fact, an easy computation gives us

$$\begin{aligned} k[\phi^2(x,t)]_{\bar{t}} &= \phi^2(x,t) - \phi^2(x,t-k) \\ &= \phi^2(x,t) - \phi(x,t)\phi(x,t-k) + \phi(x,t)\phi(x,t-k) - \phi^2(x,t-k). \end{aligned}$$

Hence,

$$[\phi^2(x,t)]_{\bar{t}} = 2\phi(x,t)\phi_{\bar{t}}(x,t) - k\phi^2_{\bar{x}\bar{t}}(x,t)$$

which proves (4.11.9).

Using the difference product rule

$$[\eta(x)\xi(x)]_{\bar{x}} = \eta(x)\xi_{\bar{x}}(x) + \xi(x-h)\eta_{\bar{x}}(x),$$

we obtain

$$\begin{aligned}\phi_{\bar{t}} a\phi_{x\bar{x}} &= [\phi_{\bar{t}} a\phi_x]_{\bar{x}} - [\phi_{\bar{t}} a]_{\bar{x}}\phi_{\bar{x}} \\ &= [\phi_{\bar{t}} a\phi_x]_{\bar{x}} - a_{\bar{x}}\phi_{\bar{t}}\phi_{\bar{x}} - \bar{a}\phi_{\bar{t}\bar{x}}\phi_{\bar{x}}.\end{aligned}$$

Thus the identity (4.11.10) now follows by replacing $\bar{a}\phi_{\bar{t}\bar{x}}\phi_{\bar{x}}$ by $\frac{\bar{a}}{2}[\phi^2_{\bar{x}}]_{\bar{t}} + \frac{\bar{a}k}{2}\phi^2_{\bar{x}\bar{t}}$ and using the identity (4.11.9). □

For any function ϕ defined on $\overline{D}$, we introduce norms $\|\phi\|_{0,t}$, $^*\|\phi\|_{0,t}$ and $\|\phi\|_{1,t}$ as follows

$$\begin{cases} \|\phi\|^2_{0,t} = h\sum_{n=1}^{N}|\phi(nh,t)|^2 = h\sum_{R(t)}\phi^2, \ (R(mk) = R_0(mk)\bigcup(1,mk)), \\ ^*\|\phi\|^2_{0,t} = h\sum_{R_0(t)}\phi^2, \ \ \|\phi\|^2_{1,t} = \|\phi\|^2_{0,t} + \|\phi_{\bar{x}}\|^2_{0,t} + \|\phi_{\bar{t}}\|^2_{0,t}. \end{cases}$$

The next result concerns the energy inequality due to Lees [358].

Theorem 4.11.1 (Lees [358]) *Suppose that $\phi(x,t)$ is a function defined on the lattice $\overline{D}$ which vanishes on $\partial^1 D$ and $\partial^2 D$. Then there exists a constant c_0 depending only on μ_i $(i = 0, 1, 2)$ and t_0 such that for all sufficiently small k,*

$$\|\phi\|^2_{1,t} \le c_0\left[E(\phi) + k\sum_{s=2k}^{t}\|L\phi\|^2_{0,s}\right], \tag{4.11.11}$$

where

$$E(\phi) = h\sum_{R(k)}[\phi^2_{\bar{x}} + \phi^2_{\bar{t}}].$$

Proof Let $\hat{D} = \overline{D}\bigcup[\partial^2 D - (1,k)]$. At each point of D, noting that ϕ vanishes on $\partial^2 D$, we have

$$\phi_{\bar{t}} L\phi = \phi_{\bar{t}}\phi_{t\bar{t}} - a\phi_{\bar{t}}\phi_{x\bar{x}}.$$

Using the identities of Lemma 4.11.1, we obtain the identity

$$\phi_{\bar{t}} L\phi = \frac{1}{2}[\phi_{\bar{t}}^2]_{\bar{t}} - [a\phi_{\bar{t}}\phi_{\bar{x}}]_{\bar{x}} + a_{\bar{x}}\phi_{\bar{t}}\phi_{\bar{x}} + \frac{1}{2}\bar{a}[\phi_{\bar{x}}^2]_{\bar{t}} + \frac{k}{2}(\phi_{\bar{t}\bar{t}}^2 + \phi_{\bar{x}\bar{t}}^2).$$

The last term on the right-hand side is non-negative since $a \geq \mu_1$ in $\overline{\Omega}$. Hence,

$$\phi_{\bar{t}} L\phi \geq \frac{1}{2}[\phi_{\bar{t}}^2]_{\bar{t}} - [a\phi_{\bar{t}}\phi_{\bar{x}}]_{\bar{x}} + a_{\bar{x}}\phi_{\bar{x}}\phi_{\bar{t}} + \frac{1}{2}\bar{a}[\phi_{\bar{x}}^2]_{\bar{t}}. \tag{4.11.12}$$

Due to

$$hk\sum_{\hat{D}}[a\phi_{\bar{t}}\phi_{\bar{x}}]_{\bar{x}} = k\sum_{s=2k}^{t_0} a\phi_{\bar{t}}\phi_{\bar{x}}\Big|_{x=0}^{x=1} = 0,$$

multiplying (4.11.12) by hk and summing over all lattice points of $\hat{D}$, we obtain

$$2hk\sum_{D}\phi_{\bar{t}} L\phi \geq hk\sum_{\hat{D}}[\phi_{\bar{t}}^2]_{\bar{t}} + 2hk\sum_{D} a_{\bar{x}}\phi_{\bar{t}}\phi_{\bar{x}} + hk\sum_{\hat{D}}\bar{a}[\phi_{\bar{x}}^2]_{\bar{t}}. \tag{4.11.13}$$

Now

$$\begin{aligned} hk\sum_{\hat{D}}[\phi_{\bar{t}}^2]_{\bar{t}} &= h\sum_{n=1}^{N}[\phi_{\bar{t}}^2(nh, t_0) - \phi_{\bar{t}}^2(nk, k)] \\ &= \|\phi_{\bar{t}}\|_{0,t_0}^2 - \|\phi_{\bar{t}}\|_{0,k}^2. \end{aligned} \tag{4.11.14}$$

Summing by parts, we obtain

$$\begin{aligned} hk\sum_{\hat{D}}\bar{a}[\phi_{\bar{x}}^2]_{\bar{t}} &= h\sum_{x=h}^{l} k\sum_{s=2k}^{t_0}\bar{a}(x,s)[\phi_{\bar{x}}^2]_{\bar{t}} \\ &= h\sum_{x=h}^{l}[\phi_{\bar{x}}^2(x,t_0)\bar{a}(x,t_0) - \phi_{\bar{x}}^2(x,k)\bar{a}(x,2k)] \\ &\quad -hk\sum_{x=h}^{l}\sum_{s=2k}^{t_0}\bar{a}_t(x,s)\phi_{\bar{x}}^2(x,s) \\ &\geq \mu_1\|\phi_{\bar{x}}\|_{0,t_0}^2 - \mu_2\|\phi_{\bar{t}}\|_{0,k}^2 - \mu hk\sum_{\hat{D}-R(t_0)}\phi_{\bar{x}}^2. \end{aligned} \tag{4.11.15}$$

It follows now from (4.11.13), (4.11.14) and (4.11.15) that

$$\|\phi_{\bar{t}}\|_{0,t_0}^2 + \mu_1\|\phi_{\bar{x}}\|_{0,t_0}^2 \leq \|\phi_{\bar{t}}\|_{0,k}^2 + \mu_1\|\phi_{\bar{x}}\|_{0,k}^2 + 2hk\sum_D \phi_{\bar{t}}L\phi$$
$$+\mu hk\sum_{D-R(t_0)}\phi_{\bar{x}}^2 + 2\mu hk\sum_D |\phi_{\bar{x}}||\phi_{\bar{t}}|. \quad (4.11.16)$$

Since

$$2\mu hk\sum_D |\phi_{\bar{x}}||\phi_{\bar{t}}| \leq k\sum_{s=2k}^{t_0}\left\{\|\phi_{\bar{x}}\|_{0,s}^2 + \|\phi_{\bar{x}}\|_{0,s}^2\right\}$$

and

$$2hk\sum_D \phi_{\bar{t}}L\phi \leq k\sum_{s=2k}^{t_0}\left\{\|\phi_{\bar{t}}\|_{0,s}^2 +^* \|L\phi\|_{0,s}^2\right\},$$

we obtain, for all k satisfying

$$(1+\mu)k \leq 1/2, \quad \mu k \leq \mu_1 - 1/2,$$

$$\|\phi_{\bar{t}}\|_{0,t}^2 + \|\phi_{\bar{x}}\|_{0,t}^2 \leq 2\max(1,\mu_2)E(\phi) + 2k\sum_{s=2k}^{l}{}^*\|L\phi\|_{0,s}^2$$
$$+2(1+2\mu)k\sum_{s=2k}^{t-k}\{\|\phi_{\bar{t}}\|_{0,s}^2 + \|\phi_{\bar{x}}\|_{0,s}^2\}. \quad (4.11.17)$$

Letting

$$\begin{cases} \omega(t) = \|\phi_{\bar{t}}\|_{0,t}^2 + \|\phi_{\bar{x}}\|_{0,t}^2, \quad c = 2(1+2\mu), \\ \rho(t) = 2\max(1,\mu_2)E(\phi) + 2k\sum_{s=2k}^{t}{}^*\|L\phi\|_{0,s}^2, \end{cases}$$

then (4.11.17) reduces to

$$\omega(t) \leq \rho(t) + ck\sum_{s=2k}^{t-k}\omega(s).$$

Applying Theorem 2.1.15 to the above inequality, we obtain

$$\omega(t) = \|\phi_{\bar{t}}\|^2_{0,t} + \|\phi_{\bar{x}}\|^2_{0,t} \leq (c_0/2)\left[E(\phi) + k\sum_{s=2k}^{t}{}^{*}\|L\phi\|^2_{0,s}\right]$$

where

$$c_0 = 4\max(1,\mu_2)\exp\left(2(1+2\mu)(t_0-2k)\right).$$

Since

$$\|\phi\|^2_{0,t} \leq \|\phi_{\bar{x}}\|^2_{0,t},$$

we obtain

$$\|\phi\|^2_{1,t} \leq c_0\left[E(\phi) + k\sum_{s=2k}^{t}{}^{*}\|L\phi\|^2_{0,s}\right]$$

which completes the proof the theorem. □

4.12 Applications of Theorems 2.1.17, 2.1.26–2.1.28 and Corollaries 2.1.13–2.1.14 to Discrete Systems

Consider first the two dimensional discrete inequalities

$$|x_i(t)| \leq |k_i| + \sum_{s=0}^{t-1}|f_i(s,x_1(s),x_2(s))| \quad (i=1,2)$$

which arises from the study of two dimensional differential systems using Euler's method, if the following conditions hold,

$$|f_i(s,x_1(s),x_2(s))| \leq b_i(t) + a_{i1}(t)|x_1(t)| + a_{i2}(t)|x_2(t)|$$

where a_{i1}, a_{i2}, b_i are non-negative functions, then it follows from Corollary 3.2 in [2] that $|x_i(t)| \leq u_i(t)$ where $u_1(t)$ and $u_2(t)$ are the solution of the following discrete system

$$\begin{cases} \Delta u_i(t) = b_i(t) + a_{i1}(t)u_1(t) + a_{i2}u_2(t), \\ u_i(0) = |k_i|. \end{cases} \tag{4.12.1}$$

Now from (4.12.1), we derive

$$u_2(t) = \prod_{s=0}^{t-1}(1 + a_{22}(s)) \left[|k_2| + \sum_{s=0}^{t-1}(b_2(s) + a_{11}(s)u_1(s)) \prod_{\tau=0}^{s} (1 + a_{22}(\tau))^{-1} \right]$$

Now substituting this in the first equation of (4.12.1), and applying Theorem 2.1.26, we can find for $u_1(t)$ the exact form as in Theorem 2.1.27.

Next following the same notations as in [452], we consider the linear stochastic discrete system

$$y_{n+1}(\omega) = A(\omega)y_n(\omega), \quad y_0(\omega) = x_0 \tag{4.12.2}$$

and the perturbed system including an operator T as

$$\begin{cases} x_{n+1}(\omega) = A(\omega)x_n(\omega) + f_n(\omega, x_n(\omega), (Tx_n)(\omega)), \\ x_0(\omega) = x_0. \end{cases} \tag{4.12.3}$$

Let $Y_n(\omega)$ denote the stochastic fundamental matrix solution of the homogeneous system (4.12.1) such that $Y_0(\omega)$ is the unit matrix.

The following modified versions of Theorems 2–4 in [452] which require weaker conditions can be readily proved using the results in Theorems 2.1.17, 2.1.26–2.1.28 and Corollaries 2.1.13–2.1.14.

Theorem 4.12.1 (Agarwal-Thandapani [17]) *Suppose that*

$$|Y_n(\omega)Y_{s+1}^{-1}(\omega)f_s(\omega, x_s(\omega), (Tx_s)(\omega))| \le a_s(\omega)|x_s(\omega)| + b_s(\omega)|(Tx_s)(\omega)|$$

where $a_n(\omega)$, $b_n(\omega)$ are non-negative random function defined for all $s \in \mathbb{N}$, $\omega \in \Omega$. Furthermore, suppose that the operator T satisfies the inequality

$$|(Tx_n)(\omega)| \le \sum_{s=0}^{n-1} c_s(\omega)|x_s(\omega)|$$

where $c_n(\omega)$ is a non-negative random function defined for all $n \in \mathbb{N}$, $\omega \in \Omega$. Then for every bounded random solution $x_n(\omega)$ of problem (4.12.1) on $\mathbb{N}$, the corresponding random solution $x_n(\omega)$ of problem (4.12.2) is bounded on $\mathbb{N}$ provided that

$$\prod_{s=0}^{+\infty}[1 + a_s(\omega) + b_s(\omega)\sum_{\tau=0}^{n-1} c_\tau(\omega)] < +\infty.$$

Theorem 4.12.2 (Agarwal-Thandapani [17]) *Assume*

$$|Y_n(\omega)Y_{s+1}^{-1}(\omega)| \le Me^{-\alpha(n-s)},$$

$$|Y_n(\omega)| \le Me^{-\alpha n},$$

$$|f_n(\omega, x_n(\omega), (Tx_n)(\omega))| \le a_n(\omega)|x_n(\omega)| + b_n(\omega)|(Tx_n)(\omega)|$$

$$|(Tx_n)(\omega)| \le e^{-\alpha n}\sum_{s=0}^{n-1} c_s(\omega)|x_s(\omega)|$$

where $M > 0$, $\alpha > 0$ are constants and $a_n(\omega)$, $b_n(\omega)$, $c_n(\omega)$ are defined in Theorem 4.12.1. Then all random solutions of problem (4.12.2) approach zero as $n \to +\infty$,

$$K = \prod_{s=0}^{+\infty}[1 + a_s(\omega) + b_s(\omega)\sum_{\tau=0}^{n-1} c_\tau(\omega)e^{-\alpha\tau}] < +\infty.$$

Theorem 4.12.3 (Agarwal-Thandapani [17]) *In Theorem 4.12.2, let $-\alpha = \epsilon$ and $K \le c$ where $c > 0$ is a constant, then the conclusion of Theorem 2.1.10 follows.*

4.13 An Application of Corollary 2.1.10 to Finite Difference Equations

To illustrate the usefulness of Corollary 2.1.10, we shall consider a simple application in the theory of finite difference equations. First, however, it is convenient to introduce some additional notations.

Let p be a positive integer and let Ω be a discrete increasing sequence of points in $[-p, +\infty)$. For each $t > -p$, let τ_t denote the largest element in Ω less than t and let $\alpha_t = \{\tau \cdot \tau_t \ge \tau \ge \tau_{t-p}, \quad \tau \in \Omega\}$. We assume that the number of points in α_t is bounded for all $t > -p$. If x is a function defined for all $t \ge -p$ with values in $\mathbb{R}^n$ (the space of n-dimensional column vectors), then $x(\alpha_t)$ denotes the vector $(x(\nu_1), x(\nu_2), \cdots, x(\nu_k))^T$ in $\mathbb{R}^{nk}$ where $\nu_1, \cdots, \nu_k$ is the largest subset of Ω with $\tau_t \ge \nu_1 > \nu_2 > \cdots > \nu_k \ge \tau_{t-p}$. $\| \;\|$ designates any appropriate norm definable on $\mathbb{R}^m$ for arbitrary m. If A is an $m \times m$ matrix, then $\|A\|$ denotes the smallest number ξ such that $\|Au\| \le \xi\|u\|$ for all u in $\mathbb{R}^m$.

Consider finite difference equations of the form for all $t \ge 0$,

$$x(t) = \sum_{t-p\le s\le t} A(\tau_t, \tau_s)x(\tau_s) + F(\tau_t, x(\alpha_t)), \tag{4.13.1}$$

where the $A(\tau_t, \tau_s)$'s are $n \times n$ matrices. $F(t, \varphi)$ is a function mapping into $\mathbb{R}^n$ and define for all $t \geq -p$ and $\varphi \in \mathbb{R}^{n(b+1)}$ where b is the maximum number of points in any α_t. Furthermore, we assume there exist a positive constant c and a function $L(t)$ such that for all $t \geq 0$ and all $\|\varphi\| < c$,

$$\|F(t, \varphi)\| \leq L(t)\|\varphi\|. \tag{4.13.2}$$

It is clear that for each specification of $x \in [-p, 0]$, there corresponds a unique solution of Eq. (4.13.1) defined for all $t \geq 0$.

Now for an arbitrary value of $t \geq 0$, let $\nu_1 > \nu_2 > \cdots > \nu_k$ denote the points in α_t. We define $A^*(\tau_t)$ to be the $(b+1) \times (b+1)$ matrix of $n \times n$ matrices $A_{ij}(\tau_t)$ where $A_{1j}(\tau_t) = A(\tau_t, \nu_j)$, $j = 1, \cdots, k$, $A_{i+1,i}(\tau_t) = I$, $i = 2, \cdots, k$, and $A_{ij}(\tau_t) = 0$ otherwise. We define the column vector $x^*(t) \in \mathbb{R}^{n(b+1)}$ as

$$(x(t), x(\nu_1), \cdots, x(\nu_k), 0, \cdots, 0)^T$$

and the vector $F^*(\tau_t, x^*(\tau_t)) \in \mathbb{R}^{n(b+1)}$ by the formula

$$F^*(\tau_t, x^*(\tau_t)) = (F(\tau_t), x(\alpha_t), 0, \cdots, 0)^T.$$

It is easily verified that Eq. (4.13.1) is equivalent to the larger system of the form for all $t \geq 0$,

$$x^*(t) = A^*(\tau_t)x^*(\tau_t) + F^*(\tau_t, x^*(\tau_t)). \tag{4.13.3}$$

We shall prove the following boundedness result.

Theorem 4.13.1 (Jones [305]) *Let $\|\prod_{\nu\in\Omega,\ t>\nu} A^*(\nu)\|$ be bounded for all $t \geq 0$. Suppose that there exists a function g such that*

$$\sup\{\prod A^*(\nu) : \nu \in \Omega,\ \ t > \nu > t_1,\ \ t \geq t_1 \geq 0\} \leq g(\tau_t) \tag{4.13.4}$$

and that $\prod_{\nu\in\Omega,\ \nu<t}(1 + g(\nu)L(\nu))$ is bounded for all $t \geq 0$. Then all solution of Eq. (4.13.1) starting with sufficiently small initial data are bounded. If $c = +\infty$, then all solution of Eq. (4.13.1) are bounded.

Proof Let $\nu_0 = 0$ if 0 is contained in Ω and otherwise let $\nu_0 = \tau_0$. Let $\nu_1 < \nu_2 < \cdots$ denote the positive elements of Ω. We observe that for all $t \in (0, \nu_1]$,

$$x^*(t) = A^*(\nu_0)x^*(\nu_0) + F(\nu_0, x^*(\nu_0)),$$

and it follows easily from induction that in general for all $t \geq 0$,

$$x^*(t) = \prod_{t>\nu_i} A^*(\nu_i)x^*(\nu_0) + \sum_{\nu_i<t}\Big(\prod_{t>\nu_j>\nu_i} A^*(\nu_j)\Big)F(\nu_i, x^*(\nu_i)). \tag{4.13.5}$$

Now let u be any real-valued function defined on $[-p, +\infty)$ such that

$$u(t) \leq \| \prod_{t>\nu_i} A^*(\nu_i)\| u(\nu_0) + g(\tau_t) \sum_{\nu_i<t} L(\nu_i)u(\nu_i). \tag{4.13.6}$$

Employing Corollary 2.1.10, we have

$$u(t) \leq \| \prod_{t<\nu_i} A^*(\nu_i)\| u(\nu_0) + g(\tau_t) \sum_{\nu_i<t}\Big(\prod_{\nu_i<\nu_j<t} (1 + g(\nu_j)L(\nu_j))L(\nu_i)\Big)\| \prod_{\nu_i>\nu_j} A^*(\nu_j)\| u(\nu_0).$$

By our hypotheses, there exists a constant $K_1 > 0$ which bounds $\| \prod_{t>\nu_i} A^*(\nu_i)\|$ for all $t \geq 0$, so we have

$$u(t) \leq K_1 \prod_{\nu_i<t}(1 + g(\nu_i)L(\nu_i))u(\nu_0).$$

But then $\prod_{\nu_i<t}(1 + g(\nu_i)L(\nu_i))$ is by hypothesis also bounded by some constant K_2, so we may conclude that for all $t > 0$,

$$u(t) \leq K_1K_2u(\nu_0). \tag{4.13.7}$$

It follows, of course, that $|u(0)| < c/(K_1K_2)$ implies $|u(t)| < c$ for all $t \geq 0$. Now returning to (4.13.5), it is clear that (4.13.2) and (4.13.4) imply that for $\|x^*(\nu_0)\| < c/(K_1K_2)$, we have

$$\|x^*(t)\| \leq \| \prod_{t>\nu_i} A^*(\nu_i)\| \; \|x^*(\nu_0)\| + g(\tau_t) \sum_{\nu_i<t} L(\nu_i)\|x^*(\nu_i)\|, \tag{4.13.8}$$

which is an inequality of the form of (4.13.6). Thus we have by (4.13.7) that $\|x^*(t)\| \leq K_1K_2\|x^*(\nu_0)\|$ for all $t \geq 0$ and the proof of our theorem is complete. □

A complementary result concerning stability which is essentially proved in the proof of Theorem 4.13.1 may be stated as follows.

Theorem 4.13.2 (Jones [305]) *Assume the hypotheses of Theorem 4.13.1, suppose Ω is not bounded, and suppose $g(t) \to 0$ as $t \to +\infty$. Then all solution of (4.13.1) staring with sufficiently small initial data tend to zero as $t \to +\infty$. If $c = +\infty$ then all solutions of (4.13.1) tend to zero as $t \to +\infty$.*

Proof In the proof of Theorem 4.13.1, it was shown that for all sufficiently small initial data and for all initial data if $c = +\infty$,

$$\|x^*(t)\| \leq \|\prod_{t>\nu_i} A^*(\nu_i)\| \ \|x^*(\nu_0)\|$$
$$+g(\tau_t)\sum_{\nu_i<t}\Big(\prod_{\nu_i<\nu_j<t}(1+g(\nu_j)L(\nu_j)L(\nu_j))L(\nu_i)\Big) \ \|\prod_{\nu_i>\nu_j} A^*(\nu_j)\| \ \|x^*(\nu_0)\|. \tag{4.13.9}$$

Since our hypotheses imply that both terms on the right-hand side of inequality (4.13.9) tend to zero as $t \to +\infty$, Theorem 4.13.2 is proved. □

4.14 An Application of Theorem 2.1.25 to an Integro-Differential Equation

In this section, we give an application of repeated integral inequality in Theorems 2.1.25. To this end, we consider the integro-differential equation

$$y'(t) = F(t, y(t), \psi(t)), \quad y(0) = y_0, \quad 0 \leq t \leq T, \tag{4.14.1}$$

where

$$\psi(t) = \int_0^t \frac{k(t, s, y(s))}{(t-s)^\alpha} ds, \quad 0 < \alpha < 1. \tag{4.14.2}$$

In the sequel, we assume that F, k are sufficiently smooth to guarantee the existence of a unique solution y which has a bounded second derivative.

We note that Brunner [114] has studied collocation methods for solving Eq. (4.14.1) where $F(t, y, \psi)$ is linear.

Applying an l-step linear multistep method to the differential part of problem (4.14.1) gives us

$$\frac{1}{h}\sum_{j=0}^{l} a_j y_{i-j} = \sum_{j=0}^{l} b_j F(t_{i-j}, y_{i-j}, z_{i-j}), \tag{4.14.3}$$

where y_i denotes an approximation to $y(t_i)$, $t = ih$, $0 \leq i \leq N$, $Nh = T$, and z_i denotes an approximation to $\psi(t_i)$.

To illustrate an application of Theorem 2.1.25, a simple product integration method is used to approximate $\psi(t_i)$,

$$z_i = h \sum_{j=0}^{i-1} w_{ij} k(t_i, v, y_j), \tag{4.14.4}$$

where

$$w_{ij} = \frac{1}{h} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_i - s)^\alpha},$$

and Euler's method is chosen as the linear multistep method of (4.14.3).

This permits us to define the discrete algorithm

$$\Phi^h(y^h) = 0, \quad \Phi^h : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1},$$

where

$$[\Phi^h(y^h)]_i = \begin{cases} y_i - \tilde{y}_i, \quad i = 0, , \\ \frac{y_i - y_{i-1}}{h} - F(t_{i-1}, y_{i-1}, z_{i-1}), \quad i = 2, 3, \ldots, N, \\ z_{i-1} = h \sum_{j=0}^{i-2} w_{i-1} k(t_{i-1}, t_j, y_j), \quad i = 2, 3, \ldots, N. \end{cases} \tag{4.14.5}$$

We assume that the required starting values $\tilde{y_0}, \ \tilde{y_1}$ are accurate of order one.

We observe that since $i - j \geq 1$,

$$|w_{ij}| = w_{ij} = \frac{1}{h^\alpha (i-j)^\alpha} \int_0^1 \frac{du}{(1 - u/(i-j))^\alpha} \leq \frac{h^{-\alpha}}{(1-\alpha)(i-j)^\alpha}. \tag{4.14.6}$$

Using the bound (4.14.6), it can be shown that the discretization (4.14.5) consists of order at least one, that is, there exist constants C_i, independent of h, such that

$$|\theta_i^h| := |\frac{y(t_i) - y(t_{i-1})}{h} - F(t_{i-1}, y(t_{i-1}), z(t_{i-1}))| \leq C_i h, \tag{4.14.7}$$

where

$$z(t_i) = h \sum_{j=0}^{i-1} w_{ij} k(t_i, t_j, v(t_j)).$$

To demonstrate the convergence of the discretization (4.14.5), we shall consider the special case

$$F(t, y(t), \psi(t)) = f(t) + \psi(t).$$

The more general case will follow in a similar manner, but notationally it is more complicated.

The error $y(t_i) - y_i$ satisfies

$$(y(t_i) - y_i) - (y(t_{i-1}) - y_{i-1}) = h(z(t_{i-1}) - z_{i-1}) + h\theta_i^h.$$

Summing over 2 to k,

$$(y(t_k) - y_k) = h\sum_{i=1}^{k-1}(z(t_i) - z_i) + h\sum_{i=2}^{k}\theta_i^h + \tilde{y}_1.$$

Using the bound (4.14.6) and consistency, and assuming that the starting values are accurate of order one, and that $k(t, s, y)$ is Lipschitz continuous with respect to y with Lipschitz constant L, $x_i = |y(t_i) - y_i|$ satisfies

$$\begin{cases} x_0 \le Ch, \\ x_i \le Ch + \dfrac{Lh^{2-\alpha}}{1-\alpha}\displaystyle\sum_{k=0}^{i-1}\sum_{j=0}^{k-1}\frac{x_j}{(k-j)^{\alpha}}, \quad i = 1, 2, \ldots, N, \end{cases}$$

for some constant $C > 0$, by applying Theorem 2.1.25, the convergence of order at least one now follows.

4.15 An Application of Theorem 2.1.30 to Finite Difference Equations

In this section, we shall use Theorem 2.1.30 to study the following two finite difference equations

$$x(n+1) = A(n)x(n) + G\Big[n, x(n), \sum_{s=n_0}^{n-1} h(n, s, x(s))\Big], \quad x(n_0) = z_0, \qquad n \in \mathbb{N} \tag{4.15.1}$$

and

$$y(n+1) = A(n)y(n), \quad y(n_0) = z_0, \quad n \in \mathbb{N}, \tag{4.15.2}$$

where $A(n)$ is an $r \times r$ matrix with $\det A(n) \ne 0$ and $x,\ y,\ G$ and h are r-dimensional vector-valued functions and z_0 is a constant r-vector.

Let $|\cdot|$ denote some convenient norm on the r-dimensional vector space $\mathbb{R}^r$ as well as a corresponding consistent matrix norm. We denote by $Y(n)$ the fundamental solution matrix of Eq. (4.15.2), and denote by Y^{-1} the inverse of Y. Here we introduce the following conception.

Definition 4.15.1 A real-valued r-vector function $F(n)$ defined on $\mathbb{N}$, here $r \geq 1$, is said to be slowly growing if the following relation holds

$$\lim_{n\to+\infty} \frac{1}{n} \sup\ ln|F(n)| \leq 0.$$

Now we assume that the following conditions hold:

(1) $|Y(n)Y^{-1}(s)| \leq f(n,s)$, for all n and $s \in \mathbb{N}$,
(2) $|G[n,x,y]| \leq a(n) + b(n)|x| + c(n)|y|$, for all $n \in \mathbb{N};\ x,\ y \in \mathbb{R}^r$,
(3) $|h(n,s,x)| \leq r(n,s) + g(n,s)|x|$, for all n and $s \in \mathbb{N}$; $x \in \mathbb{R}^{\backslash}$,

where $a(n),\ b(n),\ c(n) : \mathbb{N} \to \mathbb{R}_+$ and $f(n,s),\ g(n,s)$ and $r(n,s) : \mathbb{N}\times\mathbb{N} \to \mathbb{R}_+$ are known continuous functions. Here $f(n,s)$ and $g(n,s)$ are non-decreasing in n for $s \in \mathbb{N}$ fixed. Then by the well-known variation of constants formula, any solution of Eq. (4.15.1) may be written as

$$x(n) = Y(n)Y^{-1}(n_0)z_0 + \sum_{s=n_0}^{n-1} Y(n)Y^{-1}(s+1)G\Big[s, x(s), \sum_{s=n_0}^{n-1} h(s,t,x(t))\Big],\quad n \in \mathbb{N},\ z_0 \in \mathbb{R}^r. \tag{4.15.3}$$

Using the above conditions (1)–(3), then we obtain from (4.15.3) that for all $n \in \mathbb{N}, z_0 \in \mathbb{R}^r$,

$$\begin{aligned}
|\,x(n)\,| &= |\,Y(n)Y^{-1}(n_0)\,|\,|\,z_0\,| + \sum_{s=n_0}^{n-1} |\,Y(n)Y^{-1}(s+1)\,| \times |\,G[s,x(s), \sum_{s=n_0}^{n-1} h(s,t,x(t))]\,| \\
&\leq f(n,n_0)|z_0| + \sum_{s=n_0}^{n-1} f(n,s+1)a(s) + \sum_{s=n_0}^{n-1} f(n,s+1)c(s)\left(\sum_{s=n_0}^{n-1} r(s,t)\right) \\
&\quad + \sum_{s=n_0}^{n-1} f(n,s+1)b(s)|x(s)| + \sum_{s=n_0}^{n-1} f(n,s+1)c(s)\left(\sum_{s=n_0}^{n-1} g(s,t)|x(t)|\right).
\end{aligned}$$

Now applying Theorem 2.1.30 to the above inequality yields,

$$|\,x(n)\,| \leq \{f(n,n_0)|z_0| + I(n)\}[1+K(n)],\quad n \in \mathbb{N}, z_0 \in \mathbb{R}^r, \tag{4.15.4}$$

where

$$\begin{cases} I(n) = \sum_{s=n_0}^{n-1} f(n, s+1)\Big[a(s) + c(s)\sum_{s=n_0}^{n-1} r(s,t)\Big], \\ K(n) = \sum_{s=n_0}^{n-1} f(n, s+1)\theta(s) \prod_{s=n_0}^{n-1}\Big[1 + f(n, k+1)\theta(k) + g(n,k)\Big], \\ \theta(s) = \max[b(s), c(s)]. \end{cases}$$

As in [656], it is worth pointing out that we can easily observe from the inequality (4.15.4) that, the following conclusions hold:

1. If the functions $f(n, n_0)$, $I(n)$ and $K(n)$ are bounded on $\mathbb{N}$, then all of the solutions of the Eq. (4.15.1) are bounded.
2. If $f(n, n_0)$ and $K(n)$ are bounded on $\mathbb{N}$, and if $a(n) = c(n) \equiv 0$ or $a(n) = r(n,s) \equiv 0$ holds, then the trivial solution $x(n) \equiv 0$ to (4.15.1) is stable in the sense of Lyapunov.
3. If $a(n) = c(n) \equiv 0$ or $a(n) = r(n,s) \equiv 0$ holds, $K(n)$ is bounded on $\mathbb{N}$ and $f(n, n_0) \to 0$ as $n \to +\infty$, then the trivial solution $x(n) \equiv 0$ to (4.15.1) is asymptotically stable.
4. If $a(n) = c(n) \equiv 0$ or $a(n) = r(n,s) \equiv 0$ holds, $K(n)$ is bounded on $\mathbb{N}$ and $f(n, n_0) = 0[\exp(-\delta n)]$ as $n \to +\infty$, where $\delta > 0$ is a constant, then the trivial solution $x(n) \equiv 0$ to (4.15.1) is exponentially stable with degree δ, see [468].
5. If the functions $f(n, n_0)$, $I(n)$ and $K(n)$ are slowly growing on $\mathbb{N}$, then all solution of the Eq. (4.15.1) are slowly growing.

Remark 4.15.1 When $a(n) = c(n) \equiv 0$ or $a(n) = r(n,s) \equiv 0$ holds, then according to above conditions (2) and (3), we have, for all $n \in \mathbb{N}$,

$$G\Big[n, 0, \sum_{s=n_0}^{n-1} h(n, s, 0)\Big] = 0,$$

and hence $x(n) \equiv 0$ is a solution of the Eq. (4.15.1).

4.16 An Application of Theorem 2.1.34 to Discrete Time Control Systems

We note that the direct or second method of Lyapunov has greatly advanced the study of stability of force-free difference or discrete time systems, which has been successfully extended to the study of the bounded-input bounded-output (BIBO) stability of arbitrary, nonlinear, time-varying, discrete control systems by Lin and Varaiya [362].

In this section, we use Theorem 2.1.34 to study the stability and asymptotic stability behavior of the solutions of discrete time control systems of the form

$$x(n+1) = A(n)x(n) + f(n, x(n), \sigma(n)), x(n_0) = x_0 \tag{4.16.1}$$

with

$$\sigma(n) = u(n) + \sum_{s=n_0}^{s-1} k(n, s, x(s)) \tag{4.16.2}$$

as a perturbation of the linear system

$$y(n+1) = A(n)y(n),\ y(n_0) = x_0. \tag{4.16.3}$$

Here x, y, σ, u, f and k are the elements of $\mathbb{R}^r$, the r-dimensional vector space $A(n)$ is a $r \times r$ matrix with $det\ A(n) \neq 0$, the functions f and k are defined on $\mathbb{N} \times \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}^r$ and $\mathbb{N} \times \mathbb{N} \times \mathbb{R}^r \to \mathbb{R}^r$ respectively. The symbol $|\cdot|$ will denote some convent norm on $\mathbb{R}^r$ as well as a corresponding consistent matrix norm. We note by $x_\sigma(n, n_0, x_0)$ the solution of problem (4.16.1)–(4.16.2) with $x_\sigma(n_0, n_0, x_0) = x_0$. Let $y(n, n_0, x_0)$ be the solution of Eq. (4.16.3) and let $Y(n)$ denote the fundamental solution matrix of the system (4.16.3) such that $Y(n_0) = I$ (the identity matrix).

It is well-known that the bounded-input bounded-output (BIBO) stability and exponential stability, which may be defined for dynamical systems, are specially important. Next, we shall give their definitions.

Definition 4.16.1 The system (4.16.1)–(4.16.2) is said to be BIBO 'stable' if for any solution $x_\sigma(n, n_0, x_0)$ of problem (4.16.1)–(4.16.2) with $|\sigma(n)| < c_1\delta$ (c_1=constant), for $\delta > 0$ and $n \in \mathbb{N}$ such that $|x_0| < \delta$ implies

$$|x_\sigma(n, n_0, x_0)| < c\ \delta\ (c = constant), \quad for\ all\ n \geq n_0.$$

If, further, $\lim_{n \to +\infty} |x_\sigma(n, n_0, x_0)| = 0$, problem (4.16.1)–(4.16.2) is said to be asymptotically BIBO stable.

Definition 4.16.2 The system (4.16.1)–(4.16.2) is said to be 'exponentially asymptotically stable' if for any solution $x_\sigma(n, n_0, x_0)$ of problem (4.16.1)–(4.16.2) such that $|x_0| < \delta$ implies

$$|x_\sigma(n, n_0, x_0)| < c\delta e^{-\alpha(n-n_0)},$$

for $c \geq 0, \delta > 0, \alpha > 0$, and $n \geq n_0$.

Definition 4.16.3 The system (4.16.1)–(4.16.2) is said to be 'uniformly slowly growing' if

$$|x_\sigma(n, n_0, x_0)| < c\delta e^{\alpha(n-n_0)},$$

for $c \geq 0, \delta > 0, \alpha > 0$ and $n \geq n_0$.

Definition 4.16.4 System (4.16.3) will be called 'stable' if for any solution $y(n, n_0, x_0)$ of (4.16.3) such that $|x_0| < \delta$ implies

$$|y(n, n_0, x_0)| < c\delta \ (c = constant), \ for \ \delta > 0, \ and \ n \geq n_0.$$

If, further, $\lim_{n\to+\infty} |y(n, n_0, x_0)| = 0$, system (4.16.3) is said to be 'asymptotically stable'.

In order to deal with the subsequent discussion, we establish the following finite difference inequality, due to [475].

Theorem 4.16.1 (Pachpatte [475]) *Let $a(n, s) \geq 0$ be defined for $n, s \in \mathbb{N}$ and $W(n, x, y)$ be a non-negative function defined for all $n \in \mathbb{N}$, $0 \leq x < +\infty$, $0 \leq y < +\infty$, and monotone increasing with respect to x and y for any fixed $n \in \mathbb{N}$, and*

$$m(n) \leq h(n) + \sum_{s=n_0}^{n-1} a(n, s)W(s, m(s), \phi(s)),$$

where m, h are defined for all $n \in \mathbb{N}$. Suppose that $r_h(n)$ is the solution of the equation

$$r(n) = h(n) + \sum_{s=n_0}^{n-1} a(n, s)W(s, r(s), r(s)), \tag{4.16.4}$$

existing on $\mathbb{N}$ such that $m_h(n_0) \leq r(n_0)$, then there holds an inequality

$$m(n) \leq r_h(n), \ n \in \mathbb{N}$$

provided that

$$\phi(n) \leq r_h(n), \ n \in \mathbb{N}.$$

Proof We assume that the inequality $m(n) \leq r_h(n)$ is not satisfied for all $n \in \mathbb{N}$. Then there exists a $\beta > n_0$ in $\mathbb{N}$ such that $m(n) \leq r_h(n)$ for $n = n_0, n_0 + 1, \cdots, \beta - 1$, but $m(\beta) > r_h(\beta)$, i.e., $r_h(\beta) - m(\beta) < 0$. On the other hand,

$$r_h(\beta) - m(\beta) \geq \sum_{s=n_0}^{\beta-1} a(\beta, s) \left[W(s, r_h(s), r_h(s)) - W(s, m(s), r_h(s))\right] \geq 0.$$

We have thus obtained a contradiction, and the desired result follows. □

Definition 4.16.5 We shall take the above Eq. (4.16.4) to be stable if $h(n) < \delta$ implies $r_h(n) < c_1\delta$ $(c_1 = constant)$, for $\delta > 0$ and $n \geq n_0$.

If, further, $\lim_{n\to+\infty} r_h(n) = 0$, then this equation is said to be asymptotically stable.

The next theorem investigates that the BIBO stability and asymptotic BIBO stability behavior of solutions of (4.16.1)–(4.16.2) depends upon the stability and asymptotic stability behavior of solutions of (4.16.3) and (4.16.4).

Theorem 4.16.2 (Pachpatte [475]) *Assume that the fundamental solution matrix $Y(n)$ of (4.16.3) satisfies*

$$|Y(n)Y^{-1}(s+1)| \le a(n,s),\ 0 \le s \le n < +\infty \tag{4.16.5}$$

where $a(n,s) \ge 0$ is a real-valued function defined for all $n, s \in \mathbb{N}$. Let the function f in (4.16.1) satisfy an inequality

$$|f(n, x(n), \sigma(n))| \le W(n, |x(n)|, |\sigma(n)|),\ \textit{for all } n \in \mathbb{N}, \tag{4.16.6}$$

where $\sigma(n)$ is as given in (4.14.2) and W is the same function as defined in Theorem 4.16.1. Then BIBO stability (asymptotic BIBO stability) of system (4.16.1)–(4.16.2) follows from the stability (asymptotic stability) of system (4.16.3) and the stability (asymptotic stability) of the equation (4.16.4) with $h(n) = |y(n, n_0, x_0)|$ where $y(n, n_0, x_0)$ is any solution of system (4.16.3).

Proof Using the variation of constants formula, any solution $x_\sigma(n, n_0, x_0)$ of problem (4.16.1)–(4.16.2) is represented by

$$x_\sigma(n, n_0, x_0) = y(n, n_0, x_0) + \sum_{s=n_0}^{n-1} Y(n)Y^{-1}(s+1)f(s, x(s, n_0, x_0), \sigma(s)). \tag{4.16.7}$$

Using (4.16.5) and (4.16.6) in (4.16.7) and applying Theorem 4.16.1, we see that the inequality

$$|x_\sigma(n, n_0, x_0)| \le r_h(n), \quad n \ge n_0, \tag{4.16.8}$$

holds provided for all $n \ge n_0$,

$$|\sigma(n)| \le r_h(n), \tag{4.16.9}$$

where $r_h(n)$ is a solution of Eq. (4.16.4) with $h(n) = |y(n, n_0, x_0)|$.

Since the system (4.16.3) is stable, we have for all $n \ge n_0$,

$$|y(n, n_0, x_0)| < c\delta,$$

whenever $|x_0| < \delta$.

Further Eq. (4.16.4) is stable so that we have $r_h(n) < c_1\delta$ for $h(n) < c\delta$. Thus,

$$|\sigma(n)| < c_1\delta$$

and

$$|x_\sigma(n, n_0, x_0)| < c_1\delta, \ \textit{for all } n \geq n_0$$

whenever $|x_0| < \delta$, i.e., system (4.16.1)–(4.16.2) is BIBO stable.

From the inequality (4.16.9) it follows that if instead of (4.16.3) and (4.16.4), we have the asymptotic stability, then the system (4.16.1)–(4.16.2) will be asymptotically BIBO stable. This completes the proof of the theorem. □

Necessary and sufficient conditions for BIBO stability of a different form of (4.16.1) have been obtained using entirely different techniques by Lin and Varaiya [362]. Specially, it is shown by these authors that a discrete time control system is BIBO stable if and only if there exist Lyapunov functions possessing certain properties. We note that Theorem 4.16.2 yields sufficient conditions for BIBO stability and asymptotic BIBO stability without using a Lyapunov technique.

The next theorem shows that under some suitable conditions on the functions involved in (4.16.1)–(4.16.2), any solution of (4.16.1)–(4.16.2) is exponentially asymptotically stable.

Theorem 4.16.3 ([475]) *Assume that the fundamental solution matrix $Y(n)$ of Eq. (4.16.3) verifies*

$$|Y(n)Y^{-1}(s)| \leq Me^{-\alpha(n-s)}, \ 0 \leq s \leq n < +\infty \tag{4.16.10}$$

where M and α are positive constants. Let the functions f and k in problem (4.16.1)–(4.16.2) satisfy

$$|f(n, x(n), \sigma(n))| \leq p(n)(|x(n)| + |\sigma(n)|), \ n \in \mathbb{N}, \tag{4.16.11}$$

$$|k(n, s, x(s))| \leq e^{-\alpha n} q(s)|x(s)|, \qquad n, s \in \mathbb{N}, \tag{4.16.12}$$

where $p(n)$ and $q(n)$ are real-valued functions defined for all $n \in \mathbb{N}$ and

$$\begin{aligned} B \geq \sum_{s=n_0}^{+\infty} e^{\alpha(1-n_0)} p(s)[|u(s)|e^{\alpha s} + M|x_0|e^{\alpha n_0} \prod_{t=n_0}^{s-1} Q(t) \\ + \sum_{t=n_0}^{s-1} Mp(t)e^{\alpha(t+1)}|u(t)| \prod_{\tau=t+1}^{s-1} Q(\tau)] \end{aligned} \tag{4.16.13}$$

where $Q(n) = (1 + Me^{\alpha} p(n) + e^{-\alpha n} q(n))$, $u(n)$ is the function given in (4.16.2) and $B > 0$ is a constant. Then any solution $x_\sigma(n, n_0, x_0)$ of problem (4.16.1)–(4.16.2) with $x(n_0) = x_0$ is exponentially asymptotically stable.

Proof By using the variation of constants formula, any solution $x_\sigma(n, n_0, x_0)$ of problem (4.16.1)–(4.16.2) is represented by

$$x_\sigma(n, n_0, x_0) = Y(n)Y^{-1}(n_0)x_0 + \sum_{s=n_0}^{n-1} Y(n)Y^{-1}(s+1) \times f(s, x_\sigma(s, n_0, x_0), \sigma(s)). \tag{4.16.14}$$

Using (4.16.10), (4.16.11) and (4.16.14), we obtain

$$|x_\sigma(n, n_0, x_0)| \leq |x_0| Me^{-\alpha(n-n_0)} + \sum_{s=n_0}^{n-1} Me^{-\alpha(n-s-1)} p(s) \times [|x_\sigma(s, n_0, x_0)| + |\sigma(s)|]. \tag{4.16.15}$$

Furthermore, using (4.16.2), (4.16.12) in (4.16.15), we have

$$\begin{aligned} |x_\sigma(n, n_0, x_0)| \leq |x_0| Me^{-\alpha(n-n_0)} &+ \sum_{s=n_0}^{n-1} Me^{-\alpha(n-1)} p(s) \\ &\times [|x_\sigma(s, n_0, x_0)| e^{\alpha s} + |u(s)| e^{\alpha s}] \\ &+ \sum_{s=n_0}^{n-1} Me^{-\alpha(n-1)} p(s) \left(\sum_{\tau=n_0}^{s-1} q(\tau) e^{-\alpha\tau} |x_\sigma(\tau, n_0, x_0)| e^{\alpha r} \right). \end{aligned}$$

Multiplying both sides of the above inequality by $e^{\alpha n}$, applying Theorem 2.1.35 with $x(n) = |x_\sigma(n, n_0, x_0)| e^{\alpha n}$, then multiplying by $e^{-\alpha n}$, we obtain

$$\begin{aligned} |x_\sigma(n, n_0, x_0)| \leq Me^{-\alpha(n-n_0)} \Bigg[|x_0| &+ \sum_{s=n_0}^{n-1} p(s) e^{\alpha(1-n_0)} \Bigg\{ |u(s)| e^{\alpha s} \\ &+ M(x) e^{\alpha n_0} \prod_{t=n_0}^{s-1} Q(t) + \sum_{t=n_0}^{s-1} Mp(t) e^{\alpha(t+1)} |u(t)| \prod_{\tau=t+1}^{s-1} Q(\tau) \Bigg\} \Bigg], \end{aligned}$$

which, in view of the assumption (4.16.13), implies

$$|x_\sigma(n, n_0, x_0)| \leq Me^{-\alpha(n-n_0)} (\delta + B),$$

i.e., for all $n \geq n_0$,

$$|x_\sigma(n, n_0, x_0)| \leq \delta' M e^{-\alpha(n-n_0)},$$

whenever $|x_0| < \delta$, where $\delta' = \delta + B$. This proves the assertion of the theorem. $\square$

Theorem 4.16.4 below demonstrates that, under some suitable conditions on the functions involved in problem (4.16.1)–(4.16.2), any solution of problem (4.16.1)–(4.16.2) is slowly growing.

Theorem 4.16.4 ([475]) *Assume that the fundamental solution matrix $Y(n)$ of problem (4.16.3) verifies*

$$|Y(n)Y^{-1}(s)| \leq M e^{\alpha(n-1)}, \ 0 \leq s \leq n < +\infty \tag{4.16.16}$$

where M and α are positive constants. Let the functions f and k in (4.16.1)–(4.16.2) satisfy

$$\begin{cases} |f(n, x(n), \sigma(n))| \leq p(n)(|x(n)| + |\sigma(n)|), \ n \in \mathbb{N} & (4.16.17) \\ |k(n, s, x(s))| \leq e^{\alpha n} q(s)|x(s)|, \ n, s \in \mathbb{N} & (4.16.18) \end{cases}$$

where $p(n)$ and $q(n)$ are real-valued functions defined for all $n \in \mathbb{N}$ and

$$\begin{aligned} B \geq \sum_{s=n_0}^{+\infty} e^{-\alpha(1-n_0)} p(s) \Bigg[& |u(s)|e^{-\alpha s} + M|x_0|e^{-\alpha n_0} \prod_{t=n_0}^{s-1} R(t) \\ & + \sum_{t=n_0}^{s-1} M p(t) e^{-\alpha(t+1)} |u(t)| \prod_{\tau=t+1}^{s-1} R(\tau) \Bigg] \end{aligned} \tag{4.16.19}$$

where $R(n) = (1 + Me^{-\alpha} p(n) + e^{\alpha n} q(n))$, $u(n)$ is the function given in (4.16.2) and $B > 0$ is a constant. Then any solution $x_\sigma(n, n_0, x_0)$ of problem (4.16.1)–(4.16.2) with $x(n_0) = x_0$ is uniformly slowly growing.

The proof of this theorem follows by the similar argument as in the proof of Theorem 4.16.3 with suitable modifications, and hence we omit the details.

Now we give an interesting example to illustrate Theorem 4.16.3.

Consider the nonlinear discrete time control system (4.16.1)–(4.16.2) as a perturbation of the linear discrete time system (4.16.3) with

$$A(n) = e^{-1} \begin{pmatrix} \cos(e^{n+1} - e^n) & \sin(e^{n+1} - e^n) \\ -\sin(e^{n+1} - e^n) & \cos(e^{n+1} - e^n) \end{pmatrix}$$

and the functions f and k in (4.16.1)–(4.16.2) satisfy the hypotheses (4.16.11), (4.16.12) and (4.16.13) of Theorem 4.16.3 with $\alpha = 1$. Then the fundamental

solution matrix of (4.16.3) satisfies

$$\begin{cases} Y(n) = e^{-n} \begin{pmatrix} \sin e^n & -\cos e^n \\ \cos e^n & \sin e^n \end{pmatrix}, & (4.16.20) \\ Y^{-1}(n) = e^n \begin{pmatrix} \sin e^n & \cos e^n \\ -\cos e^n & \sin e^n \end{pmatrix}. & (4.16.21) \end{cases}$$

We may easily verify the condition (4.16.7) of Theorem 4.16.3 with respect to (4.16.20) and (4.16.21), for $\alpha = 1$, and hence we have

$$|Y(n)Y^{-1}(s)| \le Me^{-(n-s)},\ 0 \le s \le n < +\infty,\ s \in \mathbb{N}, \tag{4.16.22}$$

where $M > 0$ is a constant.

It is well known that the solution $x_\sigma(n, n_0, x_0)$ of problem (4.16.1)–(4.16.2) is represented by Eq. (4.16.14). Using (4.16.14), (4.16.22), (4.16.11), (4.16.2), (4.16.12) and following the similar argument as in the proof of Theorem 4.16.3, we obtain for all $n \ge n_0$,

$$|x_\sigma(n, n_0, x_0)| < \delta' Me^{-(n-n_0)},$$

whenever $|x_0| < \delta$, where $\delta' = \delta + B$, and hence the conclusion of Theorem 4.16.3 follows.

4.17 An Application of Theorem 2.1.52 to High Order Difference Equations

In this section, we present some applications of discrete difference inequalities in Theorem 2.1.52 due to Agarwal and Thandapani [14].

First we now consider the $k + 1$th order difference equation

$$\Delta^{k+1}y(t) = f(t, y(t), \Delta y(t), \ldots, \Delta^k y(t)) \tag{4.17.1}$$

and show that Theorem 2.1.52 is directly applicable to find the upper estimates for the solutions of (4.17.1) provided that

$$|f(t, u_0, u_1, \ldots, u_k)| \le \sum_{j=0}^{k} h_j(t)|u_j|. \tag{4.17.2}$$

In fact, any solution of Eq. (4.17.1) also satisfies

$$\Delta^k y(t) = \Delta^k y(0) + \sum_{s=0}^{t-1} f(s, y(s), \Delta y(s), \ldots, \Delta^k y(s)),$$

or

$$|\Delta^k y(t)| \le |\Delta^k y(0)| + \sum_{j=0}^{k} \sum_{s=0}^{t-1} h_j(s)|\Delta^j y(s)|.$$

Hence from Theorem 2.1.51, it follows

$$|\Delta^k y(t)| \le |\Delta^k y(0)| + \sum_{s=0}^{t-1} \phi_1^*(s) \prod_{\tau=s+1}^{t-1} [1 + \phi_2^*(\tau)],$$

where ϕ_1^* and ϕ_2^* are the same as $\phi_1(t)$ and $\phi_2(t)$ with $p(t) = |\Delta^k y(0)|$ and $q(t) = 1$.

4.18 An Application of Theorem 2.1.57 to Difference Equation

In this section, we present an application of Theorem 2.1.57 to obtain the bound on the solution of a nonlinear sum-difference equation of the norm

$$u(n) = F(n) + \sum_{s=n+1}^{+\infty} B(n, s, u(s)), \tag{4.18.1}$$

where $u,\ F : \mathbb{N}_0 \to \mathbb{R}$, $B : \mathbb{N}_0^2 \times \mathbb{R} \to \mathbb{R}$ and

$$\begin{cases} |F(n)| \le a(n), & (4.18.2) \\ |B(n, s, u(s))| \le b(s)|u(s)|, & (4.18.3) \end{cases}$$

where $a(n)$ and $b(n)$ are as in Theorem 2.1.57. Let $u(n)$ be a solution of Eq. (4.18.1). From (4.18.1)–(4.18.3), we have

$$|u(n)| \le a(n) + \sum_{s=n+1}^{+\infty} b(s)|u(s)|. \tag{4.18.4}$$

Now an application of Theorem 2.1.57 yields

$$|u(n)| \le a(n) \prod_{s=n+1}^{+\infty} [1 + b(s)]. \tag{4.18.5}$$

Thus the right-hand side of (4.18.5) gives us the bound on the solution $u(n)$ of (4.18.1) in terms of the known functions.

4.19 Applications of Theorems 2.2.4–2.2.5 to Discrete Inequalities of Gronwall Type

In this section, we shall use Theorems 2.2.4–2.2.5 to study some discrete systems.

Example 4.19.1 We consider the following sum-difference system of Volterra type

$$u_1(t) = C_1 + \sum_{s=0}^{t-1} [F_1(t, s, u_1(s), u_2(s)) + K_1(u_1(s) + u_2(s))] \tag{4.19.1}$$

and

$$u_2(t) = C_2 + \sum_{s=0}^{t-1} [F_2(t, s, u_1(s), u_2(s)) + K_2(u_1(s) + u_2(s))] \tag{4.19.2}$$

where $C_1 \geq 4, C_2 \geq 4$, and the functions $F_1, F_2, K_1, and\ K_2$ satisfy for any $t \in \mathbb{R}$,

$$\begin{cases} |F_1(t, s, u_1(s), u_2(s))| \leq e_1(s)|u_1(s)| + e_2(s)|u_2(s)|, \\ |F_2(t, s, u_1(s), u_2(s))| \leq h_1(s)|u_1(s)| + h_2(s)|u_2(s)|, \\ |K_1(u_1(s), u_2(s))| \leq e_3(s)H(|u_1(s)|) + e_4(s)H(|u_2(s)|), \\ |K_2(u_1(s), u_2(s))| \leq h_3(s)H(|u_1(s)|) + h_4(s)H(|u_2(s)|). \end{cases} \tag{4.19.3}$$

Hence we get

$$\begin{aligned} |u_1(t)| \leq C_1 &+ \sum_{s=0}^{t-1} e_1(s)|u_1(s)| + \sum_{s=0}^{t-1} e_2(s)|u_2(s)| \\ &+ \sum_{s=0}^{t-1} e_3(s)H(|u_1(s)|) + \sum_{s=0}^{t-1} e_4(s)H(|u_2(s)|) \end{aligned} \tag{4.19.4}$$

and

$$\begin{aligned} |u_2(t)| \leq C_2 &+ \sum_{s=0}^{t-1} h_1(s)|u_1(s)| + \sum_{s=0}^{t-1} h_2(s)|u_2(s)| \\ &+ \sum_{s=0}^{t-1} h_3(s)H(|u_1(s)|) + \sum_{s=0}^{t-1} h_4(s)H(|u_2(s)|). \end{aligned} \tag{4.19.5}$$

The above two inequalities are exactly of the same form as (2.2.40) and (2.2.41) in Theorem 2.2.5, where $p_1 = p_2 = p_3 = p_4 = q_1 = q_2 = q_3 = q_4 = 1$. Thus we may us Theorem 2.2.5 to find the estimates for $|u_1(t)|$ and $|u_2(t)|$ in terms of known functions. □

Example 4.19.2 Consider the following system:

$$u_1(t) = C_3(t) + \sum_{s=0}^{t-1} k_1(u_1(s), u_2(s)) \tag{4.19.6}$$

and

$$u_2(t) = C_4(t) + \sum_{s=0}^{t-1} k_1(u_1(s), u_2(s)) \tag{4.19.7}$$

where

$$K_1(u_1, u_2) \leq H(|u_1|) + H(|u_2|).$$

Hence we get

$$|u_1(t)| \leq C_3(t) + \sum_{s=0}^{t-1} H(|u_1(s)|) + \sum_{s=0}^{t-1} H(|u_2(s)|) \tag{4.19.8}$$

and

$$|u_2(t)| \leq C_4(t) + \sum_{s=0}^{t-1} H(|u_1(s)|) + \sum_{s=0}^{t-1} H(|u_2(s)|). \tag{4.19.9}$$

The above two inequalities are exactly of the same form as (2.2.29) and (2.2.30) in Theorem 2.2.4 where $a_1 = C_3, a_2 = C_4$, and $p_1 = p_2 = q_1 = q_2 = 1$. From Theorem 2.2.4, we get

$$\begin{cases} \begin{cases} A(t) = H(C_3(t) - 2) = H(C_4(t) - 2), \\ B(t) = 4H(1) = Const., \\ \psi(t) = G^{-1}\left\{G(2) + \sum_{s=0}^{t-1}(4H(1) + H(C_3(s) - 2) + H(C_4(s) - 2))\right\}, \end{cases} \\ |u_1(t)| \leq C_3(t) + \sum_{s=0}^{t-1}\{H(C_3(s) - 2) + 4H(1)\psi(s) + H(C_4(s) - 2)\}, \\ |u_2(t)| \leq C_4(t) + \sum_{s=0}^{t-1}\{H(C_3(s) - 2) + 4H(1)\psi(s) + H(C_4(s) - 2)\} \end{cases} \tag{4.19.10}$$

and

$$G(t) = \int_{r_0}^{r} \frac{ds}{s + H(s)}, \quad 0 < r_0 \leq r. \tag{4.19.11}$$

4.20 An Application of Theorem 3.1.20 to the Bellman Equations

We note that many problems within stochastic control theory lead to a so-called Bellman equation which allows us in principle to compute the minimal expected cost corresponding to the application of an optimal control policy. In case of the optimal control of one-dimensional nonconservative quasi-diffusion processes (see [237], and for classical diffusion processes [383, Sect. VI. 3]), the Bellman equation has the form

$$\begin{cases} (D_m D_p^+)(x) + \min\limits_{z \in J}\{a(x,z)^{-1}[b(x,z)(D_p^+ v)(x-0) + c(x,z)]\} = 0, \\ x \in [a,b], \quad v(a) = \eta_0, \quad (D_p^+ v)(a) = \eta_1, \end{cases} \tag{4.20.1}$$

where $D_m D_p^+$ is Feller's generalized second order differential operator with a nondecreasing right continuous function m and a (strongly) isotone continuous function p (see, [208, 236, 383]), D_p^+ stands for the right derivation with respect to the function p, J is a compact set in $\mathbb{R}$, and $a > 0$, b, c are continuous functions on $[a,b] \times J$.

In Lemma 6 of [383], it was shown that the function

$$\Psi(x,y) = -\min_{z \in J}\{a(x,z)^{-1}[b(x,z)y + c(x,z)]\} = 0, \ x \in [a,b], \quad y \in \mathbb{R}$$

is continuous and satisfies the Lipschitz condition

$$|\Psi(x,y_1) - \Psi(x,y_2)| \leq L|y_1 - y_2|, \quad x \in [a,b], \quad y_1, y_2 \in \mathbb{R}$$

where

$$L = \max\{|b(x,z)|/a(x,z);\ x \in [a,b],\ z \in J\}.$$

Integrating (4.20.1), we obtain the equation

$$(D_p^+ v)(x) = \eta_1 + \int_a^x \Psi(s, (D_p^+ v)(s-0))dm(s), \quad x \in [a,b].$$

By Theorem 3.1.20, this equation has a unique solution z. Setting

$$v(x) = \eta_0 + \int_a^x z(s)dp(s), \quad x \in [a, b],$$

then we have solved in fact the Bellman equation (4.20.1).

4.21 An Application of Theorem 3.1.23 to Differential Equations with Distributions

In this section, we shall introduce an application of the Gronwall integral inequalities which is chosen from Rao [547].

Consider the differential equation

$$Dx = F(t, x)Du, \tag{4.21.1}$$

where Dx and Du denote the derivatives of the functions x and u, respectively, in the sense of distributions, and $F : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $\mathbb{R}^n$ the real n-vector space. Let $u : [0, +\infty) \to \mathbb{R}$ be a right continuous function of bounded variation on compact subintervals of $[0, +\infty)$.

Let S be an open connected set in $\mathbb{R}^n$ and let I be an interval with left endpoint $t_0 \geq 0$. A function $x(\cdot) = x(\cdot, t_0, x_0)$ is said to be a solution of Eq. (4.21.1) through (t_0, x_0) on the interval I if $x(\cdot)$ is a right continuous function of bounded variation in S $(i.e., \; x(\cdot) \in BV(I, S)), x(t_0) = x_0$, and the distributional derivative of $x(\cdot)$ on (t_0, α) for any arbitrary $\alpha \in I$ satisfies equation (4.21.1). Assume that for each $x(\cdot) \in BV(I, S), F(t, x(t))$ is integrable with respect to the Lebesgue-Stieltjes measure du. Then, as in [545], $x(\cdot)$ is a solution of Eq. (4.21.1) through (t_0, x_0) on $J = [t_0, t_0 + b]$ if and only if it satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^t F(s, x(s))du(s) \tag{4.21.2}$$

for all $t \in J$.

Now, as an application of Theorem 3.1.23, we shall prove the uniqueness of the solution of Eq. (4.21.1) under the hypothesis that F is Lipschitzian. If possible, let $x(t)$ and $y(t)$ be two solutions of Eq. (4.21.1) through the same point (t_0, x_0), i.e., $x(t_0) = x_0 = y(t_0)$. Let $Z(t) = |x(t) - y(t)|$. Clearly $Z(t_0) = 0$, and from (4.21.2), it follows that $Z(t) \leq L \int_{t_0}^t Z(s)dv_u(s)$, where v_u is the total variation function of u, and L is the Lipschitz constant. Now applying Theorem 3.1.23, with $C \equiv 0$ and $K(t) \equiv L$, we may complete the proof.

4.22 Applications of Theorems 3.2.1–3.2.3 to a Linear Gronwall's Inequality

In this section, we shall apply Theorems 3.2.1–3.2.3 to study a linear Gronwall's inequality. The result is due to Willett [647]. To apply Theorem 3.2.3, we consider the inequality

$$u(x) \leq t + \lambda^2 t \int_0^t e^{-\lambda s} u(s) ds + \int_0^t u(s) ds, \qquad \textit{for all } t > 0,$$

where λ is a real parameter and the problem is to determine the asymptotic behavior of u as $\lambda \to +\infty$, in particular, to prove that $u = O(1)$ uniformly for t restricted to compact subintervals of $\mathbb{R}_+ \equiv [0, +\infty)$. We have

$$k(t, s) = \lambda^2 t e^{-\lambda s} + 1,$$

hence, $\partial k(t, s)/\partial t = \lambda^2 e^{-\lambda s}$. The application of Theorem 3.2.3 is straightforward and leads to the desired result. Indeed, we could have also put $v_1(t) = w_1(t) = 1$, $w_2(t) = t$ and $v_2(t) = \lambda^2 e^{-\lambda t}$ and successfully applied Theorem 3.2.2. On the other hand, the direct application of Theorem 3.2.1 in the obvious fashion,

$$k(t, s) \leq \max(1, \lambda^2 t) e^{-\lambda s}$$

or

$$k(t, s) \leq (1 + \lambda^2 t) \max(1, e^{-\lambda s}),$$

does not yield that u must be bounded as $\lambda \to +\infty$.

Chapter 5
Linear Multi-Dimensional Continuous Integral Inequalities

5.1 Linear Two-Dimensional Continuous Gronwall-Bellman Integral Inequalities

5.1.1 Linear Two-Dimensional Continuous Gronwall-Bellman Integral Inequalities and Their Generalizations

Gronwall's one dimensional inequality (Theorem 1.1.1) [239], also known in a generalized form as Bellman's lemma [61], has been extended to several independent variables by different authors. For example, in [140] Conlan and Diaz obtained a generalization of Gronwall's inequality in n variables in order to prove uniqueness of solutions of a nonlinear partial differential equation. Walter [636] gave a more natural extension of Gronwall's inequality in any number of variables by using the properties of monotone operator. By using the notion of a Riemann function, Snow [603] obtained corresponding inequalities in two independent variables for scalar and vector functions. It turns out to be that Snow's technique in the scalar case can be employed to establish Gronwall's inequality in n independent variables which coincides with the result given in [636] when a representation of the Riemann function is used.

Integral inequalities originally due to Peano and Gronwall, and their various generalizations [51, 82, 351] have been extensively used in obtaining a priori bounds for solutions of differential and integral equations. An interesting and useful but apparently neglected generalization of Gronwall's inequality in two independent variables is due to Wendroff [47]. Wendroff's inequality which has its origin in the field of partial differential equations has recently evoked lively interest as may be seen from Snow [494, 495], Young [677], Ghoshal and Masood [228], Headley [262], Chandra and Davis [128], Bondge and Pachpatte [90] and Pachpatte

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[477, 479], see also the monograph Walter [637], which are motivated by certain applications in the theory of hyperbolic partial differential and integrodifferential equations.

Two independent variables Gronwall type inequalities of considerable interest are associated with the names of Wendroff [47], Snow [603, 604], Ghoshal and Masood [227, 228], Young [677], Chandra and Davis [128], and Bondge and Pachpatte [90], which were motivated by some applications in the theory of partial differential and integral equations.

Inequalities similar to Theorem 1.1.1 but involving functions of several variables, which are originally due to Wendroff, may be found in Beckenbach and Bellman [47]. There are a number of ways to extend the above result. The following are some unpublished inequalities due to Wendroff, see also the book [47] of Beckenach and Bellman.

Lemma 5.1.1 (Kasture-Deo [312]) *If*

$$u(x,y) \leq a + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)u(s,t)dsdt, \tag{5.1.1}$$

where a is a non-negative constant for all $x \geq x_0,\ y \geq y_0$, and $k(s,t),\ u(s,t) \geq 0$, then for all $x \geq x_0,\ y \geq y_0$,

$$u(x,y) \leq a\exp\Big(\int_{x_0}^{x}\int_{y_0}^{y} k(s,t)dsdt\Big). \tag{5.1.2}$$

Proof Denote the right-hand side of (5.1.1) by v. Multiplying (5.1.1) by $k(x,y)$, rearranging, and using $v_{xy} = ku$, we obtain

$$v_{xy}/v \leq k(x,y) \leq k(x,y) + (v_x v_y/v^2).$$

Hence

$$(\partial/\partial y)(v_x/v) \leq k(x,y). \tag{5.1.3}$$

Integrating (5.1.3) with respect to y from y_0 to y, and then with respect to x from x_0 to x and using $v_x(x,0) = 0, v(0,y) = a$, we may obtain (5.1.2). □

Theorem 5.1.1 (The Wendroff Inequality [47]) *Let $u(x,y), a(x,y), k(x,y)$ be non-negative continuous functions for all $x \geq x_0, y \geq y_0$, and let $a(x,y)$ be non-decreasing in each of the variables for all $x \geq x_0, y \geq y_0$. Suppose that for all $x \geq x_0, y \geq y_0$,*

$$u(x,y) \leq a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} k(s,t)u(s,t)dsdt. \tag{5.1.4}$$

Then for all $x \geq x_0, y \geq y_0$,

$$u(x, y) \leq a(x, y) \exp\left(\int_{x_0}^{x}\int_{y_0}^{y} k(s, t)dsdt\right). \tag{5.1.5}$$

Proof Let $X > x_0$ and $Y > y_0$ be fixed. Then for all $x_0 \leq x \leq X, y_0 \leq y \leq Y$, we have

$$u(x, y) \leq a(X, Y) + \int_{x_0}^{x}\left(\int_{y_0}^{y} k(s, t)u(s, t)dt\right) ds \equiv v(x, y). \tag{5.1.6}$$

The function $v(x, y)$ is non-decreasing in each variable x, y, and satisfies, since $u(x, t) \leq v(x, t) \leq v(x, y)$,

$$\begin{cases} v(x_0, y_0) = a(X, Y), \\ \dfrac{\partial v}{\partial x}(x, y) = \displaystyle\int_{y_0}^{y} k(x, t)u(x, t)dt \leq \int_{y_0}^{y} k(x, t)dt\, v(x, y), \\ v(x, y) \leq a(X, Y) + \displaystyle\int_{x_0}^{x}\int_{y_0}^{y} k(\xi, t)v(\xi, y)dtd\xi. \end{cases} \tag{5.1.7}$$

By Lemma 5.1.1, this implies, for all $x_0 \leq x \leq X,\ y_0 \leq y \leq Y$,

$$v(x, y) \leq a(X, Y) \exp\left(\int_{x_0}^{x}\left(\int_{y_0}^{y} k(s, t)dt\right) ds\right). \tag{5.1.8}$$

Setting $x = X$ and $y = Y$ and changing notation, we arrive at (5.1.5). □

Corollary 5.1.1 (Wendroff [47]) *(i) If for all* $x \geq 0, y \geq 0$,

$$u(x, y) \leq a(x) + b(y) + \int_{y_0}^{y}\int_{x_0}^{x} k(r, s)u(r, s)drds, \tag{5.1.9}$$

where $a(x), b(y) > 0, a'(x), b'(y) \geq 0, u, k \geq 0$*, then*

$$u(x, y) \leq [a(x) + b(y)] \exp\left(\int_{y_0}^{y}\int_{x_0}^{x} k(r, s)drds\right). \tag{5.1.10}$$

In particular, for $x_0 = 0, y_0 = 0$ *and for all* $x \geq 0, y \geq 0$,

$$\begin{aligned} u(x, y) &\leq [a(x) + b(y)] + \int_{0}^{y}\int_{0}^{x} k(r, s)u(r, s)drds \\ &\leq \frac{[a(0) + b(y)][a(x) + b(0)]}{a(0) + b(0)} \exp\left(\int_{0}^{y}\int_{0}^{x} k(r, s)drds\right). \end{aligned} \tag{5.1.11}$$

(ii) If for all $x \geq 0, y \geq 0$,

$$u(x, y) \leq C + a \int_0^x u(r, y)dr + b \int_0^y u(x, s)ds, \tag{5.1.12}$$

with a constant $C \geq 0$, then for all $x \geq 0, y \geq 0$,

$$u(x, y) \leq C \exp\Big(ax + by + abxy\Big). \tag{5.1.13}$$

(iii) If for all $x \geq 0, y \geq 0$,

$$u(x, y) \leq a(x) + b(y) + a \int_0^x u(r, y)dr + b \int_0^y u(x, s)ds, \tag{5.1.14}$$

then for all $x \geq 0, y \geq 0$,

$$u(x, y) \leq Q(x, y) \tag{5.1.15}$$

where

$$Q(x, y) = \frac{[a(0) + b(0) + \int_0^y e^{-by_1} b'(y_1)dy_1][a(0) + b(0) + \int_0^x e^{-ax_1} a'(x_1)dx_1]e^{ax+by+abxy}}{a(0) + b(0)}. \tag{5.1.16}$$

The next result is to establish a useful and sharper general version of the Wendroff's inequality in Theorem 5.1.1 and the two independent variable generalization of the integral inequality established by Pachpatte [445].

Theorem 5.1.2 (Pachpatte-Walter [445]) *Let $c,\ v,\ g,\ g_x,\ g_y,\ g_{xy} \in C(R)$, where R is the first quadrant $x \geq 0,\ y \geq 0$ or a rectangle $0 \leq x \leq a,\ 0 \leq y \leq b$, and $c \geq 0,\ g_x \geq 0,\ g_y \geq 0,\ g_{xy} \geq 0$, on R, and the following inequality holds, for all $(x, y) \in R$,*

$$v(x, y) \leq g(x, y) + \int_0^x \int_0^y c(s, t)v(s, t)dsdt. \tag{5.1.17}$$

Then for all $(x, y) \in R$,

$$v(x, y) \leq g(x, y)e^{C(x,y)},\quad C(x, y) = \int_0^x \int_0^y c(s, t)dsdt. \tag{5.1.18}$$

Proof Let

$$Du = u_{xy},\ J(u) = \int_0^x \int_0^y u(s, t)dsdt,\ u^0(x, y) = u(x, 0) + u(0, y) - u(0, 0).$$

Then

$$u = u^0 + J(u), \quad u = D(J(u)).$$

Obviously, if v and w satisfy (5.1.17) and

$$w(x, y) \geq g(x, y) + \int_0^x \int_0^y c(s, t)w(s, t)dsdt \tag{5.1.19}$$

respectively; then $v \leq w$ in R. Now, let w be the function in (5.1.18), $w = ge^C, \quad C = J(c)$. Since $w^0 = g^0$ and

$$Dw = (Dg)e^C + cge^C + \{non - negative\ terms\}$$

implies

$$Dw \geq cw + Dg,$$

we get

$$w = w^0 + J(Dw) \geq g^0 + J(Dg + cw) = g + J(cw),$$

which is equivalent to (5.1.18). □

Remark 5.1.1 We note that the bound obtained in (5.1.18) is sharper than the bound obtained in Wendroff's inequality in Corollary 5.1.1 where g is given by $g = a(x) + b(y)$, since no assumption on the sign of g is made, and in Wendroff's bound, the term e^C is multiplied by

$$\frac{[a(0) + b(y)][a(x) + b(0)]}{a(0) + b(0)} \geq a(x) + b(y) = g.$$

For example, if $a(s) = b(s) = 1 + s$, then

$$\frac{(2 + y)(2 + x)}{2} = 2 + x + y + \frac{1}{2}xy > 2 + x + y.$$

Remark 5.1.2 The proof of Theorem 5.1.2 can be used to prove the n-dimensional case; see [10, p. 148]. In this case, we have to assume that

$$g_{x_1}, \ g_{x_1x_2}, \ g_{x_2x_3}, \cdots, \ Dg = g_{x_1\ldots x_n} \geq 0.$$

This remark generalizes Corollary 5.1.1 and Theorem 5.1.2 in several directions.

In particular, in the bound given in Eq. (5.1.11), the first term on the right-hand side can be replaced by the smaller term $a(x) + b(y)$.

We next establish the following two independent variable generalizations of the integral inequalities established by Pachpatte [445].

Theorem 5.1.3 (Pachpatte [445]) *Let* v, p, q, g, g_x, g_y, $g_{xy} \in C(R)$ *and* p, q, g_x, g_y, $g_{xy} \geq 0$ *on* R*, satisfy on* R*,*

$$v \leq g + J(pv) + J(pJ(qv)). \tag{5.1.20}$$

Then on R*,*

$$v \leq g + J(pge^{P+Q}),\ P = J(p),\ Q = J(q). \tag{5.1.21}$$

Proof The function $\Phi = e^{P+Q}$ satisfies (see the proof of Theorem 5.1.2)

$$\Phi = g + J((p+q)\Phi),$$

which, by repeated application, gives us

$$\begin{aligned}\Phi &\geq g + J[(p+q)(g + J((p+q)\Phi))]\\ &\geq g + J(p\Phi) + J(q(g + J(p\Phi)))\end{aligned} \tag{5.1.22}$$

or, by putting $w = g + J(p\Phi)$,

$$\Phi \geq w + J(qw).$$

If this inequality is multiplied by p, then J applied, then g added, the inequality

$$w \geq g + J(qw) + J(pJ(qw)) \tag{5.1.23}$$

follows. Inequalities (5.1.20) and (5.1.23) imply $v \leq w$ (see, e.g., [637, p. 130]), which is (5.1.21). □

Remark 5.1.3 We note that, Theorem 5.1.3 is formulated in two dimensions, the extension to dimension n is immediate.

In 1971, Nurimov [434] established the next result.

Theorem 5.1.4 (Nurimov [434]) *Assume that* $u(x, y), k(x, y), a(x, y), b(x, y)$ *are non-negative and continuous on* $D = [0, x_0] \times [0, y_0]$*. If for any* $(x, y) \in D$*, the following inequality holds*

$$u(x, y) \leq a(x, y) + b(x, y) \int_0^x \int_0^y k(t, s)u(t, s)dtds, \tag{5.1.24}$$

then for all $(x, y) \in D$,

$$u(x, y) \leq a(x, y) + b(x, y) \int_0^x \int_0^y \exp\left\{\int_t^x \int_s^y k(r, z)b(r, z)drdz\right\} a(t, s)k(t, s)dtds. \tag{5.1.25}$$

Proof The proof is similar to that of Theorem 5.1.1. □

By a reasoning similar to the proof of Theorem 5.1.2, we can easily prove the following two results.

Theorem 5.1.5 (Nurimov [434]) *Let* $u(x, y), a(x, y), k(x, y)$ *be non-negative continuous functions in* $\mathbb{R}^2_+$, *and let* $a(x, y)$ *be non-decreasing in each of the variables* x, y. *Suppose that for all* $x \geq 0,\ y \geq 0$,

$$u(x, y) \leq a(x, y) + \int_x^{+\infty} \int_y^{+\infty} k(s, t)u(s, t)dsdt, \tag{5.1.26}$$

and for all $x \geq 0,\ \ y \geq 0$,

$$\int_x^{+\infty} \int_y^{+\infty} k(s, t)dsdt < +\infty. \tag{5.1.27}$$

Then for all $x \geq 0, y \geq 0$,

$$u(x, y) \leq a(x, y) \exp\left(\int_x^{+\infty} \int_y^{+\infty} k(s, t)dsdt\right). \tag{5.1.28}$$

Theorem 5.1.6 (Nurimov [434]) *Let* $u(x, y), a(x, y), k(x, y)$ *be non-negative continuous functions in* $\mathbb{R}^2_+$, *and let* $a(x, y)$ *be non-decreasing in x, and non-increasing in y. Suppose that for all* $x \geq 0,\ y \geq 0$,

$$u(x, y) \leq a(x, y) + \int_0^x \int_y^{+\infty} k(s, t)u(s, t)dsdt, \tag{5.1.29}$$

and for all $x \geq 0,\ \ y \geq 0$,

$$\int_0^x \int_y^{+\infty} k(s, t)dsdt < +\infty. \tag{5.1.30}$$

Then for all $x \geq 0, y \geq 0$,

$$u(x, y) \leq a(x, y) \exp\left(\int_0^x \int_y^{+\infty} k(s, t)dsdt\right). \tag{5.1.31}$$

Proof The proof of Theorem 5.1.6 is carried out in terms of the following scheme.

(1) By a single integration with respect to one of the variables (say, x), we reduce the integral inequality in question to a differential inequality in which the other variable (in this case y) is treated as a constant.
(2) For the differential inequality obtained above, we apply some comparison results and arrive at the required result. We omit the detail of the proof. □

Next, we shall establish some useful integral inequalities involving functions of two independent variables which can be used as powerful tools in the analysis of certain classes of partial differential and finite difference equations.

In what follows, $\mathbb{R}_+ = [0, +\infty), \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ are the given subsets of $\mathbb{R}$. The first order partial derivatives of a function $z(x, y)$ defined for all $x, y \in \mathbb{R}$ with respect to x and y are denoted by $z_x(x, y)$ and $z_y(x, y)$ respectively. We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively. All the functions which appear in the inequalities are assumed to be real-valued and all the integrals, sums and products involved exist on the respective domains of their definitions.

Theorem 5.1.7 (Pachpatte [498]) *Let $u(x, y)$, $a(x, y)$, $b(x, y)$, $c(x, y)$ be non-negative continuous functions defined for all $x, y \in \mathbb{R}_+$.*

(a_1) *If for all $x, y \in \mathbb{R}_+$,*

$$u(x, y) \le a(x, y) + b(x, y) \int_0^x \int_y^{+\infty} c(s, t)u(s, t)dtds, \tag{5.1.32}$$

then for all $x, y \in \mathbb{R}_+$,

$$u(x, y) \le a(x, y) + b(x, y)e(x, y)\Big(\int_0^x \int_y^{+\infty} c(s, t)b(s, t)dtds\Big), \tag{5.1.33}$$

where for all $x, y \in \mathbb{R}_+$,

$$e(x, y) = \int_0^x \int_y^{+\infty} c(s, t)a(s, t)dtds, \tag{5.1.34}$$

(a_2) *If for all $x, y \in \mathbb{R}_+$,*

$$u(x, y) \le a(x, y) + b(x, y) \int_x^{+\infty} \int_y^{+\infty} c(s, t)u(s, t)dtds, \tag{5.1.35}$$

then for all $x, y \in \mathbb{R}_+$,

$$u(x, y) \le a(x, y) + b(x, y)\bar{e}(x, y) \exp\Big(\int_x^{+\infty} \int_y^{+\infty} c(s, t)b(s, t)dtds\Big), \tag{5.1.36}$$

where for all $x, y \in \mathbb{R}_+$*,*

$$\bar{e}(x, y) = \int_x^{+\infty} \int_y^{+\infty} c(s, t)a(s, t)dtds. \tag{5.1.37}$$

Proof Since the proofs resemble one another, we only give the details of the proof for (a_1), the rest of proofs can be similarly done by following the proofs of the above mentioned results with suitable changes.

Define

$$z(x, y) = \int_0^x \int_y^{+\infty} c(s, t)u(s, t)dtds. \tag{5.1.38}$$

Then (5.1.32) can be restated as

$$u(x, y) \leq a(x, y) + b(x, y)z(x, y). \tag{5.1.39}$$

From (5.1.38) and (5.1.39) it follows

$$\begin{aligned} z(x, y) &\leq \int_0^x \int_y^{+\infty} c(s, t)\big[a(s, t) + b(s, t)z(s, t)\big]dtds \\ &= e(x, y) + \int_0^{+\infty} \int_y^{+\infty} c(s, t)b(s, t)z(s, t)dtds, \end{aligned} \tag{5.1.40}$$

where $e(x, y)$ is defined by (5.1.34). Clearly, $e(x, y)$ is non-negative, continuous, non-decreasing in x and non-increasing in y for all $x, y \in \mathbb{R}_+$. First we assume that $e(x, y) > 0$ for all $x, y \in \mathbb{R}_+$. From (5.1.40) it follows that

$$\frac{z(x, y)}{e(x, y)} \leq 1 + \int_0^x \int_y^{+\infty} c(s, t)b(s, t)\frac{z(s, t)}{e(s, t)}dtds. \tag{5.1.41}$$

Define a function $v(x, y)$ by the right-hand side of (5.1.41), then $v(0, y) = v(x, +\infty) = 1$, $\frac{z(x,.y)}{e(x,y)} \leq v(x, y)$, $v(x, y)$ is non-increasing in y, $y \in \mathbb{R}_+$ and

$$\begin{aligned} v_x(x, y) &= \int_y^{+\infty} c(x, t)b(x, t)\frac{z(x, t)}{e(x, t)}dt \\ &\leq \int_y^{+\infty} c(x, t)b(x, t)v(x, t)dt \\ &\leq v(x, y)\int_y^{+\infty} c(x, t)b(x, t)dt. \end{aligned} \tag{5.1.42}$$

Now, fixing $y \in \mathbb{R}_+$ in (5.1.42), dividing both sides of (5.1.42) by $v(x, y)$, setting $x = s$ and integrating the resulting inequality from 0 to x, $x \in \mathbb{R}_+$, we get

$$v(x, y) \leq \exp\Big(\int_0^x \int_y^{+\infty} c(s, t)b(s, t)dtds\Big). \tag{5.1.43}$$

Using (5.1.43) in $\frac{z(x,y)}{e(x,y)} \leq v(x, y)$, we have

$$z(x, y) \leq e(x, y) \exp\Big(\int_0^x \int_y^{+\infty} c(s, t)b(s, t)dtds\Big). \tag{5.1.44}$$

Thus the desired inequality (5.1.33) follows from (5.1.39) and (5.1.44). If $e(x, y)$ is non-negative, we carry out the above procedure with $e(x, y) + \epsilon$ instead of $e(x, y)$, where $\epsilon > 0$ is an arbitrary small constant, and then subsequently pass to the limit as $\epsilon \to 0$ to obtain (5.1.33). □

Theorem 5.1.8 (Pachpatte [498]) *Let $u(x, y)$, $a(x, y)$, $b(x, y)$, $c(x, y)$ be non-negative continuous functions defined for all $x, y \in \mathbb{R}_+$.*

(b_1) Assume that $a(x, y)$ is non-decreasing in $x \in \mathbb{R}_+$. If for all $x, y \in \mathbb{R}_+$,

$$u(x, y) \leq a(x, y) + \int_0^x b(s, y)u(s, y)ds + \int_0^x \int_y^{+\infty} c(s, t)u(s, t)dtds, \tag{5.1.45}$$

then for all $x, y \in \mathbb{R}_+$,

$$u(x, y) \leq p(x, y)\Big[a(x, y) + A(x, y) \exp\Big(\int_0^x \int_y^{+\infty} c(s, t)p(s, t)dtds\Big)\Big], \tag{5.1.46}$$

where for all $x, y \in \mathbb{R}_+$,

$$\begin{cases} p(x, y) = \exp\Big(\int_0^x b(s, y)ds\Big), & (5.1.47) \\ A(x, y) = \int_0^x \int_y^{+\infty} c(s, t)p(s, t)a(s, t)dtds. & (5.1.48) \end{cases}$$

(b_2) Assume that $a(x, y)$ is non-increasing in $x \in \mathbb{R}_+$. If for all $x, y \in \mathbb{R}_+$,

$$u(x, y) \leq a(x, y) + \int_x^{+\infty} b(s, y)u(s, y)ds + \int_x^{+\infty} \int_y^{+\infty} c(s, t)u(s, t)dtds, \tag{5.1.49}$$

then for all $x, y \in \mathbb{R}_+$,

$$u(x, y) \leq \bar{p}\Big[a(x, y) + \bar{A}(x, y) \exp\Big(\int_x^{+\infty} \int_y^{+\infty} c(s, t)\bar{p}(s, t)dtds\Big)\Big], \tag{5.1.50}$$

where for all $x, y \in \mathbb{R}_+$,

$$\begin{cases} \bar{p}(x,y) = \exp\Big(\int_x^{+\infty} b(s,y)ds\Big), & (5.1.51)\\ \bar{A}(x,y) = \int_x^{+\infty}\int_y^{+\infty} c(s,t)\bar{p}(s,t)a(s,t)dtds. & (5.1.52)\end{cases}$$

Proof Since the proof resembles one another, we only give the detail of the proof for (b_1). The proof of (b_2) can be done in the same way as that of (b_1).

Define

$$z(x,y) = \int_0^x \int_y^{+\infty} c(s,t)u(s,t)dtds. \tag{5.1.53}$$

Then (5.1.45) can be restated as

$$u(x,y) \le z(x,y) + \int_0^x b(s,y)u(s,y)ds. \tag{5.1.54}$$

Clearly, $z(x,y)$ is a non-negative, continuous and non-decreasing function in $x, x \in \mathbb{R}_+$. Fixing $y \in \mathbb{R}_+$ in (5.1.54) and using Theorem 1.1.4 to (5.1.54), we get

$$u(x,y) \le z(x,y)p(x,y), \tag{5.1.55}$$

where $p(x,y)$ is defined by (5.1.47). From (5.1.53) and (5.1.55), it follows

$$u(x,y) \le p(x,y)[a(x,y) + v(x,y)], \tag{5.1.56}$$

where

$$v(x,y) = \int_0^x \int_y^{+\infty} c(s,t)u(s,t)dtds. \tag{5.1.57}$$

From (5.1.56) and (5.1.57), we derive

$$\begin{aligned} z(x,y) &\le \int_0^x \int_y^{+\infty} c(s,t)p(s,t)[a(s,t) + v(s,t)]dtds \\ &= A(x,y) + \int_0^x \int_y^{+\infty} c(s,t)p(s,t)v(s,t)dtds, \end{aligned} \tag{5.1.58}$$

where $A(x,y)$ is defined by (5.1.48). Clearly, $A(x,y)$ is non-negative, continuous, non-decreasing in $x, x \in \mathbb{R}_+$ and non-increasing in $y, y \in \mathbb{R}_+$. Now, similarly as the

above argument, we conclude

$$v(x, y) \leq A(x, y) \exp\left(\int_0^x \int_y^{+\infty} c(s, t)p(s, t)dtds\right). \tag{5.1.59}$$

Using (5.1.59) in (5.1.56), we get the required inequality in (5.1.46). □

The following result is the two independent variable version of the inequality used as tools in the study of terminal value problems for certain hyperbolic partial differential equations.

Theorem 5.1.9 (Pachpatte-Pachpatte [510]) *Let $u(x, y), a(x, y), b(x, y)$ be real-valued non-negative continuous functions defined for all $x, y \in \mathbb{R}_+$ and suppose that $a(x, y)$ is non-increasing in all $x, y \in \mathbb{R}_+$. If for all $x, y \in \mathbb{R}_+$,*

$$u(x, y) \leq a(x, y) + \int_x^{+\infty} \int_y^{+\infty} b(s, t)u(s, t)dtds, \tag{5.1.60}$$

then for all $x, y \in \mathbb{R}_+$,

$$u(x, y) \leq a(x, y)\exp\Big(\int_x^{+\infty} \int_y^{+\infty} b(s, t)dtds\Big). \tag{5.1.61}$$

Proof First we assume that $a(x, y) > 0$ for all $x, y \in \mathbb{R}_+$. From (5.1.60) it follows that

$$\frac{u(x, y)}{a(x, y)} \leq 1 + \int_x^{+\infty} \int_y^{+\infty} b(s, t)\frac{u(s, t)}{a(s, t)}dtds. \tag{5.1.62}$$

Define a function $z(x, y)$ by the right-hand side of (5.1.62). Then $z(x, +\infty) = z(+\infty, y) = 1$, $\frac{u(x,y)}{a(x,y)} \leq z(x, y)$ and $z_{x,y} = b(x, y)\frac{u(x,y)}{a(x,y)} \leq b(x, y)z(x, y)$.

The rest of the proof can be completed by following the proof of Theorem 4.2.1 given in [495] with suitable changes and closely looking at the proof of Theorem 5.1.4 given above. □

5.1.2 *Linear Two-Dimensional Continuous Generalization of Gronwall-Bellman Integral Inequalities*

The next result presents a generalization for systems of partial differential equations of Gronwall's classical integral inequality (i.e., Theorem 1.1.1) for ordinary differential equations. The proof is by reducing the vector integral inequality to a vector partial differential inequality and using a vector generalization of Riemann's

method to obtain the final inequality. The final inequality involves a matrix function in the integrand which is a generalization of the scalar Riemann function. The proof includes a successive approximations argument to guarantee the existence and positivity property of this matrix function.

The main result is obtained by reducing the vector integral inequality to a vector differential inequality and then integrating it by generalizing Riemann's method to apply to vector hyperbolic PDE's. As in [603], the method of proof shows the right-hand side of the final inequality is the corresponding vector Volterra equations and hence is the maximal solution of the original inequality. This method can be used to solve the corresponding type of vector Volterra integral equations.

In 1972, Snow [604, 621] gave one of the Gronwall-Bellman integral inequalities involving two independent variables. His technique was to reduce the original inequality to a second-order partial differential inequality, which is then integrated by using Riemann's method.

The function $V(s,t;x,y)$ in the next theorem is a matrix generalization of a Riemann function relative to the point $P(x,y)$ for the self-adjoint operator L. There is such a function and region D^+ on which $V \geq 0$ as proven in the following theorem.

Theorem 5.1.10 (The Snow Inequality [604]) *Let $B(s,t)$ be a continuous matrix function. Then the matrix characteristic initial value problem*

$$\begin{cases} L[V] = V_{st} - B(s,t)V = 0 & (5.1.63) \\ V(s,y) = V(x,t) = I, & (5.1.64) \end{cases}$$

where I is the identity matrix, has a unique solution $V(s,t;x,y)$ for s and t near to $P(x,y)$ and satisfying $(s-x)(t-y) \geq 0$. This solution is continuous and if $B(s,t)$ is non-negative, so is $V(s,t;x,y)$.

Proof The proof is by a successive approximation's argument. It is easily seen that the integral equation

$$V(s,t) = I + \int_x^s \int_y^t V(\xi,\eta)B(\xi,\eta)d\xi d\eta \tag{5.1.65}$$

is equivalent to Eq. (5.1.63) with conditions (5.1.64). Let T represent the transformation

$$TV = \int_x^s \int_y^t V(\xi,\eta)B(\xi,\eta)d\xi d\eta,$$

so Eq. (5.1.65) is $V = I + TV$.

Let $V_0(s,t) \equiv I$ and define $V_{n+1} = I + TV_n$. Since TV is continuous if V is continuous, it follows immediately by induction that V_n is defined and continuous

for all n. Let $||\cdot||$ be any matrix norm. Then since $(s-x)(t-y)\geq 0$, we have

$$\begin{aligned}||TV|| &= ||\int_x^s\int_y^t V(\xi,\eta)B(\xi,\eta)d\xi d\eta|| \\ &\leq \int_x^s\int_y^t ||V(\xi,\eta)||\cdot||B(\xi,\eta)||d\xi d\eta \\ &\leq \frac{1}{2}\max||V(\xi,\eta)||, \end{aligned} \tag{5.1.66}$$

if we restrict s and t to be close enough to (x,y).

Then

$$\begin{aligned}||V_{k+1}-V_k|| = ||T(V_k-V_{k-1})|| &\leq \frac{1}{2}\max||V_k-V_{k-1}|| \\ &\leq \cdots \leq 2^{-k}\max||V_1-V_0||.\end{aligned}$$

Since $V_{n+1} = V_0+\sum_{k=0}^{n}(V_{k+1}-V_k)$, V_{n+1} is the n-th partial sum of a matrix series dominated in norm by a convergent series, namely, $\max||V_1-V_0||\sum_{k=0}^{n}2^{-k}$.

Therefore the matrix sequence $\{V_n\}$ converges uniformly on the domain where (5.1.66) holds. Since each V_n is continuous, the limit function $V(s,t;x,y)$ is also. To see that V is a solution to (5.1.65), note that $I+TV = I+T(\lim V_n) = I+\lim TV_n = \lim(I+TV_n) = \lim V_{n+1} = V$. To see that V is unique, suppose W is also a solution. Then $V-W=T(V-W)$ so

$$||V-W|| = ||T(V-W)|| \leq \frac{1}{2}\max||V-W||$$

which is possible only if $||V-W||=0$; i.e., $V\equiv W$.

Now suppose $B(s,t)\geq 0$. Then, if $V\geq 0$, $TV\geq 0$, since $(s-x)(t-y)\geq 0$. Since $V_0\equiv I\geq 0$, it follows by induction that $V_n\geq 0$ for all n. But then the limit function satisfies $V(s,t;x,y)\geq 0$ also. □

The next theorem is a general form of Theorem 5.1.10.

Theorem 5.1.11 (Snow [603]) *Suppose (i) $\phi(x,y)$ and $a(x,y)$ are continuous n-vector functions on a domain D and $B(x,y)$ is a continuous symmetric, non-negative (i.e., $b_{ij}\geq 0$, all i and j), $n\times n$ matrix function on D. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two points in D such that $(x-x_0)(y-y_0)\geq 0$ and let R be the rectangular region whose diagonal is the line joining the points P_0 and P. (ii) Let $V(s,t;x,y)$ be the $n\times n$ matrix function satisfying the matrix characteristic initial value problem*

$$L[V] = V_{st}-B(s,t)V = 0, \quad V(s,y) = V(x,t) = I, \tag{5.1.67}$$

where I is the identity matrix. (iii) Let D^+ be the connected sub-domain of D containing P and on which $V(s,t;x,y)$ is non-negative. (See Fig. 5.1 and

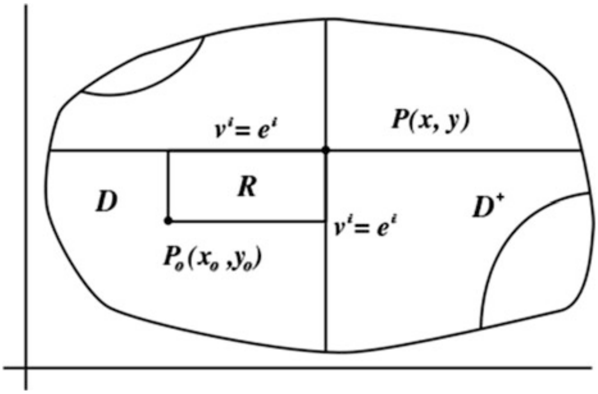

Fig. 5.1 Diagram showing the regions involved

Theorem 5.1.10.) Then if (iv) $R \subset D^+$ and $\phi(x, y)$

$$\phi(x, y) \leq a(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} B(s, t)\phi(s, t)dsdt, \tag{5.1.68}$$

where the inequality holds componentwise, then

$$\phi(x, y) \leq a(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} V^T(s, t; x, y)B(s, t)a(s, t)dsdt. \tag{5.1.69}$$

Proof Let

$$u(x, y) = \int_{x_0}^{x} \int_{y_0}^{y} B(s, t)\phi(s, t)dsdt. \tag{5.1.70}$$

Then $u_{xy} = B(x, y)\phi$ and since $B \geq 0$ and (5.1.68) holds,

$$u_{xy} = B\phi \leq B(a + u) \quad or \quad L[u] = u_{xy} - Bu \leq Ba. \tag{5.1.71}$$

This is a hyperbolic vector partial differential inequality for u. The initial conditions for u are

$$u(x_0, y) = u(x, y_0) = 0. \tag{5.1.72}$$

The operator L is self-adjoint. We note that for any $u,\ v \in C^2$,

$$v^T L[u] - u^T L[v] = v^T u_{xy} - v^T Bu - u^T v_{xy} + u^T Bv.$$

Then all term here are all scalars and since B is symmetric, the second and fourth terms on the right-hand side cancel. Thus the right-hand side is

$$\begin{aligned} &= -(u^T v_y)_x + u_x^T v_y + (v^T u_x)_y - v_y^T u_x \\ &= -(u^T v_y)_x + (v^T u_x)_y. \end{aligned} \tag{5.1.73}$$

For P_0 and P as required in the hypothesis, we label the directed sides and corners of the rectangle R as shown in Fig. 5.2.

Using s and t as the independent variables in identity (5.1.73), integrating it over R, and using Green's Theorem, we get

$$\begin{aligned} \int\int_R \{v^T L[u] - u^T L[v]\} ds dt &= \int\int_R \{-(u^T v_t)_s + (v^T u_s)_t\} ds dt \\ &= -\int_C v^T u_s ds + u^T v_t dt \\ &= -\int_{C_2+C_4} v^T u_s ds - \int_{C_1+C_3} u^T v_t dt \end{aligned} \tag{5.1.74}$$

which holds for any functions in C^2. For any $u \in C^2$ which also satisfies the initial conditions (5.1.72), $u = 0$ on C_3 and $u = u_s = 0$ on C_4. Thus (5.1.74) reduces to

$$-\int_{C_2} v^T u_s ds - \int_{C_1} u^T v_t dt. \tag{5.1.75}$$

Now suppose the vector $v^i(s,t;x,y)$ are the columns of the matrix $V(s,t;x,y)$ of Theorem 5.1.11. Then $L[v^i] = 0$ and $v^i(s,y) = v^i(x,t) = e^i$, the i-th column of

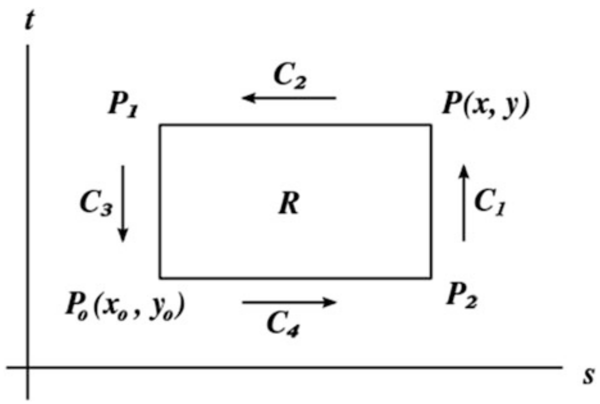

Fig. 5.2 Directed path around R

the identity matrix. Thus $v_t^i = 0$ on C_1 so (5.1.75) reduces to $\int\int_R v^{iT} L[u]dsdt = -\int_{C_2} e^i u_s ds = u_i(P)$ where the subscript refers to the component of the vector. Using the matrix V, this becomes

$$u(P) = \int\int_R V^T(s,t;x,y)L[u]dsdt.$$

For u defined by (5.1.70), since (5.1.71) holds and since $V \geq 0$ by Theorem 5.1.10, we get

$$u(P) \leq \int\int_R V^T Badsdt. \tag{5.1.76}$$

This gives us an upper bound for the integral term in (5.1.68) so that (5.1.69) follows. □

The matrix V is a generalization of a scalar Riemann function and when ϕ, a, *and* B are scalars, it reduces to the Riemann function relative to the point $P(x,y)$ for the operator L. We note that by the method of proof, if equality holds in (5.1.68), there would have been equality in (5.1.71) and (5.1.76) regardless of the non-negativity of V, so the right-hand side of (5.1.69) is the solution to the Volterra integral equation corresponding to (5.1.68). Since the right-hand side of (5.1.69) is a solution of the inequality (5.1.68) and is an upper bound for all such solutions, it is the maximal solution of (5.1.68).

Corollary 5.1.2 *If* $\phi(x,y) \leq \int_{x_0}^x \int_{y_0}^y B(s,t)\phi(s,t)dsdt$, $B \geq 0$, *and* $(x-x_0)\cdot(y-y_0) \geq 0$, *then* $\phi(x,y) \leq 0$.

Proof Let a be the zero vector in Theorem 5.1.11. □

Corollary 5.1.3 *If a is a constant vector, then* $\phi(x,y) \leq V^T(x_0,y_0;x,y)a$ *or by the symmetry of V in its variables,* $\phi(x,y) \leq V^T(x,y;x_0,y_0)a$.

Proof For a constant vector a, (5.1.69) becomes

$$\phi \leq \left\{ I + \int_{x_0}^x \int_{y_0}^y B(s,t)V^T(s,t;x,y)dsdt \right\} a.$$

By (5.1.63) and the symmetry of B, the integrand is $V_{st}^T(s,t;x,y)$. Integrating and using conditions (5.1.64) which V satisfies, we get the desired result. □

Corollary 5.1.4 *If inequality (5.1.68) is reversed, then so is inequality (5.1.69).*

Proof Since $B \geq 0$, we still have $V \geq 0$. Hence reversing inequality (5.1.68) reverses (5.1.71) and (5.1.76), so (5.1.69) is reversed. □

Lemma 5.1.2 (Ghoshal-Ghoshal-Masood [226]) *Let $a(s,t), b(s,t)$ and $H(s,t)$ be continuous matrix functions. Then the matrix characteristic initial value*

problem (5.1.68) under (5.1.82) and (5.1.83) has a unique solution $V(s,t;x,y)$ for all s and t near to $X(x,y)$ and satisfying $(s-x)\cdot(t-y)\geq 0$. The solution is continuous, and if a,b,H are non-negative, so is $V(s,t)$.

Proof Now Eq. (5.1.68) together with conditions (5.1.82) and (5.1.83), is equivalent to the Volterra integral equation

$$V(s,t)=I+\int_y^t a(x,\eta)V(x,\eta)d\eta+\int_x^s b(\xi,y)V(\xi,y)d\xi \\ +\int_x^s\int_y^t H(\xi,\eta)V(\xi,\eta)d\xi d\eta \tag{5.1.77}$$

since

$$M(V)=V_{st}-(aV)_s-(bV)_t+cV=0, H=-c.$$

□

Let T represent the transformation

$$TV=\int_y^t aVd\eta+\int_x^s bVd\xi+\int_x^s\int_y^t HVd\xi d\eta, \tag{5.1.78}$$

so that the integral equation (5.1.77) can be written as

$$V=I+TV. \tag{5.1.79}$$

Let $V_0(s,t)=I$, and define $V_{n+1}=I+TV_n$. When V is continuous, TV is also continuous under the assumptions stated in the lemma, and so by induction V_n is defined and continuous for all n. Let $\|\cdot\|$ be a matrix norm. Nothing that $(s-x)\cdot(t-y)\geq 0$, we get

$$\begin{aligned}\|TV\| &\leq \int_y^t \|a\|\cdot\|V\|d\eta+\int_x^s \|b\|\|V\|d\xi+\int_x^s\int_y^t \|H\|\cdot\|V(\xi,\eta)\|d\xi d\eta \\ &\leq \left[\int_y^t \|a\|d\eta+\int_x^s \|b\|d\xi+\int_x^s\int_y^t \|H\|d\xi d\eta\right]\cdot\max\|V(\xi,\eta)\| \\ &\leq a\cdot\max\|V(\xi,\eta)\|, \end{aligned} \tag{5.1.80}$$

where $0<a<I$ if s and t are close enough to (x,y). Then

$$\|V_{n+1}-V_n\|=\|T(V_n-V_{n-1})\|\leq a\cdot\max\|V_n-V_{n-1}\|\leq\cdots\leq a^n\cdot\max\|V_1-V_0\|.$$

Now $V_{n+1} = V_0 + \sum_{r=0}^{n}(V_{r+1} - V_r)$ is the n-th partial sum of a matrix series, which is majorized by the following convergent geometric series (in matrix norm)

$$\max \|V_1 - V_0\| \sum_{r=0}^{+\infty} a^r.$$

It is obvious that the matrix sequence $\{V_n\}$ converges uniformly in the domain where (5.1.75) holds. Each V_n is continuous and so is the limit function $V(s,t)$.

Now owing to the continuity of the operator T, we have

$$\begin{aligned} I + TV &= I + T\{\lim V_n\} = I + T \lim V_n \\ &= \lim(I + TV_n) = \lim(V_{n+1}) = V. \end{aligned}$$

So that $V = I + TV$. This implies that V is a solution of (5.1.68).

If possible, let W be any other solution; so $V - W = T(V - W)$. Hence

$$\|V - W\| = \|T(V - W)\| \le a \cdot \max \|V - W\|.$$

Thus

$$V = W,$$

which proves the uniqueness of the solution V.

Now if a, b, H are all positive non-negative, then $V \ge 0$ implies $TV \ge 0$ (since $(s - x) \cdot (t - y) \ge 0$). As $V_0 = I > 0$, it follows by induction that $V_n \ge 0$ for all n, so that the limit function $V(s,t;x,y) \ge 0$. □

Theorem 5.1.12 (Ghoshal-Ghoshal-Masood [226]) *If $f(x,y), g(x,y)$ are all continuous n-vector functions on a domain D, and $p(x,y), q(x,y), H(x,y)$ are symmetric non-negative matrix functions (matrix with non-negative elements) on D. Let $X_0(x_0,y_0)$ and $X(x,y)$ be two points in D such that $(x - x_0) \cdot (y - y_0) \ge 0$ and let R be the rectangular region whose diagonal is the line joining the points X_0 and X and let $V(s,t;x,y)$ be the $n \times n$ matrix function satisfying the matrix initial value problem*

$$M(V) = 0, \tag{5.1.81}$$

where M is the adjoint of the operator L given by

$$L(u) = u_{st} + au_s + bu_t + cu \tag{5.1.82}$$

and $a = -Hq, b = -Hp, c = -H$, with the boundary conditions:

$$V(x, y; x, y) \equiv V(x, y) \equiv V(X) = I, \tag{5.1.83}$$

$$\begin{cases} V(x,t) = & \exp\left(\int_y^t a(x,\varrho)d\varrho\right), \\ V(s,y) = & \exp\left(\int_x^s b(\varrho,y)d\varrho\right), \end{cases} \tag{5.1.84}$$

I is the identity matrix and $V(s,t;x,y) \equiv V(s,t)$ is the matrix generalization of Riemann's function relative to the point $X(x,y)$ associated with the operator L.

Let G be the connected sub-domain of D which contains X and on which $V \geq 0$. If $R \subset G$ and $f(x,y)$ satisfies

$$f(x,y) \leq g(x,y) + p(x,y)\int_{x_0}^{x} H(s,y)f(s,y)ds + \\ +q(x,y)\int_{y_0}^{y} H(x,t)f(x,t)dt + \int_{x_0}^{x}\int_{x_0}^{x}\int_{y_0}^{y} H(s,t)f(s,t)dsdt, \tag{5.1.85}$$

where the inequality holds component-wise; then $f(x,y)$ also satisfies

$$f(x,y) \leq g(x,y) + p(x,y)\int_{x_0}^{x} H(s,y)f(s,y)ds + q(x,y)\int_{y_0}^{y} H(x,t)f(x,t)dt \\ + \int_{x_0}^{x}\int_{y_0}^{y} V^T(s,t;x,y)H(s,t)g(s,t)dsdt. \tag{5.1.86}$$

Furthermore, if $q(x,y) = 0$, then

$$f(x,y) \leq g(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} V^T(s,t)H(s,t)g(s,t)dsdt \\ +p(x,y)\int_{x_0}^{x} H(s,y)\Big\{g(s,y) + \int_{s_0}^{s}\int_{y_0}^{y} V^T(\theta,t)H(\theta,t)g(\theta,t)d\theta dt\Big\} \\ \times\left[\exp\int_{s}^{x} H(\xi,y)p(\xi,y)d\xi\right]ds. \tag{5.1.87}$$

Also if $p(x, y) = 0$, *then*

$$f(x, y) \leq g(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} V^T(s, t)H(s, t)g(s, t)dsdt \\ +q(x, y) \int_{y_0}^{y} H(x, t)\Big\{g(x, t) + \int_{x_0}^{x} \int_{y_0}^{y} V^T(s, \Phi)H(s, \Phi)g(s, \Phi)dsd\Phi\Big\} \\ \times \left[\exp \int_{t}^{y} H(x, \varphi)q(x, \varphi)d\varphi\right]. \tag{5.1.88}$$

Proof Noting that since $u(x_0, y) = u(x, y_0) = 0$, we may let

$$u(x, y) = \int_{x_0}^{x} \int_{y_0}^{y} H(s, t)f(s, t)dsdt.$$

Then we obtain

$$\begin{cases} u_{xy} = H(x, y)f(x, y) \leq H(x, y)\Big(g(x, y) + pu_y + qu_x + u\Big), \\ L(u) \equiv u_{xy} + au_x + bu_y + cu \leq Hg, \end{cases}$$

where

$$a = -Hq, \quad b = -Hp, \quad c = -H.$$

This is a hyperbolic partial differential inequality for u; L is a non-self-adjoint operator.

Now for any two functions $u, V \in C^2$, we have

$$V^T L(u) - u^T M(V) = V^T[u_{xy} + au_x + bu_y + cu] \\ -u^T[V_{xy} - aV_x - bV_y + (c - a_x - b_y)V].$$

Here the relations are scalar and hold true for each column of V. If a, b, c are symmetric matrices, then we can show that the last expression is

$$\left(u^T aV + V^T \frac{u_y}{2} - \frac{u^T V_y}{2}\right)_x + \left(V^T bu + V^T \frac{u_x}{2} - u^T \frac{V_x}{2}\right)_y .$$

Taking the region R referred to in the main theorem in the form of the rectangle [232] of lemma and applying Green's theorem, we find (see Fig. 5.3), by noting that u is zero on C_1 and C_4; also ds does not vary on C_2 and dt on C_3; $u_t = 0$ on

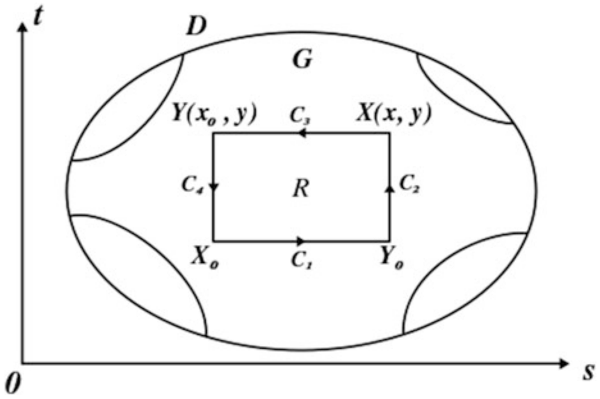

Fig. 5.3 Region and directed path around R

$C_4, u_s = 0$ on C_1,

$$\int\int_R [V^T l(u) - u^T M(V)] dsdt$$
$$= \int_{C=C_1+C_2+C_3+C_4} \left[\left(u^T aV + V^T \frac{u_t}{2} - \frac{u^T V_t}{2} \right) dt - \left(V^T bu + V^T \frac{u_s}{2} - u^T \frac{V_s}{2} \right) ds \right]$$
$$= \int_{C_2} \left[u^T aV + V^T \frac{u_t}{2} - u^T \frac{V_t}{2} \right] de - \int_{C_3} \left[V^T bu + V^T \frac{u_s}{2} - u^T \frac{V_s}{2} \right] ds$$
$$= \int_{C_2} [u^T (aV - V_t) + \frac{1}{2}(V^T u)_t] dt - \int_{C_3} [u^T (bV - V_s) + \frac{1}{2}(V^T u)_s] ds.$$

(The relations are scalar relations and hold for each column of V.) If V is the Riemann function with the initial conditions

$$\begin{cases} aV - V_t & = 0 \quad on \;\; C_2, \\ bV - V_s & = 0 \quad on \;\; C_3, \end{cases}$$

and

$$V(x, y) = I = V^T(X)$$

(which are conditions (5.1.83) and (5.1.84)) and satisfies the equation $M(V) = 0$, then we obtain

$$\int\int_R V^T L(u) dsdt = V^T(x, y)u(x, y) - \frac{1}{2}u^T(x, y_0)V(x, y_0) - \frac{1}{2}u(x_0, y)V(x_0, y_0).$$

Since

$$u(x, y) \leq \int_{x_0}^{x} \int_{y_0}^{y} V^T(s, t) H(s, t) g(s, t) ds dt,$$

$$u(X) \equiv u(x, y) = \int_{x_0}^{x} \int_{y_0}^{y} V^T(s, t) L(u) ds dt.$$

Hence

$$f(x, y) \leq g(x, y) + p(x, y) \int_{x_0}^{x} H(s, y) f(s, y) ds$$
$$+ q(x, y) \int_{y_0}^{y} H(x, t) f(x, t) dt + \int_{x_0}^{x} \int_{y_0}^{y} V^T(s, t) g(s, t) ds dt.$$

Now, let $q(x, y) = 0$ and suppose

$$\omega(x, y) = g(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} V^T(s, t) H(s, t) g(s, t) ds dt,$$

thus

$$f(x, y) \leq \omega(x, y) + p(x, y) \int_{x_0}^{x} H(s, y) f(s, y) ds.$$

This inequality may be treated as a one-dimensional Gronwall-Bellman's inequality (see, Theorem 1.1.4) for any fixed "y" between y_0 to y.

For a fixed y, let

$$\psi(x, y) = \int_{x_0}^{x} H(s, y) f(s, y) ds,$$

therefore

$$\psi(x_0, y) = 0,$$

and

$$\psi_s(s, y) = H(s, y) f(s, y) \leq H(s, y)[\omega(s, y) + p(s, y)\psi(s, y)]$$

since $H(s, y) \geq 0$, so that we have

$$\psi_s(s, y) - H(s, y) p \psi(s, y) \leq H(s, y) \omega(s, y).$$

Hence we obtain

$$\psi(x,y) \le \int_{x_0}^{x} H(s,y)\omega(s,y)e^{\int_s^x H(\xi,y)p(\xi,y)d\xi}ds,$$

so that

$$f(x,y) \le g(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} V^T(s,t)H(s,t)g(s,t)dsdt + p(x,y)$$
$$\times \int_{x_0}^{x} H(s,y)\Big[g(s,y) + \int_{x_0}^{s}\int_{y_0}^{y} V^T(\theta,t)H(\theta,t)g(\theta,t)d\theta dt\Big]e^{\int_s^x H(\xi,y)p(\xi,y)d\xi}ds.$$

Similarly, if $p(x,y) = 0$, we obtain

$$f(x,y) \le g(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} V^T(s,t)H(s,t)g(s,t)dsdt + q(x,y)\int_{y_0}^{y} H(x,t)$$
$$\times\Big[g(x,t) + \int_{x_0}^{x}\int_{y_0}^{t} V^T(s,\Phi)H(s,\Phi)g(s,\Phi)dsd\Phi\Big]e^{\int_t^y H(x,\varphi)q(x,\varphi)d\varphi}dt.$$

Thus the proof is now complete. □

Corollary 5.1.5 *Putting $p(x,y) \equiv 0 \equiv q(x,y)$ in (5.1.85), we obtain*

$$f(x,y) \le g(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} V^T(s,t)H(s,t)g(s,t)dsdt,$$

which was obtained by Snow [603]. The treatment given in [603] follows from here as a particular case.

Corollary 5.1.6 *If $f(x,y) \le \int_{x_0}^{x}\int_{y_0}^{y} H(s,t)f(s,t)dsdt$ and $H(s,t) \ge 0$ and $(x-x_0)\cdot(y-y_0) \ge 0$, then $f(x,y) \le 0$.*

Corollary 5.1.7 *If inequality (5.1.85) is reversed, then so is inequality (5.1.86) [(5.1.87) and (5.1.88)].*

Corollary 5.1.8 *If $q(x,y) = 0$ and $g(x,y) = 0$, then (5.1.85) reduces to*

$$f(x,y) \le p(x,y)\int_{x_0}^{x} H(s,y)f(s,y)ds + \int_{x_0}^{x}\int_{y_0}^{y} H(s,t)f(s,t)dsdt,$$

then by (5.1.87)

$$f(x,y) \le 0.$$

Corollary 5.1.9 *Similarly, if $p(x,y) = 0 = g(x,y)$, (5.1.85) reduces to*

$$f(x,y) \le q(x,y) \int_{y_0}^{y} H(x,t)dt + \int_{x_0}^{x} \int_{y_0}^{y} H(s,t)f(s,t)dsdt,$$

then (5.1.88) gives

$$f(x,y) \le 0.$$

The Wendroff inequality (Theorem 5.1.1) on two-variable scalar integral inequalities, quoted without proof by Beckenback and Bellman [228] have been generalized to cover a system of integral inequalities by following an approach different from that of Jagdish Chandra and Davis [604].

In 1976, Chandra and Davis [604] published their results on the generalization of Gronwall inequality to cover a system of m integral inequalities in n independent variables.

At about the same time, Shastri and Kasture [585] established a result by following the approach of differential analysis as in Snow [603]. Although their approach requires the hypothesis of differentiability not need in [604], it is a constructive approach and it has a potential for being applicable to a large class of differential and integral inequalities.

To illustrate this approach, Shastri and Kasture [585] gave the following theorem which is a generalization of the results of Wendroff (see, Theorem 5.1.1), see also [228].

Theorem 5.1.13 (Shastri-Kasture [585]) *Let $\phi(x,y)$ be a continuous, non-negative, m-vector function on a two dimensional domain D and $A(x,y)$, $B(x,y)$, $H(x,y)$ be continuous, non-negative, symmetric $m \times m$ matrix functions with $A(x,y)$, $B(x,y)$ continuously differentiable in x and y and non-increasing in y and x, respectively. If C is any non-negative constant m-vector and $\phi(x,y)$ satisfies*

$$\phi(x,y) \le C + \int_0^x A(s,y)\phi(s,y)ds + \int_0^y B(x,t)\phi(x,t)dt + \int_0^x \int_0^y H(s,t)\phi(s,t)dsdt, \tag{5.1.89}$$

then

$$\phi(x,y) \le C^T \exp\Big[\int_0^x A(\alpha,y)d\alpha + \int_0^y B(x,\beta)d\beta + \int_0^x \int_0^y \Big(A(\alpha,\beta)B(\alpha,\beta) + H(\alpha,\beta)\Big)dsd\beta\Big], \tag{5.1.90}$$

where C^T is the row vector.

Proof Let $u(x, y)$ be the solution of the integral equation corresponding to (5.1.89). The existence of $u(x, y)$ can be proved by using contraction mapping principle, as in Ghoshal and Masood [227] (see, e.g., Lemma 1). Then by Theorem III b, p.130 of Walter [636], we have

$$\phi(x, y) \leq u(x, y). \tag{5.1.91}$$

Differentiating $u(s, t)$ twice, we obtain

$$Lu = u_{st} - Bu_s - Au_t - (A_t + B_s + H)u = 0. \tag{5.1.92}$$

The equality (5.1.92) can be integrated using Riemann's method. For any twice continuously differentiable matrix function $W(s, t; x, y)$, we have as in Snow [603] (notation as in Fig. 5.4)

$$\begin{aligned} W^T(p)u(p) = u^T(p_0)W(p_0) &+ \int_{p_2}^{p} [W^T A + W_s^T]u ds \\ &+ \int_{p_1}^{p} [BW + W_t]dt + \int_0^x \int_0^y [W^T Lu - u^T MW] ds dt, \end{aligned} \tag{5.1.93}$$

where M defined by

$$MW = W_{st} + BW_s + AW_t - HW$$

is the adjoint of operator L. Now we choose W such that

(i) $W(s, t; x, y) > 0$ and $MW \geq 0,\ 0 \leq s \leq x,\ 0 \leq t \leq y$,
(ii) $W^T A + W_s^T \leq 0,\ 0 \leq s \leq x,\ t = y$,
(iii) $BW + W_t \leq 0,\ 0 \leq t \leq y,\ s = x$.

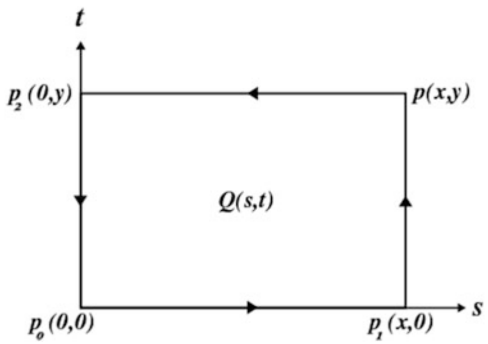

Fig. 5.4 Directed path around Q(s,t)

Then it follows from (5.1.92) and (5.1.93) that

$$W^T(p)u(p) \le u^T(p_0)W(p_0). \tag{5.1.94}$$

A function $W(s, t; x, y)$ satisfying all the requirements (i), (ii) and (iii) is easily found to be

$$\begin{aligned} W(s, t; x, y) = \exp\Big[&\int_s^x A(\alpha, t)d\alpha + \int_t^y B(s, \beta)d\beta \\ &+ \int_s^x \int_t^y \{A(\alpha, \beta)B(\alpha, \beta) + H(\alpha, \beta)\}dsd\beta \Big]. \end{aligned} \tag{5.1.95}$$

The desired conclusion now follows from (5.1.94) and (5.1.91). □

In the sequel, it is assumed that all variables are real and all functions are real-valued. The term "domain" is used in the usual sense of meaning an open, connected set.

The elementary proof given here is simply an application of the following lemma, which is established by an argument involving nothing more sophisticated than mathematical induction and integration-by parts.

Lemma 5.1.3 (Rasmussen [552]) *Let $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$ be points in the domain D such that $(x_1 - x_0) \cdot (y_1 - y_0) \ge 0$ and such that the closed rectangle R_1 with opposite vertices P_0 and P_1 is contained in D. Let $\phi(x, y)$ and $K(x, y)$, with $K(x, y)$ non-negative, be continuous functions on D. If for all (x, y) in R_1,*

$$\phi(x, y) \le \int_{x_0}^x \int_{y_0}^y K(t, s)\phi(t, s)dsdt, \tag{5.1.96}$$

then for all $(x, y) \in R_1$.

$$\phi(x, y) \le 0. \tag{5.1.97}$$

Proof Let $Q = (x_1 - x_0) \cdot (y_1 - y_0)$. If $Q = 0$, then the rectangle R_1 degenerates to a line segment and the lemma is trivially true. But if $Q > 0$, then for all (x, y) in R_1, let N be a positive upper bound on $\phi(x, y)$ and define

$$V(x, y) = \int_{x_0}^x \int_{y_0}^y K(t, s)dsdt.$$

We shall show by induction that for all natural numbers n, and for all $(x, y) \in R_1$

$$\phi(x, y) \le \frac{N \cdot V^n(x, y)}{n!}. \tag{5.1.98}$$

Indeed, obviously for $n = 1$, $\phi(x, y) \leq N$ and $K(x, y) \geq 0$ imply that $\phi(x, y) \cdot K(x, y) \leq N \cdot K(x, y)$. Then since $Q \geq 0$, substituting into (5.1.96) yields

$$\phi(x, y) \leq \int_{x_0}^{x} \int_{y_0}^{y} N \cdot K(t, s) ds dt = N \cdot V(x, y).$$

Now assume that (5.1.98) holds for $n = k$. In the same way as the preceding step, we obtain

$$\phi(x, y) \leq N \int_{x_0}^{x} \int_{y_0}^{y} K(t, s)[V^k(t, s)/k!] ds dt. \tag{5.1.99}$$

Now since differentiation of $V(x, y)$ implies $V_{12}(x, y) = K(x, y)$, integrating by parts in (5.1.99) yields

$$\begin{aligned}
\int_{y_0}^{y} K(t, s) \frac{V^k(t, s)}{k!} ds &= \int_{y_0}^{y} V_{12}(t, s) \frac{V^k(t, s)}{k!} ds \\
&= \frac{V^k(t, s)}{k!} V_1(t, s)\Big|_{y_0}^{y} - \int_{y_0}^{y} V_1(t, s) \frac{V^{k-1}(t, s)}{(k-1)!} V_2(t, s) ds \\
&= \frac{V^k(t, y)}{k!} V_1(t, y) - \int_{y_0}^{y} V_1(t, s) V_2(t, s) \frac{V^{k-1}(t, s)}{(k-1)!} ds.
\end{aligned} \tag{5.1.100}$$

For all $x \geq x_0$ and $y \geq y_0$, it is easy to verify that $K(t, s) \geq 0$ implies that $V(x, y)$, $V_1(x, y)$ and $V_2(x, y)$ are all non-negative, hence (5.1.100) implies

$$\int_{y_0}^{y} K(t, s) \frac{V^k(t, s)}{k!} ds \leq \frac{V^k(t, y)}{k!} V_1(t, y).$$

However, if $x \leq x_0$ and $y \leq y_0$, it follows that $K(t, s) \geq 0$ implies that only $V(x, y)$ is non-negative, while $V_1(x, y)$ and $V_2(x, y)$ are both non-positive. In this case, (5.1.100) implies

$$\int_{y_0}^{y} K(t, s)[V^k(t, s)/k!] ds \geq [V^k(t, y)/k!] V_1(t, y).$$

Thus in either case, substituting into (5.1.99) yields

$$\phi(x, y) \leq N \int_{x_0}^{x} \frac{V^k(t, y)}{k!} V_1(t, y) dt = N \frac{V^{k+1}(x, y)}{(k+1)!}$$

which implies that (5.1.98) holds for $n = k + 1$.

Now since R_1 is compact, $V(x, y)$ is bounded there and $N[V^n(x, y)/n!]$ approaches zero as n goes to infinity. Then noting that for all N,

$$\phi(x, y) \leq N[V^n(x, y)/n!],$$

we conclude that $\phi(x, y) \leq 0$ on R_1. □

Now applying Lemma 5.1.3, we can prove the next Snow's result [603].

Theorem 5.1.14 (Snow [603]) *Let $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$ be points in the domain D such that $(x_1 - x_0) \cdot (y_1 - y_0) \geq 0$ and such that the closed rectangle R_1 with opposite vertices P_0 and P_1 is contained in D. Let $g(x, y)$, $\phi(x, y)$ and $K(x, y)$, with $K(x, y)$ non-negative, be continuous functions on D. If for all (x, y) in R_1,*

$$\phi(x, y) \leq g(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} K(t, s)\phi(t, s)dsdt, \tag{5.1.101}$$

then

$$\phi(x, y) \leq \Phi(x, y) \tag{5.1.102}$$

on R_1, where $\Phi(x, y)$ satisfies the case of equality in (5.1.101).

Proof The argument of the theorem guarantees the existence of a continuous function $\Phi(x, y)$ satisfying the case of equality in (5.1.101), i.e., for all (x, y) in R_1,

$$\Phi(x, y) = g(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} K(t, s)\Phi(t, s)dsdt.$$

If $Q = (x_1 - x_0) \cdot (y_1 - y_0) = 0$, the theorem is trivially true on the degenerate rectangle. If $Q > 0$, define $\psi(x, y) = \phi(x, y) - \Phi(x, y)$ so that on R_1,

$$\psi(x, y) \leq \int_{x_0}^{x} \int_{y_0}^{y} K(t, s)\psi(t, s)dsdt.$$

Then by Lemma 5.1.3, $\psi(x, y) \leq 0$, or $\phi(x, y) \leq \Phi(x, y)$ on R_1. □

Let $u(x, y)$, $a(x, y)$, $b(x, y)$ be non-negative continuous functions defined for all $x, y \in \mathbb{R}_+$.

Lemma 5.1.4 (Dragomir-Kim [197]) *(i) Assume that $a(x, y)$ is non-decreasing in x and non-increasing in y for all $x, y \in \mathbb{R}_+$. If for all $x, y \in \mathbb{R}_+$,*

$$u(x, y) \leq a(x, y) + \int_{0}^{x} \int_{y}^{+\infty} b(s, t)u(s, t)dtds,$$

then for all $x, y \in \mathbb{R}_+$,

$$u(x,y) \le a(x,y) \exp\left(\int_0^x \int_y^{+\infty} b(s,t)u(s,t)dtds \right).$$

(ii) Assume that $a(x, y)$ *is non-decreasing in each of the variables* $x, y \in \mathbb{R}_+$. *If for all* $x, y \in \mathbb{R}_+$,

$$u(x,y) \le a(x,y) + \int_x^{+\infty} \int_y^{+\infty} b(s,t)u(s,t)dtds,$$

then for all $x, y \in \mathbb{R}_+$,

$$u(x,y) \le a(x,y) \exp\left(\int_x^{+\infty} \int_y^{+\infty} b(s,t)u(s,t)dtds \right).$$

The proofs of the inequalities in (i), (ii) can be completed as in [42]. Here we omit the details.

It should be noted that the uniqueness of the function $\Phi(x, y)$ is readily established by using Lemma 5.1.3 which is a special case of Theorem 5.1.14. In fact, it can be regarded as a corollary to Theorem 5.1.14.

The next several results, due to Kasture and Deo [312], will extend the results of Snow [603] in several directions. First, Kasture and Deo [312] introduced a more general inequality applicable to a larger class of Volterra integral equations in two independent variables; further, assuming some additional conditions on the functions involved in the inequality [47], they obtained different estimates which are more suitable for applications.

We first obtain further generalizations of these inequalities. When a kernel $k(x, y, s, t)$ in a Volterra integral equation is separable but consists of several functions, i.e., if $k(x, y, s, t) \le \sum_{i=1}^{n} h_i(x, y)b_i(s, t)$, then the inequalities obtained are not sufficient to accommodate such a situation. Willett [647] considered such a problem for equations containing one independent variable. Kasture and Deo [312] generalized these results in several directions and obtained the pointwise estimate.

It is noted that Theorem 5.1.11 represents the estimate of $\phi(x, y)$ in an explicit form. This inequality is the best possible in the sense that equality in (5.1.68) implies equality in (5.1.69) and thus (5.1.69) gives us an estimate of the Volterra integral equation (5.1.68). For a more general situation in which the kernel is separable in the form

$$k(x, y, s, t) \le b(s, t)h(x, y),$$

we must replace (5.1.68) by

$$u(x,y) \le a(x,y) + h(x,y)\int_{x_0}^{x}\int_{y_0}^{y} b(s,t)u(s,t)dsdt. \tag{5.1.103}$$

This leads to the following generalization of Theorem 5.1.11.

Theorem 5.1.15 (Kasture-Deo [312]) *Suppose*

(a) $u(x,y), a(x,y)$, and $b(x,y)$ are continuous functions on a domain D with $b \ge 0$ there. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two points in D such that $(x-x_0)\cdot(y-y_0) \ge 0$ and let R be the rectangular region whose opposite corners are the points P_0 and P.

(b) Let $v(s,t,x,y)$ be the solution of the characteristic initial value problem

$$v_{st} - h(s,t)b(s,t)v = 0 \tag{5.1.104}$$

subject to conditions (5.1.64). If $h(x,y) \ge 0$ for all (x,y) in D, conditions (iii) and (iv) of Theorem 5.1.11 hold with v replaced by V, and if (5.1.103) holds, then

$$u(x,y) \le a(x,y) + h(x,y)\int_{x_0}^{x}\int_{y_0}^{y} a(s,t)b(s,t)V(s,t,x,y)dsdt. \tag{5.1.105}$$

Proof The proof is similar to Theorem 5.1.11 (Snow [603]). For completeness, we give here a brief outline. Let

$$\phi(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)u(s,t)dsdt, \tag{5.1.106}$$

so that (5.1.103) becomes

$$u \le a + h\phi.$$

Then, by virtue of (5.1.103),

$$\phi_{xy} = b(x,y)u(x,y) \le b(x,y)(a(x,y) + h(x,y)\phi).$$

Hence

$$L(\phi) = \phi_{st} - h(s,t)b(s,t)\phi \le a(s,t)b(s,t). \tag{5.1.107}$$

Since L is a self-adjoint hyperbolic operator, for any twice continuously differentiable functions w and ψ, we have

$$wL\psi - \psi Lw = -(\psi w_y)_x + (w\psi_x)_y. \tag{5.1.108}$$

Setting $w = V$ and $\psi = \phi$ in (5.1.108), integrating on R, and using the Green's theorem and the conditions on ϕ and V, we obtain

$$\phi(x, y) = \int_R VL(\phi)dsdt \leq \int_R Vabdsdt, \tag{5.1.109}$$

since $V > 0$, and (5.1.107) holds. Conclusion (5.1.105) follows from (5.1.109) and (5.1.106). The continuity of h and b on D guarantee the existence and continuity of V (see, e.g., [157]). □

Remark 5.1.4 (i) If the inequality in (5.1.103) is replaced by an equality, then the inequality in (5.1.105) is replaced by an equality. In this sense, (5.1.105) is the best inequality for the theorem.
(ii) If inequality (5.1.103) is reversed, then so is inequality (5.1.105).

While obtaining estimate (5.1.105) for inequality (5.1.103), we have imposed only the continuity assumptions on functions a, h, b. If we add some more assumptions on a and h, we are led to different estimates than (5.1.105).

Theorem 5.1.16 (Kasture-Deo [312]) *If, in Theorem 5.1.15, a is a positive constant and h is non-decreasing on D in both the variables, then (5.1.105) is replaced by*

$$u(x, y) \leq a\frac{h(x, y)}{h(x_0, y_0)}V(x_0, y_0, x, y) \tag{5.1.110}$$

where $x - x_0 > 0$ and $y - y_0 > 0$.

Proof From (5.1.105) it follows after the integration and use of initial conditions on V,

$$\begin{aligned} u(x, y) &\leq a + h(x, y)a \int\int_R bVdsdt \\ &= a + h(x, y)a \int\int_R \frac{V_{st}}{h(s, t)}dsdt \\ &\leq a + a\frac{h(x, y)}{h(x_0, y_0)} \int\int_R V_{st}dsdt \\ &= a\frac{h(x, y)}{h(x_0, y_0)}V(x_0, y_0, x, y) - a\left(\frac{h(x, y)}{h(x_0, y_0)} - 1\right). \end{aligned}$$

Conclusion (5.1.110) is now clear in view of the monotonicity of h. □

Theorem 5.1.17 (Kasture-Deo [312]) *If, in Theorem 5.1.15, h is non-increasing on D in both the variables, and a is a positive constant, then conclusion (5.1.105) is*

replaced by

$$u(x,y) \le aV(x_0, y_0, x, y). \tag{5.1.111}$$

Proof The proof is similar to that of Theorem 5.1.16. □

Corollary 2 in [603] is a special case of Theorem 5.1.16 or 5.1.17 when $h = 1$.

We may use Theorem 5.1.16 to prove the following theorem, which majorizes u in Theorem 5.1.15.

Theorem 5.1.18 (Kasture-Deo [312]) *If, in Theorem 5.1.15, let a and h be pointwise and non-decreasing in both the variables on D, then for all $x > x_0, y > y_0$, (5.1.105) is replaced by*

$$u(x,y) \le \Big(a(x,y)h(x,y)/h(x_0,y_0)\Big)V(x_0,y_0,x,y). \tag{5.1.112}$$

Proof Since a is positive, dividing (5.1.103) by $a(x,y)$, and using the monotonicity of a, we obtain

$$\overline{u}(x,y) \le 1 + h(x,y)\int\int_R b(s,t)\overline{u}(s,t)dsdt, \tag{5.1.113}$$

where

$$\overline{u}(x,y) = u(x,y)/a(x,y). \tag{5.1.114}$$

Applying Theorem 5.1.16 to (5.1.113) and using of (5.1.114) yields the desired result (5.1.112). □

Theorem 5.1.19 (Kasture-Deo [312]) *If, in Theorem 5.1.15, let $h(x,y) \ge 1$ for all $(x,y) \in D$ and let $a(x,y)$ be positive and non-decreasing on D, then (5.1.105) in Theorem 5.1.15 is replaced by*

$$u(x,y) \le a(x,y)h(x,y)\big(V(x_0,y_0,x,y) - 1\big). \tag{5.1.115}$$

Proof Since $a(x,y)$ is positive and non-decreasing, (5.1.113) holds. Since $h \ge 1$, we have from (5.1.113),

$$\overline{u}(x,y) \le h(x,y)\left[1 + \int\int_R b\overline{u}dsdt\right]. \tag{5.1.116}$$

Let

$$\phi(x,y) = 1 + \int\int_R b\overline{u}dsdt,$$

so that

$$\phi_{xy} = b\bar{u} \leq bh\phi,$$

or

$$L\phi = \phi_{xy} - bh\phi \leq 0. \tag{5.1.117}$$

Now following the method of proof of Theorem 5.1.15, we obtain

$$\phi(x, y) + 1 - V(x_0, y_0, x, y) = \int \int_R VL\phi dsdt \leq 0.$$

Hence

$$\phi(x, y) \leq V(x_0, y_0, x, y) - 1.$$

Therefore from (5.1.115), the desired result (5.1.115) follows immediately. □

This theorem is an alternative to Theorem 5.1.16 with $h(x, y) = 1$. Therefore, it can be applied in place of Theorem 5.1.16 to derive alternative results corresponding to Theorems 5.1.17 and 5.1.19.

Theorem 5.1.20 (Kasture-Deo [312]) *Let all the assumptions of Theorem 5.1.17 hold. Then*

$$u(x, y) \leq a \exp\Big(h(x_0, y_0) \int_{x_0}^{x} \int_{y_0}^{y} b(s, t) dsdt\Big). \tag{5.1.118}$$

Proof Since h is non-increasing on D in both the variables, it follows from (5.1.103) that

$$u(x, y) \leq a + \int_{x_0}^{x} \int_{y_0}^{y} h(x_0, y_0) b(s, t) u(s, t) dsdt \tag{5.1.119}$$

Applying Lemma 5.1.1 to (5.1.119) yields the desired result (5.1.118). □

The next theorem is an alternative to Theorem 5.1.19.

Theorem 5.1.21 (Kasture-Deo [312]) *Let all the assumptions of Theorem 5.1.19 hold. Then*

$$u(x, y) \leq a(x, y) h(x, y) \exp\Big(\int_{x_0}^{x} \int_{y_0}^{y} h(s, t) b(s, t) dsdt\Big). \tag{5.1.120}$$

Proof We apply Wendroff's method in place of the method of Snow [603] to integrate inequality (5.1.117) obtained in Theorem 5.1.19. In the notation of

Theorem 5.1.19, we have from (5.1.117)

$$(\phi_{xy}/\phi) - (\phi_x \phi_y/\phi^2) \leq h(x, y)b(x, y),$$

since u_x and u_y are non-negative on D. Therefore

$$(\partial^2/\partial x \partial y) \log \phi \leq hb.$$

Integrating on R, we obtain

$$\phi(x, y) \leq \exp\Big(\int_{x_0}^{x}\int_{y_0}^{y} h(s, t)b(s, t)dsdt\Big). \tag{5.1.121}$$

Thus the desired conclusion (5.1.120) now follows from (5.1.116) and (5.1.121). □

Next, we discuss the generalizations to separable kernels of order n. In the study of differential and integral equations, we often have to deal with the following inequalities

$$\phi(x, y) \leq a(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} k(s, t, x, y)\phi(s, t)dsdt, \tag{5.1.122}$$

where $a(x, y)$ and $k(s, t, x, y)$ are known functions and ϕ is an unknown function. It has been studied in Theorem 5.1.15 the particular case of this inequality when

$$k(s, t, x, y) \leq h(x, y)b(s, t).$$

In the following theorems, we majorize the function ϕ in (5.1.122) when the function $k(s, t, x, y)$ is separable in the form

$$k(s, t, x, y) \leq \sum_{i=1}^{n} h_i(x, y)b_i(s, t).$$

Note that results of this type when ϕ, a, h_i, b_i are functions of one independent variable were obtained by Willett [647].

Theorem 5.1.22 (Kasture-Deo [312]) *Suppose that*

(i) $u(x, y), a(x, y), h_i(x, y)$ *and* $b_i(x, y)$ *are real-valued continuous non-negative functions on a domain D,* $i = 1, 2, \cdots, n$.
(ii) For some (x, y) *and* (x_0, y_0) *in D, let R be the rectangular region:*

$$R = \{(s, t) : x_0 \leq s \leq x, y_0 \leq t \leq y\}.$$

(iii) *Define the operators* E_i $(i = 0, 1, 2, \cdots, n)$ *inductively as the composition of* $i + 1$ *functional operators as follows*

$$\begin{cases} E_i = D_i D_{i-1} \cdots D_0, \\ D_0 w = w, \\ D_j w = w + E_{j-1} h_j \displaystyle\int_R b_j v_j w;\ j = 1, 2, \cdots, n, \end{cases}$$

where v_j *is the solution of the characteristic initial value problem*

$$\begin{cases} v_{st} - (E_{j-1} h_j) b_j v = 0, \\ v(s, y) = v(x, t) = 1. \end{cases} \tag{5.1.123}$$

(iv) *Let* D^+ *be a connected sub-domain of* D *on which all the* v_i $(i = 1, \cdots, n)$ *are positive, and let* $R \subset D^+$. *If for all* $(x, y) \in R$,

$$u(x, y) \leq a(x, y) + \sum_{i=1}^{n} h_i(x, y) \int_R b_i u dx dy, \tag{5.1.124}$$

then for all $(x, y) \in R$,

$$u(x, y) \leq E_n(a). \tag{5.1.125}$$

Proof The proof is by finite induction. For $n = 1$, the theorem reduces to Theorem 5.1.15 and hence is true. Suppose n is given and $n > 1$. Assume that the following statements (A) and (B) hold for $i = k$, where k is some fixed integer with $0 \leq k \leq n - 1$.

(A) $E_i w$ is continuous on D^+ for any w that is continuous on D^+.
(B)

$$u \leq E_i a + \sum_{m=i+1}^{n} (E_i h_m) \int_R b_m u dx dy.$$

Then we show that (A) and (B) hold for $i = k + 1$; and that if (B) holds for $i = n - 1$, then (5.1.125) follows. Then since (A) and (B) hold by assumption for $i = 0$, the theorem is proved.

Let (A) hold for $i = k$ where k is a fixed integer with $0 \leq k \leq n - 1$. Then since $E_k h_{k+1}$ and b_k are continuous, it follows that v_{k+1} is continuous and hence from (5.1.123) $D_{k+1} w$ is continuous on D^+. Replacing w by $E_k w$, we find that $E_{k+1} w$ is also continuous on D^+. Then since $E_i w$ is continuous by assumption for $i = 0$, the existence and continuity of all $E_i w$ and v_i is established by finite induction for $i = 1, 2, \cdots, n$.

Now suppose (B) holds for $i = k$, where k is a fixed integer with $0 \le k \le n-2$. Then

$$u \le \phi^* + E_k h_{k+1} \int_R b_{k+1} u dxdy, \tag{5.1.126}$$

where

$$u^* = E_k a + \sum_{m=k+2}^{n} E_k h_m \int_R b_m u dxdy.$$

Applying Theorem 5.1.15 to (5.1.126), rearranging, and using the fact that all the functions are non-negative, we conclude

$$\begin{aligned} u &\le D_{k+1}(E_k a) + \sum_{m=k+2}^{n} D_{k+1}(E_k h_m) \int_R b_m u dxdy \\ &= E_{k+1}(a) + \sum_{m=(k+1)+1}^{n} E_{k+1} h_m \int_R b_m u dxdy. \end{aligned}$$

Thus (B) holds for $i = k+1$. As (B) holds by assumption for $i = 0$, it follows by finite induction that (B) holds for $i = n-1$. Applying Theorem 5.1.15 to (B) with $i = n-1$, we obtain the desired result (5.1.125). □

Theorems 5.1.18 and 5.1.19 may also be generalized by assuming that (5.1.124) holds. The generalization of Theorem 5.1.22 is the following theorem.

Theorem 5.1.23 (Kasture-Deo [312]) *Suppose that conditions (i), (ii), and inequality (5.1.124) of Theorem 5.1.22 hold. Further, let $a(x,y)$ and $h_j(x,y)$ be non-decreasing on D with $h_i(x_0,y_0) > 0$. For $j = 0, 1, \cdots, n$, define a sequence of functions $\alpha_i(x,y)$ on D inductively as follows*

$$\begin{cases} \alpha_0(x,y) = 1, \\ \alpha_i(x,y) = \alpha_{j-1}^2(x,y) \dfrac{h_j(x,y) V_j(x,y)}{h_{j0}}, \end{cases}$$

where $v_j(x,y)$ is the solution of the characteristic initial value problem

$$\begin{cases} v_{st} - \alpha_{j-1}(s,t) h_j(s,t) b_j(s,t) v(s,t) = 0, \\ v(s,y) = v(x,t) = 1, \end{cases}$$

and $h_{j_0} = h_j(x_0,y_0)$. Let D^+ be a connected sub-domain of D on which all the v_j are positive and let R of Theorem 5.1.22 be contained in D^+. Then for all $(x,y) \in R$,

$$u(x,y) \le a(x,y)\alpha_n(x,y).$$

Proof From the definition of v_j, the existence and continuity of α_{j-1} implies the existence and continuity of v_j on D. Hence the existence and continuity of α_j on D follows from the α_{j-1}. But by definition, α_0 is continuous on D. Hence by finite induction, all α_j and v_j $(j = 1, 2, \cdots, n)$ are continuous on D.

For $n = 1$, Theorem 5.1.23 reduces to Theorem 5.1.18 and hence is true. Suppose the theorem holds for $n = k$; i.e., for (5.1.125) holds for $n = k$, then

$$u \le a\alpha_k(x, y).$$

Now assume that (5.1.125) holds for $n = k + 1$, so that

$$u \le u^* + \sum_{i=1}^{k} h_i \int_R b_i u dxdy,$$

where

$$u^* = a + h_{k+1} \int_R b_{k+1} u dxdy.$$

Then since u^* is non-negative and non-decreasing and Theorem 5.1.21 holds for $n = k$, we have

$$u \le u^* \alpha_k(x, y) = \alpha_k(x, y)\left(a + h_{k+1} \int_R b_{k+1} u dxdy\right). \tag{5.1.127}$$

Applying Theorem 5.1.21 to (5.1.127), we have

$$\phi(x, y) \le a(x, y)\alpha_{k+1}(x, y).$$

Thus Theorem 5.1.23 holds for $n = k + 1$. Since the theorem holds for $n = 1$, it follows by finite induction that it holds for all n. □

The next theorem is a generalization of Theorem 5.1.19.

Theorem 5.1.24 (Kasture-Deo [312]) *Let conditions (i), (ii), and inequality (5.1.124) of Theorem 5.1.22 hold. Further, let $a(x, y)$ and $h_i(x, y)$ be non-decreasing on D with $h_i \ge 1$ there. For $j = 0, 1, 2, \cdots, n$, define a sequence of function $\psi_j(x, y)$ inductively on D as follows,*

$$\begin{cases} \psi_0 = 1, \\ \psi_i(x, y) = \psi_{j-1}^2(x, y) h_j(x, y) v_j(x, y), \ j = 1, \cdots, n, \end{cases} \tag{5.1.128}$$

where $v_j(x, y)$ *is either the solution of the characteristic initial value problem*

$$\begin{cases} v_{st} - \psi_{j-1}(s,t)b_j(s,t)v(s,t) = 0, \\ v(s,y) = v(x,t) = 1, \end{cases}$$

or

$$v_j(x,y) = \exp\Big(\int_R \psi_{j-1}(s,t)h_j(s,t)b_j(s,t)dsdt\Big). \tag{5.1.129}$$

Then

$$u(x,y) \le a(x,y)\psi_n(x,y). \tag{5.1.130}$$

Proof The proof of this theorem, with v_j defined by (5.1.129), is similar to that of Theorem 5.1.23 by using Theorem 5.1.19 in place of Theorem 5.1.18. If v_j is defined by (5.1.129), the proof follows on the lines of Theorem 5.1.23 by using Theorem 5.1.21. □

Theorem 5.1.25 (Pachpatte [477]) *Suppose* $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$ *and* $\sigma(x,y)$ *are non-negative continuous functions on a domain D. Let* $P_0(x_0, y_0)$ *and* $P(x, y)$ *be two points in D such that* $(x - x_0)\cdot(y - y_0) > 0$ *and let R be the rectangular region whose opposite corners are the points* P_0 *and P. Let* $v(s,t;x,y)$ *be the solution of the characteristic initial value problem*

$$L[v] = v_{st} - [b(s,t) + c(s,t)]v = 0,\ v(s,y) = v(x,t) = 1, \tag{5.1.131}$$

and let D^+ *be a connected sub-domain of D which contains P and on which* $v > 0$*, Then if* $R \subset D^+$*, and* $u(x,y)$ *satisfies*

$$\begin{aligned} u(x,y) \le\ & a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)u(s,t)dsdt \\ & + \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\Big[\sigma(s,t) + \int_{x_0}^{s}\int_{y_0}^{t} c(\xi,\eta)u(\xi,\eta)d\xi d\eta\Big]dsdt, \end{aligned} \tag{5.1.132}$$

then

$$\begin{aligned} u(x,y) \le\ & a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\Big[a(s,t) + \sigma(s,t) \\ & + \int_{x_0}^{s}\int_{y_0}^{t} \Big\{a(\xi,\eta)c(\xi,\eta) + b(\xi,\eta)[a(\xi,\eta) + \sigma(\xi,\eta)]\Big\} \\ & \times v(\xi,n;s,t)d\xi d\eta\Big]dsdt. \end{aligned} \tag{5.1.133}$$

Proof Define a function $\phi(x, y)$ such that

$$\begin{cases} \phi(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)u(s,t)dsdt \\ \qquad\qquad + \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\left[\sigma(s,t) + \int_{x_0}^{s}\int_{y_0}^{t} c(\xi,\eta)u(\xi,\eta)d\xi d\eta\right]dsdt, \\ \phi(x_0,y) = \phi(x,y_0) = 0, \end{cases} \tag{5.1.134}$$

then

$$\phi_{xy}(x,y) = b(x,y)\left[u(x,y) + \sigma(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} c(\xi,\eta)u(\xi,\eta)d\xi d\eta\right], \tag{5.1.135}$$

which, in view of (5.1.132), implies

$$\begin{aligned} \phi_{xy}(x,y) \le\ & b(x,y)\Big[a(x,y) + \sigma(x,y) + \phi(x,y) \\ & + \int_{x_0}^{x}\int_{y_0}^{y} c(\xi,\eta)[a(\xi,\eta) + \phi(\xi,\eta)]d\xi d\eta\Big]. \end{aligned} \tag{5.1.136}$$

If we put

$$\begin{cases} \psi(x,y) = \phi(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} c(\xi,\eta)\,[a(\xi,\eta) + \phi(\xi,\eta)]\,d\xi d\eta, \\ \psi(x_0,y) = \psi(x,y_0) = 0, \end{cases} \tag{5.1.137}$$

then

$$\psi_{xy}(x,y) = \phi_{xy}(x,y) + c(x,y)[a(x,y) + \phi(x,y)]. \tag{5.1.138}$$

Using $\phi_{xy}(x, y) \le b(x, y)[a(x, y) + \sigma(x, y) + \psi(x, y)]$ from (5.1.136) and $\phi(x, y) \le \psi(x, y)$ from (5.1.137) in (5.1.138), we have

$$\begin{aligned} \psi_{xy}(x,y) \le\ & [b(x,y) + c(x,y)]\psi(x,y) + a(x,y)c(x,y) \\ & + b(x,y)[a(x,y) + \sigma(x,y)], \end{aligned}$$

i.e.,

$$\begin{aligned} L(\psi) =\ & \psi_{xy}(x,y) - [b(x,y) + c(x,y)]\psi(x,y) \\ & \le a(x,y)c(x,y) + b(x,y)[a(x,y) + \sigma(x,y)]. \end{aligned} \tag{5.1.139}$$

The operator L is self-adjoint and hyperbolic. For any twice continuously differentiable ψ and v, the operator L satisfies the identity

$$vL[\psi] - \psi L[v] = -(\psi v_y)_x + (v\psi_x)_y.$$

Let P_0 and P be any points as in Theorem 5.1.11 and label the directed sides and corners of the rectangle R as shown in Fig. 5.2.

Using s and t as the independent variables, integrating the identity over R and using Green's theorem, we obtain

$$\begin{aligned}\int_R\int (vL[\psi] - \psi L[v])dsdt &= -\int_{C_1+C_2+C_3+C_4}(v\psi_s ds + \psi v_t dt)\\ &= -\int_{C_1+C_4} v\psi_s ds - \int_{C_2+C_3}\psi v_t dt\end{aligned}$$

which holds for any functions in C^2.

For the particular function ψ defined earlier, we have $\psi = 0$ on C_3 and $\psi = \psi_s = 0$ on C_4; so the right-hand side reduces to

$$-\int_{C_1} v\psi_s ds - \int_{C_2}\psi v_t dt. \tag{5.1.140}$$

Now suppose v satisfies

$$\begin{cases} L[v] = v_{st} - [b(s,t) + c(s,t)]v = 0,\\ \quad v = 1 \ \text{ on } \ C_1,\\ \quad v_t = 0 \ \text{ on } \ C_2.\end{cases} \tag{5.1.141}$$

Then it follows from (5.1.141) that

$$v = 1 \ \text{ on } \ C_2. \tag{5.1.142}$$

Since $v \geq 0$ on R and $\psi(P_1) = 0$, by using (5.1.139), identity (5.1.140) becomes

$$\psi(P) \leq \int\int_R v[a(s,t)c(s,t) + b(s,t)(a(s,t) + \sigma(s,t))]dsdt,$$

i.e.,

$$\psi(x,y) \leq \int_{x_0}^{x}\int_{y_0}^{y}[a(s,t)c(s,t) + b(s,t)(a(s,t) + \sigma(s,t))]v(s,t;x,y)dsdt.$$

Substituting this value of $\psi(x, y)$ in (5.1.136), we obtain

$$\phi_{xy}(x, y) \leq b(x, y)\Big[a(x, y) + \sigma(x, y) + \int_{x_0}^{x}\int_{y_0}^{y}[a(s, t)c(s, t) + b(s, t)(a(s, t) + \sigma(s, t))v(s, t; x, y)]dsdt\Big],$$

which implies

$$\phi_{xy}(x, y) \leq \int_{x_0}^{x}\int_{y_0}^{y} b(s, t)\Big[a(s, t) + \sigma(s, t) + \int_{x_0}^{s}\int_{y_0}^{t}[a(\xi, \eta)c(\xi, \eta) + b(\xi, \eta)(a(\xi, \eta) + \sigma(\xi, \eta))v(\xi, \eta; s, t)d\xi, \eta]\Big]dsdt.$$

Now substituting this value of $\phi(x, y)$ in (5.1.132), we derive the desired bound in (5.1.133). □

The method of proof of this theorem is along the line given for the one variable case and involves a second order partial differential inequality which is integrated by using Riemann's method (see, [602]). The generalization in (5.1.133) of the exponential function in (1.2.118) is the Riemann function $v(s, t; x, y)$ relative to the point $P(x, y)$ for the self-adjoint operator L, whose existence is well-known. There is a sub-domain D^+ containing P on which $v > 0$ since $v = 1$ on the vertical and horizontal lines through P and since v is continuous.

We note that as in [603], Theorem 5.1.25 in the special case when $a = 0$, $\sigma = 0$ and $a =$ constant can be used is some applications.

A slightly different version of Theorem 5.1.25 which can be used in some applications is embodied in the following theorem.

Theorem 5.1.26 (Pachpatte [477]) *Suppose $u(x, y)$, $a(x, y)$, $b(x, y)$, $c(x, y)$ and $k(x, y)$ are non-negative continuous functions on a domain D. Let $P_0(x_0, y_0)$ and $P(x, y)$ be two points in D such that $(x - x_0)(y - y_0) > 0$ and let R be the rectangular region whose opposite corners are the points P_0 and P. Let $v(s, t; x, y)$ and $w(s, t; x, y)$ be the solutions of the characteristic initial value problems*

$$L[v] = v_{st} - [b(s, t) + c(s, t) + k(s, t)]v = 0, \ v(s, y) = v(x, t) = 1,$$

and

$$M[w] = w_{st} - [b(s, t) - c(s, t)]w = 0, \ w(s, y) = w(x, t) = 1$$

respectively, and let D^+ be a connected sub-domain of D which contains P and on which $v > 0$ and $w > 0$. Then, if $R \subset D^+$ and $u(x, y)$ satisfies

$$u(x, y) \leq a(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} b(s, t)u(s, t)dsdt + \int_{x_0}^{x} \int_{y_0}^{y} c(s, t) \left(\int_{x_0}^{s} \int_{y_0}^{t} k(\xi, \eta)u(\xi, \eta)d\xi d\eta \right) dsdt, \quad (5.1.143)$$

then

$$u(x, y) \leq a(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} w(s, t; x, y)\Big[a(s, t)b(s, t) + c(s, t) \times \int_{x_0}^{s} \int_{y_0}^{t} a(\xi, \eta)(b(\xi, \eta) + k(\xi, \eta))u(\xi, \eta; s, t)d\xi d\eta\Big]dsdt. \quad (5.1.144)$$

Proof Define a function $\phi(x, y)$ such that

$$\begin{cases} \phi = \int_{x_0}^{x} \int_{y_0}^{y} b(s, t)u(s, t)dsdt + \int_{x_0}^{x} \int_{y_0}^{y} c(s, t) \left(\int_{x_0}^{s} \int_{y_0}^{t} k(\xi, \eta)u(\xi, \eta)d\xi d\eta \right) dsdt, \\ \phi(x, y_0) = \phi(x_0, y) = 0. \end{cases}$$

Then we have

$$\phi_{xy}(x, y) = b(x, y)u(x, y) + c(x, y) \left(\int_{x_0}^{x} \int_{y_0}^{y} k(\xi, \eta)u(\xi, \eta)d\xi d\eta \right),$$

which, in view of (5.1.143), implies

$$\phi_{xy}(x, y) \leq b(x, y)[a(x, y) + \phi(x, y)] + c(x, y) \left(\int_{x_0}^{x} \int_{y_0}^{y} k(\xi, \eta)[a(\xi, \eta) + \phi(\xi, \eta)]d\xi d\eta \right).$$

Adding $c(x, y)\phi(x, y)$ to both sides of the above inequality, we have

$$\phi_{xy}(x, y) + c(x, y)\phi(x, y) \leq b(x, y)[a(x, y) + \phi(x, y)] + c(x, y)\Big[\phi(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} k(\xi, \eta)[a(\xi, \eta) + \phi(\xi, \eta)]d\xi d\eta\Big]. \quad (5.1.145)$$

If we put

$$\begin{cases} \psi(x, y) = \phi(x, y) + \displaystyle\int_{x_0}^{x} \int_{y_0}^{y} k(\xi, \eta)[a(\xi, \eta) + \phi(\xi, \eta)]d\xi d\eta, \\ \psi(x_0, y) = \psi(x, y_0) = 0, \end{cases} \tag{5.1.146}$$

then we obtain

$$\psi_{xy}(x, y) = \phi_{xy}(x, y) + k(x, y)[a(x, y) + \phi(x, y)]. \tag{5.1.147}$$

Using $\phi_{xy}(x, y) \leq b(x, y)[a(x, y) + \phi(x, y)] + c(x, y)\psi(x, y)$ from (5.1.145) and $\phi(x, y) \leq \psi(x, y)$ from (5.1.146) in (5.1.147), we have

$$\psi_{xy}(x, y) \leq [b(x, y) + c(x, y) + k(x, y)]\psi(x, y) + a(x, y)[b(x, y) + k(x, y)],$$

i.e.,

$$\begin{aligned} L[\psi] &= \psi_{xy}(x, y) - [b(x, y) + c(x, y) + k(x, y)]\psi(x, y) \\ &\leq a(x, y)[b(x, y) + k(x, y)]. \end{aligned}$$

Now following the same argument as in the proof of Theorem 5.1.25, we can obtain

$$\Psi(x, y) \leq \int_{x_0}^{x} \int_{y_0}^{y} a(s, t)[b(s, t) + k(s, t)]v(s, t; x, y)dsdt$$

which, substituted in (5.1.145), gives us

$$\begin{aligned} M[\phi] &= \phi_{xy}(x, y) - [b(x, y) - c(x, y)]\phi(x, y) \\ &\leq a(x, y)b(x, y) + c(x, y) \int_{x_0}^{x} \int_{y_0}^{y} a(s, t)[b(s, t) + k(s, t)]v(s, t; x, y)dsdt. \end{aligned}$$

Again following the similar argument as in the proof of Theorem 5.1.25, we obtain

$$\begin{aligned} \phi(x, y) \leq & \int_{x_0}^{x} \int_{y_0}^{y} w(s, t; x, y)\Big[a(s, t)b(s, t) + c(s, t) \\ & \times \int_{x_0}^{s} \int_{y_0}^{t} a(\xi, \eta)(b(\xi, \eta) + k(\xi, \eta))v(\xi, \eta; s, t)d\xi d\eta\Big]dsdt \end{aligned}$$

which, substituted in (5.1.143), gives us the desired bound in (5.1.144). □

In the special case when $b = 0$, the inequality established in Theorem 5.1.26 reduces to another inequality which can be used in some applications.

Theorem 5.1.27 (Pachpatte [477]) *Suppose $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$ and $h(x,y)$ are non-negative continuous functions on a domain D. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two points in D such that $(x-x_0)\cdot(y-y_0) > 0$ and let R be the rectangular region whose opposite corners are the points P_0 and P. Let $v(s,t;x,y)$ and $w(s,t;x,y)$ be the solutions of the characteristic initial value problem*

$$L[v] = v_{st} - [b(s,t) + c(s,t) + h(s,t)]v = 0,\ v(s,y) = v(x,t) = 1, \tag{5.1.148}$$

and

$$M[w] = w_{st} - b(s,t)w = 0,\ w(s,y) = w(x,t) = 1, \tag{5.1.149}$$

respectively, and let D^+ be a connected sub-domain of D which contains P and on which $v > 0$ and $w > 0$. Then if $R \subset D^+$ and $u(x,y)$ satisfies

$$\begin{aligned} u(x,y) \le a(x,y) &+ \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)u(s,t)dsdt \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} c(\xi,\eta)u(\xi,\eta)d\xi d\eta\right)dsdt + \int_{x_0}^{x}\int_{y_0}^{y} b(s,t) \\ &\times\left(\int_{x_0}^{s}\int_{y_0}^{t} c(\xi,\eta)\left(\int_{x_0}^{\xi}\int_{y_0}^{\eta} h(\alpha,\beta)u(\alpha,\beta)d\alpha d\beta\right)d\xi d\eta\right)dsdt, \end{aligned} \tag{5.1.150}$$

then

$$\begin{aligned} u(x,y) \le a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\Big[a(s,t) &+ \int_{x_0}^{s}\int_{y_0}^{t} w(\xi,\eta;s,t) \\ \times\Big[a(\xi,\eta)b(\xi,\eta) &+ c(\xi,\eta)\{a(\xi,\eta) + \int_{x_0}^{\xi}\int_{y_0}^{\eta} v(\alpha,\beta;\xi,\eta) \\ &\times a(\alpha,\beta)[b(\alpha,\beta)+c(\alpha,\beta)+h(\alpha,\beta)]d\alpha d\beta\}\Big]d\xi d\eta\Big]dsdt. \end{aligned} \tag{5.1.151}$$

Proof The proof of Theorem 5.1.27 follows from those of Theorem 5.1.25 with suitable modifications. Thus we omit the details. □

Theorem 5.1.28 (Pachpatte [477]) *Suppose $u(x,y)$, $a(x,y)$, $b(x,y)$, $c(x,y)$, $h(x,y)$, $p(x,y)$ and $q(x,y)$ are non-negative continuous functions on a domain D. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two points in D such that $(x-x_0)(y-y_0) > 0$ and R be the*

rectangular region whose opposite corners are the points P_0 and P. Let $v(s,t;x,y)$ and $w(s,t;x,y)$, and $e(s,t;x,y)$ be the solutions of the characteristic initial value problems

$$\begin{cases} L[v] = v_{st} - [b(s,t)+c(s,t)+h(s,t)+p(s,t)+q(s,t)]v = 0, \\ v(s,y) = v(x,t) = 1; \\ M[w] = w_{st} - [b(s,t)+c(s,t)+h(s,t)-p(s,t)]w = 0, \\ w(s,y) = w(x,t) = 1; \end{cases} \tag{5.1.152}$$

and

$$N[e] = e_{st} - [b(s,t)-c(s,t)]e = 0, \; e(s,y) = e(x,t) = 1, \tag{5.1.153}$$

respectively, and let D^+ be a connected sub-domain of D which contains P and on which $v > 0$ and $w > 0$. Then if $R \subset D^+$ and $u(x,y)$ satisfies

$$\begin{aligned} u(x,y) \le a(x,y) &+ \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)u(s,t)dsdt \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} h(\xi,\eta)u(\xi,\eta)d\xi d\eta\right)dsdt + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t) \\ &\times\left(\int_{x_0}^{s}\int_{y_0}^{t} p(\xi,\eta)\left(\int_{x_0}^{\xi}\int_{y_0}^{\eta} q(\alpha,\beta)u(\alpha,\beta)d\alpha d\beta\right)d\xi d\eta\right)dsdt, \end{aligned} \tag{5.1.154}$$

then

$$\begin{aligned} u(x,y) \le a(x,y) &+ \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)e(s,t;x,y)\Big[a(s,t)b(s,t) + c(s,t)\int_{x_0}^{s}\int_{y_0}^{t} w(\xi,\eta;s,t) \\ &\times\Big[a(\xi,\eta)(b(\xi,\eta)+h(\xi,\eta)) + p(\xi,\eta)\int_{x_0}^{\xi}\int_{y_0}^{\eta} v(\alpha,\beta;\xi,\eta)a(\alpha,\beta) \\ &\times[b(\alpha,\beta)+h(\alpha,\beta)+q(\alpha,\beta)]d\alpha d\beta\Big]d\xi d\eta\Big]dsdt. \end{aligned} \tag{5.1.155}$$

Proof The proof of Theorem 5.1.28 follows from those of Theorems 5.1.25 and 5.1.26 with suitable modifications. Thus we omit the details. □

We note that, in Theorems 5.1.25–5.1.28, the functions $v(s,t;x,y)$, $w(s,t;x,y)$ and $e(s,t;x,y)$ are the Riemann functions relative to the point $P(x,y)$ for the self-adjoint operators L, M, and N respectively. The existence and continuity of the Riemann function is well-known and may be demonstrated by the method of

successive approximation (see, [157]). Properties and specific examples of Riemann function are discussed by Copson in [149].

Now we begin to introduce hyperbolic integro-differential inequalities which can be used in the analysis of a class of hyperbolic integro-differential equations.

Theorem 5.1.29 (Pachpatte [477]) *Suppose that $u(x,y)$, $u_{xy}(x,y)$, $a(x,y)$ and $b(x,y)$ are non-negative continuous functions on a domain D. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two points in D such that $(x-x_0)\cdot(y-y_0) > 0$ and let R be the rectangular region whose opposite corners are the points P_0 and P. Let $v(s,t;x,y)$ be the solution of the characteristic initial value problem*

$$L[v] = v_{st} - [1 + b(s,t)]v = 0,\ v(s,y) = v(x,t) = 1, \tag{5.1.156}$$

and let D^+ be a connected sub-domain of D which contains P and on which $v > 0$. Then if $R \subset D^+$, and $u(x,y)$ satisfies

$$u_{xy}(x,y) \le a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\Big(u(s,t) + u_{st}(s,t)\Big)dsdt, \tag{5.1.157}$$

then

$$\begin{aligned} u(x,y) \le h(x,y) + \int_{x_0}^{x}\int_{y_0}^{y}\Bigg\{ & a(s,t) + \int_{x_0}^{s}\int_{y_0}^{t} b(s,t) \\ & \times\Bigg[a(\xi,\eta) + h(\xi,\eta) + \int_{x_0}^{\xi}\int_{y_0}^{\eta} b(\xi,\eta)v(\xi,\eta;s,t) \\ & \times\Bigg(b(\alpha,\beta)\Big(a(\alpha,\beta) + h(\alpha,\beta)\Big) + a(\alpha,\beta)\Bigg)d\alpha d\beta\Bigg]d\xi d\eta\Bigg\}\, dsdt, \end{aligned} \tag{5.1.158}$$

where $h(x,y) = u(x,y_0) + u(x_0,y) - u(x_0,y_0)$ is a non-negative continuous functions on D.

Proof Define

$$\begin{cases} \phi_{xy} = \displaystyle\int_{x_0}^{x}\int_{y_0}^{y} b(s,t)[u(s,t) + u_{st}(s,t)]dsdt, \\ \phi(x,y_0) = \phi(x_0,y) = 0, \end{cases}$$

then we have

$$\phi_{xy}(x,y) = b(x,y)[u(x,y) + u_{xy}(x,y)]. \tag{5.1.159}$$

Using the definition of $\phi(x, y)$, (5.1.157) can be restated as

$$\phi_{xy} \leq a(x, y) + \phi(x, y). \tag{5.1.160}$$

Integrating both sides of (5.1.160) on R, we obtain

$$u(x, y) \leq h(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} [a(s, t) + \phi(s, t)]dsdt. \tag{5.1.161}$$

Using (5.1.161) in (5.1.159), we derive

$$\begin{aligned}\phi_{xy}(x, y) \;\leq\; & b(x, y)\Big[a(x, y) + h(x, y) + \phi(x, y) \\ & + \int_{x_0}^{x} \int_{y_0}^{y} [a(s, t) + \phi(s, t)]dsdt\Big].\end{aligned} \tag{5.1.162}$$

Define

$$\begin{cases} \psi(x, y) = \phi(x, y) + \displaystyle\int_{x_0}^{x} \int_{y_0}^{y} [a(s, t) + \phi(s, t)]dsdt, \\ \psi(x_0, y) = \psi(x, y_0) = 0, \end{cases} \tag{5.1.163}$$

then

$$\psi_{xy}(x, y) = \phi_{xy}(x, y) + a(x, y) + \phi(x, y). \tag{5.1.164}$$

Using $\phi_{xy}(x, y) \leq b(x, y)[a(x, y) + h(x, y) + \psi(x, y)]$ from (5.1.162) and $\phi(x, y) \leq \psi(x, y)$ from (5.1.163) in (5.1.164), we have

$$\psi_{xy}(x, y) \leq b(x, y)[a(x, y) + h(x, y) + \psi(x, y)] + a(x, y) + \psi(x, y),$$

i.e.,

$$\begin{aligned} L[\psi] &= \psi_{xy}(x, y) - [1 + b(x, y)]\psi(x, y) \\ &\leq b(x, y)[a(x, y) + h(x, y)] + a(x, y). \end{aligned} \tag{5.1.165}$$

Now following the same steps as in the proof of Theorem 5.1.25, we can obtain

$$\psi(x, y) \leq \int_{x_0}^{x} \int_{y_0}^{y} v(s, t; x, y)[b(s, t)(a(s, t) + h(s, t)) + a(s, t)]dsdt.$$

Substituting the above inequality in (5.1.162), we obtain

$$\phi_{xy}(x,y) \leq b(x,y)\Big[a(x,y) + h(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} v(s,t;x,y) \\ \times[b(s,t)[a(s,t) + h(s,t)] + a(s,t)]dsdt\Big]$$

which implies

$$\phi(x,y) \leq \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)\Big[a(s,t) + h(s,t) + \int_{x_0}^{s}\int_{y_0}^{t} v(\xi,\eta;s,t) \\ \times[b(\xi,\eta)[a(\xi,\eta) + h(\xi,\eta)] + a(\xi,\eta)]d\xi d\eta\Big]dsdt.$$

Now substituting the above inequality in (5.1.160) and integrating both sides on R, we can obtain (5.1.158). □

Theorem 5.1.30 (Pachpatte [477]) *Suppose $u(x,y)$, $u_{xy}(x,y)$, $a(x,y)$, $c(x,y)$ and $p(x,y)$ are non-negative continuous functions on a domain D. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two points in D such that $(x-x_0)\cdot(y-y_0) > 0$ and let R be the rectangular region whose opposite corners are the points P_0 and P. Let $v(s,t;x,y)$ and $w(s,t;x,y)$ be the solutions of the characteristic initial value problems*

$$L[v] = v_{st} - [1 + b(s,t) + c(s,t) + p(s,t)]v = 0,\ v(s,y) = v(x,t) = 1, \tag{5.1.166}$$

and

$$M[w] = w_{st} - [1 + b(s,t) - c(s,t)]w = 0,\ w(s,y) = w(x,t) = 1 \tag{5.1.167}$$

respectively, and let D^+ be a connected sub-domain of D which contains P and on which $v > 0$ and $w > 0$. Then if $R \subset D^+$ and $u(x,y)$ satisfies

$$u_{xy}(x,y) \leq a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)[u(s,t) + u_{st}(s,t)]dsdt + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t) \\ \times\left[\int_{x_0}^{s}\int_{y_0}^{t} p(\xi,\eta)[u(\xi,\eta) + u_{\xi\eta}(\xi,\eta)]d\xi d\eta)\right]dsdt, \tag{5.1.168}$$

then

$$u(x,y) \leq h(x,y) + \int_{x_0}^{x}\int_{y_0}^{y}\Big[a(s,t) + \int_{x_0}^{s}\int_{y_0}^{t}[b(s,t)(a(\xi,\eta) + c(\xi,\eta) + Q(\xi,\eta)) \\ \times c(\xi,\eta)\int_{x_0}^{\xi}\int_{y_0}^{\eta} p(\alpha,\beta)(a(\alpha,\beta) + h(\alpha,\beta) + Q(\alpha,\beta))d\alpha d\beta]d\xi d\eta\Big]dsdt \tag{5.1.169}$$

where

$$Q(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} w(s,t;x,y)\Big[a(s,t) + b(s,t)(a(s,t) + h(s,t))$$
$$+c(s,t)\int_{x_0}^{s}\int_{y_0}^{t} v(\xi,\eta;s,t)[(a(\xi,\eta) + h(\xi,\eta))$$
$$\times(b(\xi,\eta) + p(\xi,\eta)) + a(\xi,\eta)]d\xi d\eta\Big]dsdt$$

and $h(x,y) = u(x,y_0) + u(x_0,y) - u(x_0,y_0)$ *is a non-negative continuous function on D.*

Proof The proof follows from the proofs of Theorems 5.1.29 and 5.1.26 with suitable modifications, and we leave the details to the reader. □

Note that if in Theorems 5.1.29 and 5.1.30, a,b,c,d and $u(x,y_0), u(x_0,y)$ and $u(x_0,y_0)$ are known and $u(x,y)$ and $u_{xy}(x,y)$ are unknown functions; i.e., the inequalities established in Theorems 5.1.29 and 5.1.30 gives us the bounds in terms of the known functions which majores $u(x,y)$.

Ghoshal and Masood [227] obtained a further generalization of the inequality established by Snow [605] (see Theorem 5.1.12) which can be used in the analysis of a class of non-self-adjoint partial differential equations of the parabolic type. In the next theorem, we shall introduce a further generalization of this inequality (i.e., Theorem 5.1.12) which can be used in investigating the behavior of solutions of a class of non-self-adjoint partial integro-differential equations of the parabolic type.

Theorem 5.1.31 (Pachpatte [477]) *Suppose that* $u(x,y)$, $a(x,y)$,$b(x,y)$,$c(x,y)$, $p(x,y)$, $q(x,y)$ *and* $r(x,y)$ *are non-negative continuous functions on a domain D. Let* $P_0(x_0,y_0)$ *and* $P(x,y)$ *be two points in D such that* $(x-x_0)\cdot(y-y_0) > 0$ *and let R be the rectangular region whose opposite corners are the points* P_0 *and P. Let* $v(s,t;x,y)$ *be the solution of the characteristic initial value problem*

$$M[v] = 0 \tag{5.1.170}$$

where M is the adjoint operator of the operator L defined by

$$L[\psi] = \psi_{st} + a_1\psi_s + b_1\psi_t + c_1\psi \tag{5.1.171}$$

in which $a_1 = -bq$, $b_1 = -bp$, $c_1 = -(br + b + c)$. *Let* $w(s,t;x,y)$ *be the solution of the characteristic initial value problem*

$$N[w] = 0 \tag{5.1.172}$$

where N is the adjoint operator of the operator T defined by

$$T[\phi] = \phi_{st} + a_2\phi_s + b_2\phi_t + c_2\phi \tag{5.1.173}$$

in which $a_2 = -bq, b_2 = -bp, c_2 = -[rb - b]$. *The functions* $v(s,t;x,y)$ *and* $v(s,t;x,y)$ *are called the well-known Riemann functions for the partial differential operators L and T respectively and satisfy all the properties of Riemann functions for operators with continuous coefficients. Let* D^+ *be a connected sub-domain of D which contains P and on which* $v \geq 0$ *and* $w \geq 0$. *If* $R \subset D^+$ *and* $u(x,y)$ *satisfies*

$$u(x,y) \leq a(x,y) + p(x,y)\int_{x_0}^{x} b(s,y)u(s,y)ds$$
$$+q(x,y)\int_{y_0}^{y} b(x,t)u(x,t)dt + r(x,y)\int_{x_0}^{y}\int_{y_0}^{y} b(s,t)u(s,t)dsdt$$
$$+\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} b(\xi,\eta)u(\xi,\eta)d\xi d\eta\right)dsdt, \qquad (5.1.174)$$

then

$$u(x,y) \leq a(x,y) + p(x,y)\int_{x_0}^{x} b(s,y)u(s,y)ds + q(x,y)\int_{y_0}^{y} b(x,t)u(x,t)dt$$
$$+r(x,y)Q(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)Q(x,y)dsdt, \qquad (5.1.175)$$

where

$$Q(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} w(s,t;x,y)b(s,t)\Big[a(s,t)$$
$$+\int_{x_0}^{s}\int_{y_0}^{t} v(\xi,\eta;s,t)a(\xi,\eta)b(\xi,\eta)d\xi d\eta\Big]dsdt. \qquad (5.1.176)$$

Furthermore, if $q(x,y) = 0$, *then*

$$u(x,y) \leq a(x,y) + r(x,y)Q(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)Q(x,y)dsdt$$
$$+p(x,y)\Bigg[\int_{x_0}^{x} b(s,y)\Bigg(a(s,y) + r(s,y)Q(s,y)$$
$$+\int_{x_0}^{s}\int_{y_0}^{y} c(\xi,t)Q(\xi,t)d\xi dt\Bigg) \times \exp\left(\int_{s}^{x} b(\xi,y)p(\xi,y)d\xi\right)ds\Bigg]. \qquad (5.1.177)$$

Also, if $p(x,y)=0$*, then*

$$
\begin{aligned}
u(x,y) \le a(x,y) + r(x,y)Q(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} c(\xi,\eta)Q(\xi,\eta)d\xi d\eta \\
+p(x,y)\Bigg[\int_{y_0}^{y} b(x,t)\Bigg(a(x,t) + r(x,t)Q(x,t) \\
+\int_{x_0}^{x}\int_{y_0}^{t} c(\xi,\eta)Q(\xi,\eta)d\xi d\eta\Bigg)\exp\Bigg(\int_{t}^{y} b(x,\eta)p(x,\eta)d\eta\Bigg)dt\Bigg].
\end{aligned}
\tag{5.1.178}
$$

The function $Q(x,y)$ *involved in (5.1.177) and (5.1.178) is defined by (5.1.176).* □

Proof Define a function $\phi(x,y)$ such that

$$
\begin{cases}
\phi(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} b(s,t)u(s,t)dsdt, \\
\phi(x,y_0) = \phi(x_0,y) = 0,
\end{cases}
$$

then we have

$$\phi_{xy}(x,y) = b(x,y)u(x,y),$$

which, in view of (5.1.174), implies

$$
\begin{aligned}
\phi_{xy}(x,y) \le b(x,y)\Big[a(x,y) + p(x,y)\phi_y(x,y) + q(x,y)\phi_x(x,y) \\
+r(x,y)\phi(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\phi(s,t)dsdt\Big].
\end{aligned}
\tag{5.1.179}
$$

Adding $b(x,y)\phi(x,y)$ to both sides of the above inequality, we get

$$
\begin{aligned}
\phi_{xy}(x,y) + b(x,y)\phi(x,y) \le b(x,y)\Big[a(x,y) + p(x,y)\phi_y(x,y) + q(x,y)\phi_x(x,y) \\
+r(x,y)\phi(x,y) + \phi(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\phi(s,t)dsdt\Big].
\end{aligned}
\tag{5.1.180}
$$

If we put

$$
\begin{cases}
\psi(x,y) = \phi(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\phi(s,t)dsdt, & (5.1.181) \\
\psi(x_0,y) = \psi(x,y_0) = 0, & (5.1.182)
\end{cases}
$$

then

$$\psi_{xy}(x, y) = \phi_{xy}(x, y) + c(x, y)\psi(x, y). \tag{5.1.183}$$

Using

$$\phi_{xy}(x, y) \le b(x, y)\Big[a(x, y) + p(x, y)\phi_y(x, y) + q(x, y)\phi_x(x, y) \\ + r(x, y)\phi(x, y) + \psi(x, y)\Big]$$

from (5.1.179) and $\phi(x, y) \le \psi(x, y)$ from (5.1.181), we obtain

$$\psi_{xy}(x, y) \le b(x, y)\Big[a(x, y) + p(x, y)\psi_y(x, y) + q(x, y)\psi_x(x, y) \\ + r(x, y)\psi(x, y) + \psi(x, y)\Big] + c(x, y)\psi(x, y),$$

i.e.,

$$\begin{aligned} L[\psi] &= \psi_{xy}(x, y) + a_1(x, y)\psi_x(x, y) + b_1(x, y)\psi_y(x, y) + c_1(x, y)\psi(x, y) \\ &\le a(x, y)b(x, y) \end{aligned} \tag{5.1.184}$$

where

$$a_1 = -bq, \quad b_1 = -bp, \quad c_1 = -(br + b + c).$$

Now for any two twice continuously differentiable functions ψ and v, the operators L and M satisfy the identity

$$vL[\psi] - \psi M[v] = (a_1\psi v + \frac{v\psi_y}{2} - \frac{v_y\psi}{2})_x + (b_1\psi v + \frac{v\psi_x}{2} - \frac{\psi v_x}{2})_y \tag{5.1.185}$$

where M is the adjoint operator of L.

Let R be a rectangular region with corners $P_0(x_0, y_0)$, $P_1(x, y_0)$, $P(x, y)$ and $P_2(x_0, y)$ so that P_0P is the diagonal, as shown in Fig. 5.3.

Using s and t as the independent variables, integrating the identity (5.1.185) over R and using Green's theorem, we may obtain

$$\begin{aligned} &\int_R\int [vL[\psi] - \psi M[v]]dsdt \\ = &\int_{C_1+C_2+C_3+C_4} (a_1\psi v + \frac{v\psi_t}{2} - \frac{\psi v_t}{2}) - (b_1\psi v + \frac{v\psi_s}{2} - \frac{\psi v_s}{2})ds. \end{aligned}$$

Since ψ is zero on C_1 and C_4 and also ds does not very on C_2 and dt does not vary on C_3, we get

$$\int_R \int [vL[\psi] - \psi M[v]] dsdt$$
$$= \int_{C_2} (a_1 \psi v + \frac{v\psi_t}{2} - \frac{\psi v_t}{2}) - \int_{C_3} (b_1 \psi v + \frac{v\psi_s}{2} - \frac{\psi v_s}{2}) ds.$$

Integrating right-hand side by parts along the characteristic segments C_2 and C_3 to eliminate partial derivatives of ψ, we obtain

$$\int \int_R \Big(vL[\psi] - \psi M[v] \Big) dsdt = \int_{C_2} (a_1 v - v_t) \psi dt - \int_{C_3} (b_1 v - v_s) \psi ds$$
$$+ \psi(P)v(P) - \frac{\psi(P_2)v(P_2)}{2} - \frac{\psi(P_1)v(P_1)}{2}$$
$$= \int_{C_2} (a_1 v - v_t) \psi dt - \int_{C_3} (b_1 v - v_s) \psi ds + \psi(P)v(P). \tag{5.1.186}$$

Now since $v(s, t; x, y)$ is the solution of the characteristic initial value problem $M[v] = 0$, it is by definition the Riemann function $v(s, t; x, y) = v(s, t)$ associated with the partial differential equation $L[\psi] = 0$ such that

$$\begin{cases} v(x, y; x, y) = v(x, y) = v(P) = 1, & (5.1.187) \\ v_t = a_1 v \ on \ C_2, \ \ v(x, t) = \exp\left(\int_y^t a_1(x, \eta) d\eta \right), & (5.1.188) \end{cases}$$

and

$$v_s = b_1 v \ on \ \ C_3, \ \ v(s, y) = \exp\left(\int_x^s b_1(\xi, y) d\xi \right). \tag{5.1.189}$$

So we get from the identity (5.1.186)

$$\psi(x, y) = \int_{x_0}^x \int_{y_0}^y v(s, t; x, y) L[\psi] dsdt$$

or

$$\psi(x, y) \le \int_{x_0}^x \int_{y_0}^y v(s, t; x, y) a(s, t) b(s, t) dsdt$$

which gives us

$$\begin{aligned} T[\phi] &= \phi_{xy}(x,y) + a_2(x,y)\psi_x(x,y) + b_2(x,y)\psi_y(x,y) + c_2(x,y)\psi(x,y) \\ &\leq b(x,y)\left[a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} v(s,t;x,y)a(s,t)b(s,t)dsdt\right], \end{aligned}$$

where

$$a_2 = -bq,\ b_2 = -bp,\ c_2 = -[rb - b].$$

Now following the same argument as above, we can obtain

$$\begin{aligned} \phi(x,y) \ &\leq \ \int_{x_0}^{x}\int_{y_0}^{y} w(s,t;x,y)b(s,t) \\ &\quad \times\Big[a(s,t) + \int_{x_0}^{s}\int_{y_0}^{t} v(\xi,\eta;s,t)a(\xi,\eta)b(\xi,\eta)d\xi d\eta\Big]dsdt \\ &= Q(x,y). \end{aligned}$$

Thus substituting this value of $\phi(x,y)$ in (5.1.174), we obtain (5.1.175).

Now, let $q(x,y) = 0$ and

$$h(x,y) = a(x,y) + r(x,y)Q(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)Q(s,t)dsdt, \quad (5.1.190)$$

then inequality (5.1.175) reduces to

$$u(x,y) \leq h(x,y) + p(x,y)\int_{x_0}^{x} b(s,y)u(s,y)ds. \quad (5.1.191)$$

The inequality (5.1.191) may be treated as an one-dimensional Bellmon-Gronwall inequality for any fixed y between y_0 to y, which implies

$$u(x,y) \leq h(x,y) + p(x,y)\left[\int_{x_0}^{x} b(s,y)h(s,y)\exp\Big(\int_{s}^{x} b(\xi,y)p(\xi,y)d\xi\Big)ds\right]. \quad (5.1.192)$$

Therefore (5.1.177) follows from (5.1.192) and (5.1.190).

Further, substituting $p = 0$ in (5.1.175) and following the similar argument as above, we finally obtain (5.1.178). □

We note that when $p(x,y) = q(x,y) = 0$, the inequality established in Theorem 5.1.31, reduces to another new inequality which can be used in the analysis of a class of nonlinear self-adjoint hyperbolic integrodifferential equations. If $a(x,y) = 0$ in (5.1.177) or (5.1.178), then we obtain $u(x,y) = 0$.

The next result, due to Pachpatte [480], is to present a partial integral inequality involving two independent variables.

Theorem 5.1.32 (Pachpatte [480]) *Suppose that the following assumptions* (H_1) *and* (H_2) *are true.*

(H_1) $u(x,y), a(x,y), b(x,y), c(x,y), p(x,y)$ *and* $q(x,y)$ *are real-valued non-negative continuous functions defined on a domain D.*

(H_2) $P_0(x_0,y_0)$ *and* $P(x,y)$ *are two points in D such that* $(x-x_0)(y-y_0)>0$ *and R the rectangular region whose opposite corners are the points* P_0 *and P.*

Let $v(s,t;x,y)$ *and* $w(s,t;x,y)$ *be the solutions of the characteristic initial value problems*

$$\begin{cases} L[v] = v_{st} - [p(s,t) + b(s,t)(c(s,t) + q(s,t))]v = 0, \\ v(s,y) = v(x,t) = 1, \end{cases} \tag{5.1.193}$$

and

$$\begin{cases} M[w] = w_{st} - [b(s,t)c(s,t) - p(s,t)]w = 0, \\ w(s,y) = w(x,t) = 1, \end{cases} \tag{5.1.194}$$

respectively, and let D^+ *be a connected sub-domain of D which contains P and on which* $v>0$ *and* $w>0$. *Then if* $R \subset D^+$ *and* $u(x,y)$ *satisfies*

$$\begin{aligned} u(x,y) \le a(x,y) + b(x,y)\Big[&\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)u(s,t)dsdt \\ &+ \int_{x_0}^{x}\int_{y_0}^{y} p(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} q(\xi,\eta)u(\xi,\eta)d\xi d\eta\right)dsdt\Big], \end{aligned} \tag{5.1.195}$$

then

$$\begin{aligned} u(x,y) \le a(x,y) + b(x,y)\Big[&\int_{x_0}^{x}\int_{y_0}^{y} w(s,t;x,y)\Big(a(s,t)c(s,t) + p(s,t)\int_{x_0}^{s}\int_{y_0}^{t} a(\xi,\eta) \\ &\times [c(\xi,\eta) + q(\xi,\eta)]v(\xi,\eta;s,t)d\xi d\eta\Big)dsdt\Big]. \end{aligned} \tag{5.1.196}$$

Proof Define a function $\phi(x,y)$ such that

$$\begin{cases} \phi(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} c(s,t)u(s,t)dsdt + \int_{x_0}^{x}\int_{y_0}^{y} p(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} q(\xi,\eta)u(\xi,\eta)d\xi d\eta\right)dsdt, \\ \phi(x_0,y) = \phi(x,y_0) = 0, \end{cases} \tag{5.1.197}$$

then we have

$$\phi_{xy}(x, y) = c(x, y)u(x, y) + p(x, y) \int_{x_0}^{x} \int_{y_0}^{y} q(\xi, \eta)u(\xi, \eta)d\xi d\eta,$$

which, in view of (5.1.195), implies

$$\begin{aligned}\phi_{xy}(x, y) = c(x, y)\,[a(x, y) + b(x, y)\phi(x, y)] \\ +p(x, y)\left(\int_{x_0}^{x} \int_{y_0}^{y} q(\xi, \eta)[a(\xi, \eta) + b(\xi, \eta)\phi(\xi, \eta)]d\xi d\eta\right).\end{aligned}$$

Adding $p(x, y)\phi(x, y)$ to both sides of the above inequality, we have

$$\begin{aligned}&\phi_{xy}(x, y) + p(x, y)\phi(x, y) \\ &\leq c(x, y)\,[a(x, y) + b(x, y)\phi(x, y)] \\ &\quad +p(x, y)\left[\phi(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} q(\xi, \eta)[a\xi, \eta) + b(\xi, \eta)\phi(\xi, \eta)]d\xi d\eta\right].\end{aligned} \tag{5.1.198}$$

If we put

$$\begin{cases} \psi(x, y) =\phi(x, y) + \displaystyle\int_{x_0}^{x} \int_{y_0}^{y} q(\xi, \eta)[a\xi, \eta) + b(\xi, \eta)\phi(\xi, \eta)]d\xi d\eta, \\ \psi(x_0, y) =\psi(x, y_0) = 0, \end{cases} \tag{5.1.199}$$

then we obtain

$$\psi_{xy}(x, y) = \phi_{xy}(x, y) + q(x, y)[a(x, y) + b(x, y)\phi(x, y)]. \tag{5.1.200}$$

Using $\phi_{xy}(x, y) \leq c(x, y)[a(x, y) + b(x, y)\phi(x, y)] + p(x, y)\psi(x, y)$ from (5.1.198) and $\phi(x, y) \leq \psi(x, y)$ from (5.1.199) in (5.1.200), we obtain

$$\begin{aligned}\psi_{xy} \leq a(x, y)[c(x, y) + q(x, y)] \\ +[p(x, y) + b(x, y)(c(x, y) + q(x, y))]\psi(x, y),\end{aligned}$$

i.e.,

$$\begin{aligned}L[\psi] = \psi_{xy}(x, y) - [p(x, y) + b(x, y)(c(x, y) + q(x, y))]\psi(x, y) \\ \leq a(x, y)[c(x, y) + q(x, y)].\end{aligned} \tag{5.1.201}$$

The operator L is self-adjoint and hyperbolic. For any twice continuously differentiable ψ and v, the operator L satisfies the identity

$$vL[\psi] - \psi L[v] = -(\psi v_y)_x + (v\psi_x)_y. \tag{5.1.202}$$

Let P and P_0 be any points as in theorem and label the directed sides and corners of the rectangle R as shown in Fig. 5.1.

Using s and t as the independent variables, integrating the identity (5.1.200) over R and using Green's theorem, we obtain

$$\begin{aligned}\int\int_R (vL[\psi] - \psi L[v])dsdt &= -\int_{C_1+C_2+C_3+C_4} (v\psi_s ds + \psi v_t dt)\\ &= -\int_{C_1+C_4} v\psi_s ds - \int_{C_2+C_3} \psi v_t dt.\end{aligned}$$

This holds for any functions in C^2.

For the particular function ψ defined earlier, we have $\psi = 0$ on C_3 and $\psi = \psi_s = 0$ on C_4, so the right-hand side in the above identity reduces to

$$-\int_{C_1} v\psi_s ds - \int_{C_2} \psi v_t dt. \tag{5.1.203}$$

Now suppose v satisfies

$$\begin{cases} L[v] = v_{st} - [p(s,t) + b(s,t)(c(s,t) + q(s,t)]v = 0, & (5.1.204)\\ v = 1 \quad \text{on} \quad C_1, & (5.1.205)\\ v_t = 0 \quad \text{on} \quad C_2. & (5.1.206)\end{cases}$$

Then it follows from (5.1.205) and (5.1.206) that

$$v = 1 \quad \text{on} \quad C_2. \tag{5.1.207}$$

Since $v \geq 0$ on R and $\psi(P_1) = 0$, by using (5.1.201), identity (5.1.203) becomes

$$\psi(P) \leq \int\int_R v[a(s,t)[c(s,t) + q(s,t)]]dsdt,$$

i.e.,

$$\psi(x,y) \leq \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)[c(s,t) + q(s,t)]v(s,t;x,y)dsdt.$$

Substituting this bound on $\psi(x, y)$ in (5.1.198), we obtain

$$M[\phi] = \phi_{xy}(x, y) - [b(x, y)c(x, y) - p(x, y)]\phi(x, y)$$
$$\leq \left[a(x, y)c(x, y) + p(x, y)\int_{x_0}^{x}\int_{y_0}^{y} a(s, t)[c(s, t) + q(s, t)]v(s, t; x, y)dsdt\right].$$

Again following the same argument as above, we conclude

$$\phi(x, y) \leq \int_{x_0}^{x}\int_{y_0}^{y} w(s, t; x, y)\Big[a(s, t)c(s, t)$$
$$+p(s, t)\int_{x_0}^{s}\int_{y_0}^{t} a(\xi, \eta)[c(\xi, \eta) + q(\xi, \eta)]v(\xi, \eta; x, y)d\xi d\eta\Big]dsdt.$$

Therefore substituting this bound on $\phi(x, y)$ in (5.1.195), we can derive (5.1.196). □

Another interesting and useful generalization, due to Pachpatte [481], is the following theorem.

Theorem 5.1.33 (Pachpatte [480]) *Suppose* (H_1) *and* (H_2) *are true. Let* $v(s, t; x, y)$ *and* $w(s, t; x, y)$ *be the solutions of the characteristic initial value problem*

$$\begin{cases} L[v] = v_{st} - b(s, t)[p(s, t) + c(s, t) + q(s, t))]v = 0, \\ v(s, y) = v(x, t) = 1, \end{cases} \tag{5.1.208}$$

and

$$\begin{cases} M[w] = w_{st} - b(s, t)c(s, t)w = 0, \\ w(s, y) = w(x, t) = 1, \end{cases} \tag{5.1.209}$$

respectively, and let D^+ *be a connected sub-domain of* D *which contains* P *and on which* $v > 0$ *and* $w > 0$*. Then, if* $R \subset D^+$ *and* $u(x, y)$ *satisfies*

$$u(x, y) \leq a(x, y) + b(x, y)\Big[\int_{x_0}^{x}\int_{y_0}^{y} c(s, t)u(s, t)dsdt$$
$$+\int_{x_0}^{x}\int_{y_0}^{y} p(s, t)\left(u(s, t) + b(s, t)\int_{x_0}^{s}\int_{y_0}^{t} q(\xi, \eta)u(\xi, \eta)d\xi d\eta\right)dsdt\Big], \tag{5.1.210}$$

then

$$u(x,y) \le a(x,y) + b(x,y)\Big[\int_{x_0}^{x}\int_{y_0}^{y} w(s,t;x,y)\Big(a(s,t)c(s,t) + p(s,t) + b(s,t)p(s,t) \times \int_{x_0}^{s}\int_{y_0}^{t} a(\xi,\eta)[c(\xi,\eta) + p(\xi,\eta) + q(\xi,\eta)]v(\xi,\eta;s,t)d\xi d\eta\Big)dsdt\Big]. \tag{5.1.211}$$

Proof The proof of this theorem follows by the similar argument to that in the proof of Theorem 5.1.32 with suitable modifications. Hence we omit the details. ☐

The next theorem is also due to Pachpatte [481].

Theorem 5.1.34 (Pachpatte [481]) *Suppose the following assumptions (H_1)–(H_3) are true.*

(H_1) $u(x,y), a(x,y), b(x,y), c(x,y), p(x,y), q(x,y), r(x,y), h(x,y)$, and $g(x,y)$ are real-valued non-negative continuous functions defined on a domain D.
(H_2) $P_0(x_0,y_0)$ and $P(x,y)$ are two points in D such that $(x-x_0)/(y-y_0) \ge 0$ and R is the rectangular region whose opposite corners are points P_0 and P.
(H_3) The function $V(s,t;x,y)$ and $W(s,t;x,y)$ are the Riemann functions for the partial differential operators L and T, respectively, and satisfy all the properties of Riemann functions for operators with continuous coefficients.

Let $V(s,t;x,y)$ be the solution of the characteristic initial value problem

$$M[V] = 0, \tag{5.1.212}$$

where M is the adjoint operator of the operator L defined by

$$L[\Psi] = \Psi_{st} + a_1\Psi_s + a_2\Psi_t + a_3\Psi \tag{5.1.213}$$

in which $a_1 = -bcq,\ a_2 = -bcp,\ a_3 = -[g + bc(r+h)]$. Let $W(s,t;x,y)$ be the solution of the characteristic initial value problem

$$N[W] = 0, \tag{5.1.214}$$

where N is the adjoint operator of the operator T defined by

$$T[\phi] = \phi_{st} + b_1\phi_s + b_2\phi_t + b_3\phi \tag{5.1.215}$$

in which $b_1 = -bcq,\ b_2 = -bcp,\ b_3 = -bc(r-h)$. Let D^+ be a connected subdomain of D which contains P and on which $V \ge 0$ and $W \ge 0$. If $R \subset D^+$ and

$u(x, y)$ satisfies

$$u(x,y) \le a(x,y) + b(x,y)\Big[p(x,y)\int_{x_0}^{x} c(s,y)u(s,y)ds$$
$$+q(x,y)\int_{y_0}^{y} c(x,t)u(x,t)dt + r(x,y)\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)u(s,t)dsdt$$
$$+h(x,y)\int_{x_0}^{x}\int_{y_0}^{y} g(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} c(\xi,\eta)u(\xi,\eta)d\xi d\eta\right)ds\, dt\Big], \tag{5.1.216}$$

then

$$u(x,y) \le a(x,y) + b(x,y)\Big[p(x,y)\int_{x_0}^{x} c(s,y)u(s,y)ds + q(x,y)\int_{y_0}^{y} c(x,t)u(x,t)dt$$
$$+r(x,y)Q(x,y) + h(x,y)\int_{x_0}^{x}\int_{y_0}^{y} g(s,t)Q(s,t)ds\, dt\Big], \tag{5.1.217}$$

where

$$Q(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} W(s,t;x,y)c(s,t)\Big\{a(s,t) + b(s,t)h(s,t)$$
$$\times\left(\int_{x_0}^{s}\int_{y_0}^{t} V(\xi,\eta,s,t)a(\xi,\eta)c(\xi,\eta)d\xi\, d\eta\right)\Big\}ds\, dt. \tag{5.1.218}$$

Further, if $q(x, y) = 0$, then

$$u(x,y) \le f(x,y)$$
$$+b(x,y)p(x,y)\Big[\int_{x_0}^{x} c(s,y)f(s,y)\exp\left(\int_{s}^{x} c(\xi,y)b(\xi,y)p(\xi,y)d\xi\right)ds\Big], \tag{5.1.219}$$

where

$$f(x,y) = a(x,y) + b(x,y)\left[r(x,y)Q(x,y) + h(x,y)\int_{x_0}^{x}\int_{y_0}^{y} g(s,t)Q(s,t)ds\, dt\right]. \tag{5.1.220}$$

Again, if $p(x, y) = 0$*, then*

$$u(x, y) \leq f(x, y) + b(x, y)q(x, y)\Big[\int_{y_0}^{y} c(x, t)f(x, t) \times \exp\left(\int_{t}^{y} c(x, \eta)b(x, \eta)q(x, \eta)d\eta\right) dt\Big], \tag{5.1.221}$$

where $f(x, y)$ *is as defined in (5.1.220) in which the function* $Q(x, y)$ *is as defined in (5.1.218).*

Proof Define a function $\phi(x, y)$ such that

$$\phi(x, y) = \int_{x_0}^{x}\int_{y_0}^{y} c(\xi, \eta)u(\xi, \eta)d\xi\, d\eta, \quad \phi(x_0, y) = \phi(x, y_0) = 0,$$

then we have

$$\phi_{xy}(x, y) = c(x, y)u(x, y),$$

which, in view of the definition of $\phi(x, y)$ and (5.1.216), implies

$$\phi_{xy}(x, y) \leq c(x, y)\Big[a(x, y) + b(x, y)\Big(p(x, y)\phi_y(x, y) + q(x, y)\phi_x(x, y) + r(x, y)\phi(x, y) + h(x, y)\int_{x_0}^{x}\int_{y_0}^{y} g(s, t)\phi(s, t)ds\, dt\Big)\Big].$$

Adding $b(x, y)c(x, y)h(x, y)\phi(x, y)$ to both sides of the above inequality, we have

$$\begin{aligned}
&\phi_{xy}(x, y) + b(x, y)c(x, y)h(x, y)\phi(x, y)\\
&\leq a(x, y)c(x, y) + b(x, y)c(x, y)\Big[p(x, y)\phi_y(x, y) + q(x, y)\phi_x(x, y)\\
&\quad + r(x, y)\phi(x, y) + h(x, y)\{\phi(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s, t)\phi(s, t)ds\, dt\}\Big].
\end{aligned} \tag{5.1.222}$$

If we put

$$\Psi(x, y) = \phi(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s, t)\phi(s, t)ds\, dt, \quad \Psi(x_0, y) = \Psi(x, y_0) = 0, \tag{5.1.223}$$

then

$$\Psi_{xy}(x, y) = \phi_{xy}(x, y) + g(x, y)\phi(x, y). \tag{5.1.224}$$

Using

$$\phi_{xy}(x,y) \le a(x,y)c(x,y) + b(x,y)c(x,y)\Big\{p(x,y)\phi_y(x,y) + q(x,y)\phi_x(x,y)$$
$$+ r(x,y)\phi(x,y) + h(x,y)\Psi(x,y)\Big\}$$

from (5.1.222) and $\phi(x,y) \le \Psi(x,y),\ \phi_x(x,y) \le \Psi_x(x,y),\ \phi_y(x,y) \le a(x,y)c(x,y)$ from (5.1.223) in (5.1.224), we have

$$L[\Psi] = \Psi_{xy}(x,y) + a_1\Psi_x(x,y) + a_2\Psi_y(x,y) + a_3\Psi(x,y) \le a(x,y)c(x,y), \tag{5.1.225}$$

where a_1, a_2 and a_3 are as defined in (5.1.213).

Now for any two twice continuously differentiable functions Ψ and V, the operators L and M satisfy the identity

$$VL[\Psi] - \Psi M[V] = (a_1\Psi V + \frac{1}{2}V\Psi_y - \frac{1}{2}V_y\Psi)_x + (a_2\Psi V + \frac{1}{2}V\Psi_x - \frac{1}{2}\Psi V_x)_y, \tag{5.1.226}$$

where M is the adjoint operator of L. Let R be a rectangular region with corners $P_0(x_0, y_0),\ P_1(x, y_0), P(x,y)$ and $P_2(x_0, y)$, so that P_0P is the diagonal as shown in Fig. 5.5.

Using s and t as the independent variables, we integrate identity (5.1.226) over R and use Green's theorem to obtain

$$\int\int_R (VL[\Psi] - \Psi M[V])ds\,dt$$
$$= \int_{C_1+C_2+C_3+C_4} (a_1\Psi V + \frac{1}{2}V\Psi_t - \frac{1}{2}\Psi V_t)dt - (a_2\Psi V + \frac{1}{2}V\Psi_s - \frac{1}{2}\Psi V_s)ds.$$

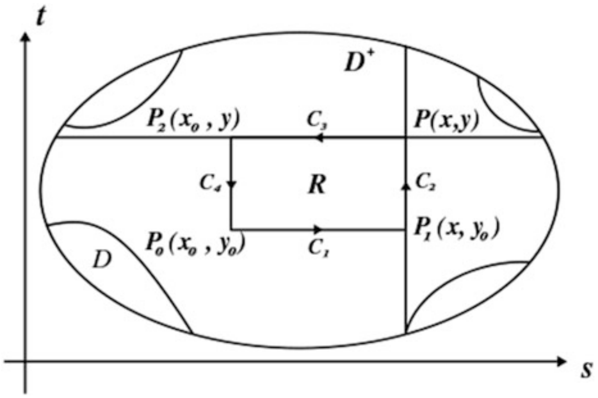

Fig. 5.5 Region and directed path around R

Since Ψ is zero on C_1 and C_4, and also ds does not vary on C_2, and dt does not vary on C_3, we get

$$\begin{aligned}&\int\int_R (VL[\Psi]-\Psi M[V])ds\,dt\\&=\int_{C_2}(a_1\Psi V+\frac{1}{2}V\Psi_t-\frac{1}{2}\Psi V_t)dt-\int_{C_3}(a_2\Psi V+\frac{1}{2}V\Psi_s-\frac{1}{2}\Psi V_s)ds.\end{aligned}$$

Integrating the right-hand side by parts along the characteristic segments C_2 and C_3 to eliminate partial derivatives of Ψ, we obtain

$$\begin{aligned}\int\int_R (VL[\Psi]-\Psi M[V])ds\,dt&=\int_{C_2}(a_1V-V_t)\Psi dt-\int_{C_3}(a_2V-V_s)\Psi ds\\&\quad+\Psi(P)V(P)-\frac{1}{2}\Psi(P_1)V(P_1)-\frac{1}{2}\Psi(P_2)V(P_2)\\&=\int_{C_2}(a_1V-V_t)\Psi dt\\&\quad-\int_{C_3}(a_2V-V_s)\Psi ds+\Psi(P)V(P).\end{aligned}\tag{5.1.227}$$

Now since $V(s,t;x,y)$ is the solution of the characteristic initial value problem $M[V]=0$, it is by definition the Riemann function $V(s,t;x,y)=V(s,t)$ associated with the partial differential equation $L[\Psi]=0$ such that

$$V(x,y;x,y)=V(x,y)=V(P)=1,$$

$$V_t=a_1V\quad on\quad C_2,\quad V(x,t)=\exp\left(\int_y^t a_1(x,\eta)d\eta\right),$$

and

$$V_s=a_2V\quad on\quad C_3,\quad V(s,y)=\exp\left(\int_x^s a_2(\xi,y)d\xi\right).$$

So we get from identity (5.1.227) and inequality (5.1.225)

$$\Psi(x,y)\le\int_{x_0}^x\int_{y_0}^y V(s,t;x,y)a(s,t)c(s,t)ds\,dt.$$

Now substituting this bound on $\Psi(x,y)$ in (5.1.222), we have

$$\begin{aligned}T[\phi]&=\phi_{xy}(x,y)+b_1\phi_x(x,y)+b_2\phi_y(x,y)+b_3\phi(x,y)\\&\le c(x,y)\Big(a(x,y)+b(x,y)h(x,y)\left(\int_{x_0}^x\int_{y_0}^y V(s,t;x,y)a(s,t)c(s,t)ds\,dt\right)\Big).\end{aligned}$$

Now following the same steps as above, we obtain

$$\phi(x,y) \le \int_{x_0}^{x}\int_{y_0}^{y} W(s,t;x,y)c(s,t)\Big\{a(s,t)+b(s,t)h(s,t) \\ \times \left(\int_{x_0}^{s}\int_{y_0}^{t} V(\xi,\eta;s,t)a(\xi,\eta)c(\xi,\eta)d\xi\, d\eta\right)\Big\}ds\, dt = Q(x,y).$$

Now substituting this bound on $\phi(x,y)$ in (5.1.216), we obtain (5.1.217).
Now let $q(x,y)=0$ in (5.1.217) and define

$$f(x,y) = a(x,y) + b(x,y)\Big[r(x,y)Q(x,y) + h(x,y)\int_{x_0}^{x}\int_{y_0}^{y} g(s,t)Q(s,t)ds\, dt\Big]. \tag{5.1.228}$$

Then inequality (5.1.217) reduces to

$$u(x,y) \le f(x,y) + b(x,y)p(x,y)\int_{x_0}^{x} c(s,y)u(s,y)ds. \tag{5.1.229}$$

Inequality (5.1.229) may be treated as one-dimensional Gronwall's inequality for any fixed y between y_0 to y, which implies the estimate for $u(x,y)$ such that

$$u(x,y) \le f(x,y) + b(x,y)p(x,y) \\ \times \left[\int_{x_0}^{x} c(s,y)f(s,y)\exp\left(\int_{s}^{x} c(\xi,y)b(\xi,y)d\xi\right)ds\right], \tag{5.1.230}$$

which is (5.1.219). Further, substituting $p(x,y)=0$ in (5.1.217) and following the similar argument as above, we obtain (5.1.221). □

In 1982, Corduneanu [152] proved the following result.

Theorem 5.1.35 (Corduneanu [152]) *If the continuous function $u=u(x,y)$ satisfies the inequality, for all $x,y\ge 0$,*

$$u(x,y) \le f(x,y) + \int_{0}^{x}\int_{0}^{y} a(s,t)u(s,t)dsdt, \tag{5.1.231}$$

where

(a) $f=f(x,y)$ is continuous for all $x,y\ge 0$ and monotone non-decreasing with respect to each variable,

(b) $a = a(x, y)$ is continuous and non-negative for all $x, y \geq 0$,

then for all $x, y \geq 0$,

$$u(x, y) \leq f(x, y)\left[1 + \int_0^x \int_0^y a(s, t) \exp\left(\int_s^x \int_t^y a(\xi, \eta) d\xi d\eta\right) dsdt\right]. \quad (5.1.232)$$

Proof As in the case of one variable, from the inequality, for all $x, y \geq 0$,

$$u(x, y) \leq f(x, y) + \int_0^x \int_0^y k(x, y, s, t) u(s, t) dsdt, \quad (5.1.233)$$

in which k is continuous and non-negative, we deduce that, for all $x, y \geq 0$,

$$u(x, y) \leq f(x, y) + \int_0^x \int_0^y r(x, y, s, t) f(s, t) dsdt, \quad (5.1.234)$$

where

$$r(x, y, s, t) = \sum_{n=0}^{+\infty} k_n(x, y, s, t), \quad (5.1.235)$$

the kernels k being given by the equation

$$\begin{cases} k_0 = k, \\ k_n(x, y, s, t) = \displaystyle\int_s^x \int_t^y k_{n-1}(\xi, \eta, s, t) k(\xi, \eta, s, t) d\xi d\eta, & \textit{for all } n \geq 1. \end{cases} \quad (5.1.236)$$

In particular, if $k(x, y, s, t) = a(s, t) \geq 0$, then we have after a simple computation that

$$k_n(x, y, s, t) = \left(\frac{1}{n!}\right) a(s, t) \left(\int_s^x \int_t^y a(\xi, \eta) d\xi d\eta\right)^n, \qquad \textit{for all } n \geq 1, \quad (5.1.237)$$

and consequently, in this case

$$r(x, y, s, t) \leq a(s, t) \exp\left(\int_s^x \int_t^y a(\xi, \eta) d\xi d\eta\right)^n, \qquad \textit{for all } n \geq 1. \quad (5.1.238)$$

If we observe that condition (a) implies that $f(s, t) \leq f(x, y)$ for all $0 \leq s \leq x$, $0 \leq t \leq y$, then from (5.1.234) and (5.1.238), we easily prove (5.1.232), which concludes the proof. □

Remark 5.1.5 In the case $f(x, y) \geq 0$, the above inequality (5.1.232) is stronger than those obtained in Theorem 5.1.35 of [95], because we have for all $x, y \geq 0$,

$$1 + \int_0^x \int_0^y a(s,t) \exp\left(\int_s^x \int_t^y a(\xi,\eta)d\xi d\eta\right) dsdt \leq \exp\left(\int_0^x \int_0^y a(\xi,\eta)d\xi d\eta\right). \tag{5.1.239}$$

To prove (5.1.239), it suffices to pose for fixed $x,\ y > 0$,

$$F(s,t) = \exp\left(\int_0^x \int_0^y a(\xi,\eta)d\xi d\eta\right) \tag{5.1.240}$$

and to observe that

$$F''_{st}(s,t) = a(s,t) \exp\left(\int_0^x \int_0^y a(\xi,\eta)d\xi d\eta\right) + \text{non-negative term}. \tag{5.1.241}$$

Integrating (5.1.241) on the rectangle $0 \leq s \leq x,\ 0 \leq t \leq y$, we obtain (5.1.239).

Remark 5.1.6 If $a(x, y) > 0$ for all $x,\ y > 0$, then the inequality (5.1.239) is strict for all $x,\ y > 0$.

Example 5.1.1 If $a(x, y) \equiv a > 0$ where $a =$ constant, then (5.1.239) becomes for all $x, y > 0$,

$$1 + \sum_{n=1}^{+\infty} (axy)^n/n! < 1 + \sum_{n=1}^{+\infty} (axy)^n/n!. \tag{5.1.242}$$

Remark 5.1.7 We also note that in Theorems 1 and 2 of [95], the conditions imposed on the function f (there denoted by g) may be weakened. It suffices to suppose only that f is continuous and monotone non-decreasing with respect to each variable. Then in the proof of Theorem 2 of [95], the following bound will be useful,

$$1 + \int_0^x \int_0^y a(s,t) \exp\left(\int_0^s \int_0^t a(\xi,\eta)d\xi d\eta\right) dsdt \leq \exp\left(\int_0^x \int_0^y a(\xi,\eta)d\xi d\eta\right) \tag{5.1.243}$$

for all $x, y \geq 0$, where $a = a(x, y)$ is continuous and non-negative.

Remark 5.1.8 Theorem 5.1.35 may be extended to the n-dimensional case without major modifications.

The differential and integral inequalities occupy a very privileged position in the theory of differential and integral equations. In recent years, these inequalities

have been greatly enriched by the recognition of their potential and intrinsic worth in many applications of the applied sciences. Since the appearance of Gronwall's fundamental paper [239] in 1919, an enormous amount of effort has been devoted to the discovery of new types of inequalities, and to the application of inequalities in many parts of analysis. In [203], we obtained several new integral inequalities of Gronwall-Bellman type of single independent variable. These inequalities are directly useful in studying several properties of the solutions of ordinary differential equations.

A more general version of the inequality in Theorem 5.1.25, due to Pachpatte [477]), may be stated as follows.

Theorem 5.1.36 (El-Owaidy-Ragab-Abdeldaim [202]) *Let $u(x,y), f(x,y)$, are real-valued non-negative continuous functions defined on a domain D, and $h(x,y), q(x,y)$, are real-valued positive continuous functions defined on a domain D, and u_0 is a non-negative constant. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two points in D such that $(x-x_0)(y-y_0) > 0$ and let R be the rectangular region whose opposite corners are the points P_0 and P. Let $v(s,t;x,y)$ be the solution of the characteristic initial value problem*

$$L(v) = v_{st}(s,t) + (f(s,t) + q(s,t))v(s,t) = 0; \quad v(s,y) = v(x,t) = 1,$$

and let D^+ be a connected sub-domain of D which contains P and which $v > 0$. Then if $R \subset D^+$ and it holds that

$$u(x,y) \le u_0 + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)u(s,t)dsdt + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)u(s,t) \Big(u(s,t) + \int_{x_0}^{x}\int_{y_0}^{y} q(\xi,\eta)u(\xi,\eta)d\xi d\eta\Big)dsdt, \tag{5.1.244}$$

then

$$u(x,y) \le u_0 \exp\Big(\int_{x_0}^{x}\int_{y_0}^{y}\Big[f(s,t) + \frac{h(s,t)}{\int_s^{x_0}\int_t^{y_0} h(\xi,\eta)v(\xi,\eta;s,t)d\xi d\eta}\Big]dsdt\Big). \tag{5.1.245}$$

Proof Define a function $n(x,y)$ be the right-hand side of (5.1.244). Then

$$\begin{cases} n_{xy}(x,y) = f(x,y)u(x,y) + h(x,y)u(x,y)\Big(u(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} q(s,t)u(s,t)dsdt\Big), \\ n(x_0,y) = n(x,y_0) = u_0; \quad n_x(x,y_0) = n_y(x_0,y) = 0, \end{cases}$$

which, in view in (5.1.244), implies

$$n_{xy}(x,y) \le f(x,y)n(x,y) + h(x,y)n(x,y)\Big(n(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} q(s,t)n(s,t)dsdt\Big). \tag{5.1.246}$$

Let

$$m(x,y) = n(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} q(s,t)n(s,t)dsdt, \quad m(0,y) = m(x,0) = u_0.$$

Then

$$m_{xy}(x,y) = n_{xy}(x,y) + q(x,y)n(x,y).$$

From (5.1.246) and the fact that $n(x,y) \le m(x,y)$, it follows

$$m_{xy}(x,y) \le \Big(f(x,y) + q(x,y)\Big)m(x,y) + h(x,y)m^2(x,y).$$

The above inequality can be written as

$$m^{-2}(x,y)m_{xy}(x,y) - (f(x,y) + q(x,y))m^{-1}(x,y) \le h(x,y), \tag{5.1.247}$$

which yields

$$m^{-2}(x,y)m_{xy}(x,y) - \Big(f(x,y) + q(x,y)\Big)m^{-1}(x,y) \le h(x,y) + \frac{2m_x(x,y)m_y(x,y)}{m^3(x,y)},$$

i.e.,

$$m^{-2}(x,y)m_{xy}(x,y) - \frac{2m_x(x,y)m_y(x,y)}{m^3(x,y)} - (f(x,y) + q(x,y))m^{-1}(x,y) \le h(x,y).$$

Let $m^{-1}(x,y) = a(x,y)$, so that

$$m^{-2}(x,y)m_{xy}(x,y) - \frac{2m_x(x,y)m_y(x,y)}{m^3(x,y)} = -a_{xy}(x,y).$$

Then we have

$$-a_{xy}(x,y) - (f(x,y) + q(x,y))a(x,y) \le h(x,y),$$

i.e.,

$$L(a) = a_{xy}(x, y) + (f(x, y) + q(x, y))a(x, y) \geq -h(x, y).$$

Now following the same steps as in the proof of Theorem 5.1.25, we obtain

$$a(x, y) \geq \int_x^{x_0} \int_y^{y_0} h(s, t)v(s, t; x, y)dsdt.$$

Now substituting $a(x, y) = m^{-1}(x, y)$ in the above inequality, we have

$$m(x, y) \leq \frac{1}{\int_x^{x_0} \int_y^{y_0} h(s, t)v(s, t; x, y)dsdt}, \tag{5.1.248}$$

then from (5.1.244) it follows

$$n_{xy}(x, y) \leq f(x, y)n(x, y) + h(x, y)n(x, y)m(x, y),$$

i.e.,

$$\frac{n_{xy}(x, y)}{n(x, y)} \leq f(x, y) + h(x, y)m(x, y). \tag{5.1.249}$$

Thus from (5.1.248) and (5.1.249) it follows

$$\frac{n_{xy}(x, y)}{n(x, y)} \leq f(x, y) + \frac{h(x, y)}{\int_x^{x_0} \int_y^{y_0} h(s, t)v(s, t; x, y)dsdt}, \tag{5.1.250}$$

which implies

$$\frac{n(x, y)n_{xy}(x, y)}{n^2(x, y)} \leq f(x, y) + \frac{h(x, y)}{\int_x^{x_0} \int_y^{y_0} h(s, t)v(s, t; x, y)dsdt} + \frac{n_x(x, y)n_y(x, y)}{n^2(x, y)}.$$

Then

$$\frac{\partial}{\partial y}\left(\frac{n_x(x, y)}{n(x, y)}\right) \leq f(x, y) + \frac{h(x, y)}{\int_x^{x_0} \int_y^{y_0} h(s, t)v(s, t; x, y)dsdt}.$$

Now integrating both sides of the above inequality with respect to y from y_0 to y, we have

$$\frac{n_x(x, y)}{n(x, y)} - \frac{n_x(x, y_0)}{n(x, y_0)} \leq \int_{y_0}^{y} \left(f(x, t) + \frac{h(x, t)}{\int_x^{x_0} \int_t^{y_0} h(s, \eta)v(s, \eta; x, t)dsd\eta}\right)dt.$$

Since $n_x(x, y_0) = 0$, we get

$$\ln \frac{n(x,y)}{n(0,y)} \leq \int_{x_0}^{x}\int_{y_0}^{y}\Big(f(s,t) + \frac{h(s,t)}{\int_s^{x_0}\int_t^{y_0} h(\xi,\eta)v(\xi,\eta;s,t)d\xi d\eta}\Big)dsdt,$$

but $n(x_0, y) = u_0$, then

$$n(x,y) \leq u_0 \exp\Big\{\int_{x_0}^{x}\int_{y_0}^{y}\Big(f(s,t) + \frac{h(s,t)}{\int_s^{x_0}\int_t^{y_0} h(\xi,\eta)v(\xi,\eta;s,t)d\xi d\eta}\Big)dsdt\Big\}.$$

Now substituting the value of $n(x, y)$ in (5.1.244), we obtain (5.1.245). This completes the proof. □

By setting $f(x, y) = 0$ in Theorem 5.1.36, we arrive at the following integral inequality.

Corollary 5.1.10 (El-Owaidy-Ragab-Abdeldaim [202]) *Let $u(x, y), f(x, y)$, are real-valued non-negative continuous functions defined on a domain D, and $h(x, y), q(x, y)$, be real-valued positive continuous functions defined on a domain D, and u_0 is a non-negative constant. Let $P_0(x_0, y_0)$ and $P(x, y)$ be two points in D such that $(x - x_0)(y - y_0) > 0$ and let R be the rectangular region whose opposite corners are the points P_0 and P. Let $v(s, t; x, y)$ be the solution of the characteristic initial value problem*

$$L(v) = v_{st}(s,t) + q(s,t)v(s,t) = 0; \quad v(s,y) = v(x,t) = 1,$$

and let D^+ be a connected sub-domain of D which contains P and which $v > 0$. Then, if $R \subset D^+$ and it holds that

$$u(x,y) \leq u_0 + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)u(s,t)\Big(u(s,t) + \int_{x_0}^{x}\int_{y_0}^{y} q(\xi,\eta)u(\xi,\eta)d\xi d\eta\Big)dsdt,$$

then

$$u(x,y) \leq u_0 \exp\Big(\int_{x_0}^{x}\int_{y_0}^{y}\Big[\frac{h(s,t)}{\int_s^{x_0}\int_t^{y_0} h(\xi,\eta)v(\xi,\eta;s,t)d\xi d\eta}\Big]dsdt\Big).$$

Now, by setting $h(x, y) = 1$ in the Theorem 5.1.36, we arrive at the following integral inequality.

Corollary 5.1.11 (El-Owaidy-Ragab-Abdeldaim [202]) *Let $u(x, y), f(x, y)$, are real-valued non-negative continuous functions defined on a domain D, and $q(x, y)$, are real-valued positive continuous functions defined on a domain D, and u_0 is a non-negative constant. Let $P_0(x_0, y_0)$ and $P(x, y)$ be two points in D such that $(x - x_0)(y - y_0) > 0$ and let R be the rectangular region whose opposite corners are the points P_0 and P. Let $v(s, t; x, y)$ be the solution of the characteristic initial value*

problem

$$L(v) = v_{st}(s,t) + (f(s,t) + q(s,t))v(s,t) = 0; \quad v(s,y) = v(x,t) = 1,$$

and let D^+ *be a connected sub-domain of* D *which contains* P *and which* $v > 0$. *Then, if* $R \subset D^+$ *and it holds that*

$$u(x,y) \le u_0 + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)u(s,t)dsdt \\ + \int_{x_0}^{x}\int_{y_0}^{y} u(s,t)\Big(u(s,t) + \int_{x_0}^{x}\int_{y_0}^{y} q(\xi,\eta)u(\xi,\eta)d\xi d\eta\Big)dsdt,$$

then

$$u(x,y) \le u_0 \exp\Big(\int_{x_0}^{x}\int_{y_0}^{y}\Big[f(s,t) + \frac{1}{\int_s^{x_0}\int_t^{y_0} v(\xi,\eta;s,t)d\xi d\eta}\Big]dsdt\Big).$$

Theorem 5.1.37 (El-Owaidy-Ragab-Abdeldaim [202]) *Let* $u(x,y), h(x,y)$ *and* $f(x,y)$ *be real-valued non-negative continuous functions defined on* $\mathbb{R}_+ \times \mathbb{R}_+$, *suppose* $0 \le p < 1$, *and* u_0 *is a non-negative constant, and suppose further that the following inequality holds for all* $(x,y) \in \mathbb{R}_+^2$,

$$u(x,y) \le h(x,y)\Big(u_0 + \int_0^x\int_0^y f(s,t)u^p(s,t)dsdt\Big). \tag{5.1.251}$$

Then for all $(x,y) \in \mathbb{R}_+^2$,

$$u(x,y) \le h(x,y)\Big(u_0^q + q\int_0^x\int_0^y f(s,t)u^p(s,t)dsdt\Big)^{1/q}, \tag{5.1.252}$$

where $p + q = 1$.

Proof The inequality (5.1.251) can be written as

$$u(x,y) \le h(x,y)m(x,y), \tag{5.1.253}$$

where

$$\begin{cases} m(x,y) = u_0 + \int_0^x\int_0^y f(s,t)u^p(s,t)dsdt; \\ m(0,y) = m(x,0) = u_0; \quad m_x(x,0) = m_y(0,y) = 0, \end{cases} \tag{5.1.254}$$

thus

$$m_{xy}(x,y) = f(x,y)u^p(x,y).$$

Then from (5.1.253) it follows

$$m_{xy}(x,y) \leq f(x,y)h^p(x,y)m^p(x,y).$$

Using the non-deceasing nature of $m(x,y)$, we find

$$\frac{m_{xy}(x,y)}{m^p(x,y)} \leq f(x,y)h^p(x,y), \tag{5.1.255}$$

which gives us

$$\frac{m^p(x,y)m_{xy}(x,y)}{m^{2p}(x,y)} \leq f(x,y)h^p(x,y) + \frac{2pm_x(x,y)m_x(x,y)}{m(x,y)},$$

i.e.,

$$\frac{m^p(x,y)m_{xy}(x,y)}{m^{2p}(x,y)} - \frac{2pm_x(x,y)m_x(x,y)}{m(x,y)} \leq f(x,y)h^p(x,y),$$

or

$$\frac{\partial}{\partial y}\frac{m_x(x,y)}{m^p(x,y)} \leq f(x,y)h^p(x,y).$$

Now integrating both sides of the above inequality with respect to y from 0 to y, we get

$$\frac{m_x(x,y)}{m^p(x,y)} - \frac{m_x(x,0)}{m^p(x,0)} \leq \int_0^y f(x,t)h^p(x,t)dt,$$

but $m_x(x,0) = 0$, hence

$$\frac{m_x(x,y)}{m^p(x,y)} \leq \int_0^y f(x,t)h^p(x,t)dt.$$

Integrating both sides of the above inequality with respect to x from 0 to x, we obtain

$$\frac{1}{q}(m^q(x,y) - m^q(0,y)) \leq \int_0^x \int_0^y f(s,t)h^p(s,t)dtds,$$

but $m(0,y) = u_0$, then $m^q(0,y) = u_0^q$, hence

$$\frac{1}{q}(m^q(x,y) - u_0^q) \leq \int_0^x \int_0^y f(s,t)h^p(s,t)dtds,$$

i.e.,

$$m^q(x,y) - u_0^q \le q \int_0^x \int_0^y f(s,t)h^p(s,t)dtds,$$

then

$$m(x,y) \le \Big(u_0^q + q \int_0^x \int_0^y f(s,t)h^p(s,t)dtds\Big)^{1/q}.$$

Now substituting this bound on $m(x,y)$ in (5.1.251), we can obtain (5.1.250). This completes the proof. □

By setting $h(x,y) = 1$, in Theorem 5.1.37, we arrive at the following integral inequality.

Corollary 5.1.12 (El-Owaidy-Ragab-Abdeldaim [202]) *Let $u(x,y)$ and $f(x,y)$ be real-valued non-negative continuous functions defined on $\mathbb{R}_+ \times \mathbb{R}_+$, suppose $0 \le p < 1$, and u_0 is a non-negative constant, and suppose further that the following inequality holds, for all $(x,y) \in \mathbb{R}_+^2$,*

$$u(x,y) \le u_0 + \int_0^x \int_0^y f(s,t)u^p(s,t)dsdt,$$

Then for all $(x,y) \in \mathbb{R}_+^2$,

$$u(x,y) \le \Big(u_0^q + q \int_0^x \int_0^y f(s,t)u^p(s,t)dsdt\Big)^{1/q},$$

where $p + q = 1$.

We now apply Corollary 5.1.12 to establish the following integral inequality.

Corollary 5.1.13 (El-Owaidy-Ragab-Abdeldaim [202]) *Let $u(x,y)$ and $f(x,y)$ be real-valued non-negative continuous functions defined on $\mathbb{R}_+ \times \mathbb{R}_+$ and $a(x,y)$ be a positive, monotonic, non-deceasing, continuous function defined on $\mathbb{R}_+ \times \mathbb{R}_+$, satisfying for all $(x,y) \in \mathbb{R}_+^2$,*

$$u(x,y) \le a(x,y) + \int_0^x \int_0^y f(s,t)a^q(s,t)u^p(s,t)dsdt. \tag{5.1.256}$$

Then for all $(x,y) \in \mathbb{R}_+^2$,

$$u(x,y) \le a(x,y)\Big(1 + q \int_0^x \int_0^y f(s,t)dsdt\Big)^{1/q}, \tag{5.1.257}$$

where $0 \le p < 1, p + q = 1$.

Proof Since $a(x, y)$ is a positive, monotonic, non-decreasing, continuous function, from (5.1.256) it follows that

$$\frac{u(x, y)}{a(x, y)} \leq 1 + \int_0^x \int_0^y f(s, t) \frac{a^q(s, t)u^p(s, t)}{a(x, y)} dsdt,$$

i.e.,

$$\frac{u(x, y)}{a(x, y)} \leq 1 + \int_0^x \int_0^y f(s, t) \frac{u(s, t)}{a(s, t)} dsdt.$$

Define a function $m(x, y)$ by

$$m(x, y) = \frac{u(x, y)}{a(x, y)}, \tag{5.1.258}$$

hence

$$m(x, y) \leq 1 + \int_0^x \int_0^y f(s, t)m^p(s, t)dsdt.$$

Applicating Corollary 5.1.12, we have

$$m(x, y) \leq \Big(1 + q \int_0^x \int_0^y f(s, t)dsdt\Big)^{1/q}. \tag{5.1.259}$$

Thus (5.1.257) follows from (5.1.258) and (5.1.259). This completes the proof. □

Theorem 5.1.38 (El-Owaidy-Ragab-Abdeldaim [202]) *Let $u(x, y)$, $f(x, y)$, $h(x, y)$ and $b(x, y)$ are real-valued non-negative continuous functions defined on a domain D, $0 \leq p < 0$, and u_0 is a non-negative constant. Let $P_0(x_0, y_0)$ and $P(x, y)$ be two points in D such that $(x - x_0)(y - y_0) > 0$ and let R be the rectangular region whose opposite corners are the points P_0 and P. Let $v(s, t; x, y)$ be the solution of the characteristic initial value problem*

$$L(v) = v_{st}(s, t) - qh(s, t)b(s, t)v(s, t) = 0; \quad v(s, y) = v(x, t) = 1,$$

and let D^+ be a connected sub-domain of D which contains P and which $v > 0$. Then, if $R \subset D^+$ and $u(x, y)$ satisfies

$$u(x, y) \leq b(x, y)\Big(u_0 + \int_{x_0}^x \int_{y_0}^y h(s, t)u(s, t)dsdt + \int_{x_0}^x \int_{y_0}^y f(s, t)u^p(s, t)dsdt\Big), \tag{5.1.260}$$

then

$$u(x,y) \leq b(x,y)\left(q\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)b^p(s,t)v(s,t;x,y)dsdt\right)^{1/q}, \qquad (5.1.261)$$

where $p+q=1$.

Proof The inequality (5.1.260) can be written as

$$u(x,y) \leq b(x,y)n(x,y), \qquad (5.1.262)$$

where

$$n(x,y) = u_0 + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)u(s,t)dsdt + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)u^p(s,t)dsdt.$$

Thus,

$$n_{xy}(x,y) = h(x,y)u(x,y) + f(x,y)u^p(x,y),$$

then from (5.1.262) it follows that

$$n_{xy}(x,y) = h(x,y)(b(x,y)n(x,y)) + f(x,y)(b(x,y)n(x,y))^p.$$

Using the non-decreasing nature of $n(x,u)$, we find

$$n^{-1}(x,y)n_{xy}(x,y) - h(x,y)b(x,y)n^q(x,y) \leq f(x,y)b^p(x,y), \qquad (5.1.263)$$

which gives

$$n^{-1}(x,y)n_{xy}(x,y) - h(x,y)b(x,y)n^q(x,y) \leq f(x,y)b^p(x,y) + \frac{pn_x(x,y)n_y(x,y)}{n^{p+1}(x,y)},$$

i.e.,

$$n^{-1}(x,y)n_{xy}(x,y) - \frac{pn_x(x,y)n_y(x,y)}{n^{p+1}(x,y)} - h(x,y)b(x,y)n^q(x,y) \leq f(x,y)b^p(x,y).$$

Let $n^p(x,y) = m(x,y)$, so that

$$n^{-1}(x,y)n_{xy}(x,y) - \frac{pn_x(x,y)n_y(x,y)}{n^{p+1}(x,y)} = \frac{1}{q}m_{xy}(x,y),$$

then

$$\frac{1}{q}m_{xy}(x,y) - h(x,y)b(x,y)m(x,y) \le f(x,y)b^p(x,y),$$

i.e.,

$$L(m) = m_{xy}(x,y) - qh(x,y)b(x,y)m(x,y) \le qf(x,y)b^p(x,y).$$

Now following the same steps as in the proof of Theorem 5.1.25, we obtain

$$m(x,y) \le q\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)b^p(s,t)v(s,t;x,y)dsdt.$$

Now substituting $m(x,y) = n^q(x,y)$ in the above inequality, we have

$$n(x,y) \le \left(q\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)b^p(s,t)v(s,t;x,y)dsdt\right)^{1/q}. \tag{5.1.264}$$

Thus (5.1.261) follows from (5.1.264) and (5.1.262). This completes the proof. □

In the special case, when $b(x,y) = 1$, Theorem 5.1.38 takes the following form.

Corollary 5.1.14 (El-Owaidy-Ragab-Abdeldaim [202]) *Let $u(x,y), f(x,y)$, $h(x,y)$ are real-valued non-negative continuous functions defined on a domain D, $0 \le p < 1$, and u_0 is a non-negative constant. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two points in D such that $(x-x_0)(y-y_0) > 0$ and let R be the rectangular region whose opposite corners are the points P_0 and P. Let $v(s,t;x,y)$ be the solution of the characteristic initial value problem*

$$L(v) = v_{st}(s,t) - qh(s,t)v(s,t) = 0; \quad v(s,y) = v(x,t) = 1,$$

and let D^+ be a connected sub-domain of D which contains P and which $v > 0$. Then, if $R \subset D^+$ and $u(x,y)$ satisfies

$$u(x,y) \le u_0 + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)u(s,t)dsdt + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)u^p(s,t)dsdt,$$

then

$$u(x,y) \le \left(q\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)v(s,t;x,y)dsdt\right)^{1/q},$$

where $p+q=1$.

We now may apply Corollary 5.1.14 to establish the following integral inequality.

Corollary 5.1.15 (El-Owaidy-Ragab-Abdeldaim [202]) *Let $u(x,y), f(x,y)$, $h(x,y)$ are real-valued non-negative continuous functions defined on a domain D, $0 \le p < 1$, and $a(x,y)$ be a positive, monotonic, non-decreasing, continuous function defined on a domain D. Let $P_0(x_0,y_0)$ and $P(x,y)$ be two points in D such that $(x-x_0)(y-y_0) > 0$ and let R be the rectangular region whose opposite corners are the points P_0 and P. Let $v(s,t;x,y)$ be the solution of the characteristic initial value problem*

$$L(v) = v_{st}(s,t) - qh(s,t)v(s,t) = 0; \quad v(s,y) = v(x,t) = 1,$$

and let D^+ be a connected sub-domain of D which contains P and which $v > 0$. Then, if $R \subset D^+$ and it holds

$$u(x,y) \le a(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)u(s,t)dsdt + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)a^q(s,t)u^p(s,t)dsdt, \tag{5.1.265}$$

then

$$u(x,y) \le a(x,y)\Big(q\int_{x_0}^{x}\int_{y_0}^{y} f(s,t)v(s,t;x,y)dsdt\Big)^{1/q}, \tag{5.1.266}$$

where $p+q=1$.

Proof Since $a(x,y)$ is a positive, monotonic, non-decreasing, continuous function, we derive from (5.1.265) that

$$\frac{u(x,y)}{a(x,y)} \le 1 + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)\frac{u(s,t)}{a(x,y)}dsdt + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\frac{a^q(s,t)u^p(s,t)}{a(x,y)}dsdt,$$

i.e.,

$$\frac{u(x,y)}{a(x,y)} \le 1 + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)\frac{u(s,t)}{a(s,t)}dsdt + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)\left(\frac{u(s,t)}{a(s,t)}\right)^p dsdt.$$

Define a function $m(x,y)$ by

$$m(x,y) = \frac{u(x,y)}{a(x,y)}, \tag{5.1.267}$$

hence

$$m(x,y) \le 1 + \int_{x_0}^{x}\int_{y_0}^{y} h(s,t)m(s,t)dsdt + \int_{x_0}^{x}\int_{y_0}^{y} f(s,t)m^p(s,t)dsdt.$$

Applying Corollary 5.1.14 to the above inequality, we have

$$m(x,y) \le \left(q \int_{x_0}^{x} \int_{y_0}^{y} f(s,t) v(s,t;x,y) ds dt \right)^{1/q}. \tag{5.1.268}$$

Thus the conclusion of the theorem follows from (5.1.267) and (5.1.268). This completes the proof. □

Remark 5.1.9 We note that the integral inequalities in Theorems 5.1.37–5.1.38 allow us to study the stability, boundedness and asymptotic behavior of the solutions of a class of more general partial differential and integral equations similar to those obtained in [91, 227, 477, 480].

The following result, due to Bondge and pachpatte[91], is to establish two-independent-variable generalizations of the integral inequalities recently established by Gollwitzer [231], Langenhop [352], and Pachpatte [443, 444].

A useful two-independent-variable generalization of Gollwitzer's inequality [231], is stated in the following theorem.

Theorem 5.1.39 (Bondge-Pachpatte [91]) *Let* $\phi(s,t)$, $a(s,t)$, $b(s,t)$ *be real-valued non-negative continuous functions defined on* $\mathbb{R}_+ \times \mathbb{R}_+$, *let* $u(s,t)$ *be a positive real-valued continuous function defined on* $\mathbb{R}_+ \times \mathbb{R}_+$*; and suppose further that the following inequality holds for all* $0 \le x \le s < +\infty$, $0 \le y \le t < +\infty$,

$$u(s,t) \ge \phi(x,y) - a(s,t) \left(\int_x^s \int_y^t b(m,n)\phi(m,n)\, dm\, dn \right), \tag{5.1.269}$$

then for all $0 \le x \le s < +\infty$, $0 \le y \le t < +\infty$,

$$u(s,t) \ge \phi(x,y) \exp\left(-a(s,t) \int_x^s \int_y^t b(m,n)\, dm\, dn \right). \tag{5.1.270}$$

Proof We may rewrite (5.1.269) as

$$\phi(x,y) \le u(s,t) + a(s,t) \left(\int_x^s \int_y^t b(m,n)\phi(m,n)\, dm\, dn \right). \tag{5.1.271}$$

For fixed s and t in the interval $\mathbb{R}_+$, we define for all $0 \le x \le s$, $0 \le y \le t$,

$$\begin{cases} r(x,y) = u(s,t) + a(s,t) \left(\int_x^s \int_y^t b(m,n)\phi(m,n)\, dm\, dn \right), \\ r(x,t) = r(s,y) = u(s,t), \end{cases} \tag{5.1.272}$$

then we derive from (5.1.272)

$$r_{xy}(x,y) = \ a(s,t)\ b(x,y)\phi(x,y),$$

which, combined with (5.1.271), implies

$$r_{xy}(x,y) \leq \ a(s,t)\ b(x,y)\ r(x,y),$$

i.e.,

$$\frac{r_{xy}(x,y)}{r(x,y)} \leq \ a(s,t)\ b(x,y). \tag{5.1.273}$$

From (5.1.273) we derive

$$\frac{r(x,y)\ r_{xy}(x,y)}{r^2(x,y)} \leq \ a(s,t)\ b(x,y) + \frac{r_x(x,y)\ r_y(x,y)}{r^2(x,y)},$$

i.e.,

$$\frac{\partial}{\partial y}\Big(\frac{r_x(x,y)}{r(x,y)}\Big) \leq \ a(s,t)\ b(x,y).$$

Now integrating both sides of the above inequality with respect to y from y to t, we have

$$\frac{r_x(x,t)}{r(x,t)} - \frac{r_x(x,y)}{r(x,y)} \leq \ a(s,t) \int_y^t \ b(x,n)\ dn.$$

Integrating both sides of the above inequality with respect to x from x to s, we get

$$r(x,y) \leq \ u(s,t) \exp\Bigg(\ a(s,t)(\int_x^s \int_y^t \ b(m,n)\ dm\ dn)\Bigg). \tag{5.1.274}$$

Thus (5.1.270) follows from (5.1.271) and (5.1.272) since s and t are arbitrary in the interval $\mathbb{R}_+$. □

Theorem 5.1.40 (Bondge-Pachpatte [91]) *Let $\phi(s,t)$, $a(s,t)$, $b(s,t)$, and $c(s,t)$ be real-valued non-negative continuous functions defined on $\mathbb{R}_+ \times \mathbb{R}_+$; let $u(s,t)$ be a positive real-valued continuous function defined on $\mathbb{R}_+ \times \mathbb{R}_+$; and suppose,*

further, that the following inequality holds for all $0 \le x \le s < +\infty, \quad 0 \le y \le t < +\infty$,

$$u(s,t) \ge \phi(x,y) - a(s,t)\Big[\int_x^s \int_y^t b(m,n)\phi(m,n)\,dm\,dn + \int_x^s \int_y^t b(m,n)\left(\int_m^s \int_n^t c(\xi,\zeta)\phi(\xi,\zeta)\,d\xi\,d\zeta\right) dm\,dn\Big]. \tag{5.1.275}$$

Then for all $0 \le x \le s < +\infty,\ 0 \le y \le t < +\infty$,

$$u(s,t) \ge \phi(x,y)\Big[1 + a(s,t)\Big(\int_x^s \int_y^t b(m,n) \times \exp\left(\int_m^s \int_n^t [\,a(s,t)\,b(\xi,\zeta) + c(\xi,\zeta)]\,d\xi\,d\zeta\right) dm\,dn\Big)\Big]^{-1}. \tag{5.1.276}$$

Proof We may rewrite (5.1.275) as

$$\phi(x,y) \le u(s,t) + a(s,t)\Big[\int_x^s \int_y^t b(m,n)\phi(m,n)\,dm\,dn + \int_x^s \int_y^t b(m,n)\left(\int_m^s \int_n^t c(\xi,\zeta)\phi(\xi,\zeta)\,d\xi\,d\zeta\right) dm\,dn\Big]. \tag{5.1.277}$$

For fixed s and t in the interval $\mathbb{R}_+$, we define for all $0 \le x \le s,\ 0 \le y \le t$,

$$\begin{cases} r(x,y) = u(s,t) + a(s,t)\Big[\displaystyle\int_x^s \int_y^t b(m,n)\phi(m,n)\,dm\,dn \\ \qquad + \displaystyle\int_x^s \int_y^t b(m,n)\left(\int_m^s \int_n^t c(\xi,\zeta)\phi(\xi,\zeta)\,d\xi\,d\zeta\right) dm\,dn\Big], \\ r(x,t) = r(s,y) = u(s,t), \end{cases} \tag{5.1.278}$$

then we derive from (5.1.278),

$$r_{xy}(x,y) = a(s,t)\,b(x,y)\left[\phi(x,y) + \int_x^s \int_y^t c(\xi,\zeta)\phi(\xi,\zeta)\,d\xi\,d\zeta\right],$$

which, along with (5.1.277), implies

$$r_{xy}(x,y) \le a(s,t)\, b(x,y) \left[r(x,y) + \int_x^s \int_y^t c(\xi,\zeta)\, r(\xi,\zeta)\, d\xi\, d\zeta \right]. \tag{5.1.279}$$

Define

$$\begin{cases} v(x,y) = r(x,y) + \displaystyle\int_x^s \int_y^t c(\xi,\zeta)\, r(\xi,\zeta)\, d\xi\, d\zeta, \\ v(s,y) = v(x,t) = u(s,t), \end{cases} \tag{5.1.280}$$

then we deduce from (5.1.280),

$$v_{xy}(x,y) = r_{xy}(x,y) + c(x,y)\, r(x,y),$$

which, by using (5.1.279) and the inequality $r(x,y) \le v(x,y)$, implies

$$v_{xy}(x,y) \le [\, a(s,t)\, b(x,y) + c(x,y)]\; v(x,y).$$

Hence, the following an argument similar to that in the proof of Theorem 5.1.39, we get

$$v(x,y) \le u(s,t) \exp\left(\int_x^s \int_y^t [\, a(s,t)\, b(\xi,\zeta) + c(\xi,\zeta)]\, d\xi\, d\zeta \right).$$

Substituting this bound on $v(x,y)$ in (5.1.279), we get

$$r_{xy}(x,y) \le a(s,t)\, b(x,y)\, u(s,t) \exp\left(\int_x^s \int_y^t [\, a(s,t)\, b(\xi,\zeta) + c(\xi,\zeta)]\, d\xi\, d\zeta \right).$$

Now integrating both sides of the above inequality with respect to y from y to t, we obtain

$$\begin{aligned} &r_x(x,t) - r_x(x,y) \\ &\le a(s,t)\, u(s,t) \int_y^t b(x,n) \exp\left(\int_x^s \int_n^t [\, a(s,t)\, b(\xi,\zeta) + c(\xi,\zeta)]\, d\xi\, d\zeta \right) dn. \end{aligned}$$

Integrating both sides of the above inequality with respect to x from x to s, we conclude

$$\begin{aligned} r(x,y) \le u(s,t)\Big[1 + a(s,t) \Big(&\int_x^s \int_y^t b(m,n) \\ &\times \exp\left(\int_x^s \int_n^t [\, a(s,t)\, b(\xi,\zeta) + c(\xi,\zeta)]\, d\xi\, d\zeta \right) dm\, dn \Big) \Big]. \end{aligned} \tag{5.1.281}$$

Thus (5.1.276) follows from (5.1.277) and (5.1.281) since s and t are arbitrary in the interval $\mathbb{R}_+$. □

Now we introduce the following two-independent-variable generalization of the integral inequality established by Langenhop [351].

Theorem 5.1.41 (Langenhop [351]) *Let $u(s,t)$, $a(s,t)$, and $b(s,t)$ be as defined in Theorem 5.1.39; let $W(r)$ be a positive continuous, monotonic, non-decreasing function for all $r > 0$, $W(0) = 0$, and $(\partial/\partial y)\, W(r(x,y)) = W_y(r(x,y)) \geq 0$; and suppose further that the inequality*

$$u(s,t) \geq u(x,y) - a(s,t)\left(\int_x^s \int_y^t b(m,n)\, W(u(m,n))\, dm\, dn\right), \tag{5.1.282}$$

is satisfied for all $0 \leq x \leq s < +\infty$, $0 \leq y \leq t < +\infty$. Then for $s_1, t_1 \in I$, $0 \leq x \leq s \leq s_1$, $0 \leq y \leq t \leq t_1$,

$$u(s,t) \geq \Omega^{-1}[\Omega(u(x,y)) - a(s,t)\left(\int_x^s \int_y^t b(m,n)\, dm\, dn\right)], \tag{5.1.283}$$

where

$$\Omega(r) = \int_{r_0}^{r} \left(\frac{ds}{W(s)}\right), \ r \geq r_0 > 0, \tag{5.1.284}$$

Ω^{-1} is the inverse function of Ω, and

$$\Omega(u(x,y)) - a(s,t)\left(\int_y^t b(m,n)\, dm\, dn\right) \in Dom(\Omega^{-1}),$$

for all $0 \leq x \leq s < +\infty$, $0 \leq y \leq t < +\infty$.

Proof We may rewrite (5.1.282) as

$$u(x,y) \leq u(s,t) + a(s,t)\left(\int_y^t b(m,n)\, W(u(m,n))\, dm\, dn\right). \tag{5.1.285}$$

For fixed s and t in the interval I, we define for $0 \leq x \leq s$, $0 \leq y \leq t$,

$$\begin{aligned} r(x,y) &= u(s,t) + a(s,t)\left(\int_y^t b(m,n)\, W(u(m,n))\, dm\, dn\right), \\ r(x,y) &= r(s,y) = u(s,t), \end{aligned} \tag{5.1.286}$$

then from (5.1.286) it follows

$$r_{xy}(x,y) = a(s,t)\, b(x,y)\, W(u(x,y)),$$

which, in view of (5.1.285), implies

$$r_{xy}(x,y) \leq \ a(s,t)\ b(x,y)\ W(\ r(x,y)),$$

i.e.

$$\frac{r_{xy}(x,y)}{W(\ r(x,y))} \leq \ a(s,t)\ b(x,y). \tag{5.1.287}$$

From (5.1.287) we see that

$$\frac{W(\ r(x,y))(\ r_{xy}(x,y)}{W^2(\ r(x,y))} \leq \ a(s,t)\ b(x,y) + \frac{W_y(r(x,y))(r_x(x,y)}{W^2(\ r(x,y))},$$

i.e.,

$$\frac{\partial}{\partial y}(\frac{r_x(x,y)}{W(\ r(x,y))} \leq \ a(s,t)\ b(x,y).$$

Now integrating both sides of the above inequality with respect to y from y to t, we have

$$\frac{r_x(x,t)}{W(\ r(x,t))} - \frac{r_x(x,y)}{W(\ r(x,y))} \leq \ a(s,t) \int_y^t \ b(x,n)\ dn. \tag{5.1.288}$$

From (5.1.284) and (5.1.288), we observe that

$$\Omega_x(\ r(x,y)) - \Omega_x(\ r(x,y)) \leq \ a(s,t) \int_y^t \ b(x,n)\ dn.$$

Integrating both sides of the above inequality with respect to x from x to s, we have

$$\Omega(\ r(x,y)) \leq \Omega(\ u(s,t)) + \ a(s,t)(\int_x^s \int_y^t \ b(m,n)\ dm\ dn),$$

which implies

$$\Omega(\ u(s,t)) \geq \Omega(\ u(x,y)) - \ a(s,t)(\int_x^s \int_y^t \ b(m,n)\ dm\ dn). \tag{5.1.289}$$

Thus (5.1.283) follows from (5.1.289). The intervals of real numbers s and t are obvious. □

The next result deals with the two-independent-variable generalization of the integral inequality recently established in Theorem 1 of Pachpatte [456].

Theorem 5.1.42 (Pachpatte [456]) *Let $\phi(s,t)$, $a(s,t)$, $b(s,t)$, and $c(s,t)$ be real-valued non-negative continuous functions defined on $I \times I$; let $u(s,t)$ be a positive real-valued continuous function defined on $I \times I$; and suppose further that the inequality*

$$u(s,t) \geq \phi(x,y) - a(s,t)[\int_x^s \int_y^t b(m,n)\phi(m,n)\,dm\,dn + \int_x^s \int_y^t b(m,n)(\int_m^s \int_n^t c(\xi,\zeta)\phi(\xi,\zeta)\,d\xi\,d\zeta)\,dm\,dn], \tag{5.1.290}$$

is satisfied for all $0 \leq x \leq s < +\infty,\ 0 \leq y \leq t < +\infty$. *Then*

$$u(s,t) \geq \phi(x,y)[1 + a(s,t)(\int_x^s \int_y^t b(m,n) \times \exp(\int_m^s \int t_n[a(s,t)\,b(\xi,\zeta) + c(\xi,\zeta)]\,d\xi\,d\zeta)\,dm\,dn)]^{-1}, \tag{5.1.291}$$

for all $0 \leq x \leq s < +\infty,\ 0 \leq y \leq t < +\infty.$

Proof We may rewrite (5.1.290) as

$$\phi(x,y) \leq u(s,t) + a(s,t)[\int_x^s \int_y^t b(m,n)\phi(m,n)\,dm\,dn + \int_x^s \int_y^t b(m,n)(\int_m^s \int_n^t c(\xi,\zeta)\phi(\xi,\zeta)\,d\xi\,d\zeta)\,dm\,dn]. \tag{5.1.292}$$

For fixed s and t in the interval I, we define for all $0 \leq x \leq s,\ 0 \leq y \leq t$,

$$\begin{aligned} r(x,y) =\ & u(s,t) + a(s,t)[\int_x^s \int_y^t b(m,n)\phi(m,n)\,dm\,dn \\ & + \int_x^s \int_y^t b(m,n)(\int_m^s \int_n^t c(\xi,\zeta)\phi(\xi,\zeta)\,d\xi\,d\zeta)\,dm\,dn], \\ r(x,t) =\ & r(s,y) = u(s,t), \end{aligned} \tag{5.1.293}$$

then from (5.1.293) it follows

$$r_{xy}(x,y) = a(s,t)\,b(x,y)[\phi(x,y) + \int_x^s \int_y^t c(\xi,\zeta)\phi(\xi,\zeta)\,d\xi\,d\zeta],$$

which, in view of (5.1.292), implies

$$r_{xy}(x,y) \leq a(s,t)\, b(x,y)[\, r(x,y) + \int_x^s \int_y^t c(\xi,\zeta)\, r(\xi,\zeta)\, d\xi\, d\zeta]. \tag{5.1.294}$$

Define

$$\begin{aligned} v(x,y) &= r(x,y) + \int_x^s \int_y^t c(\xi,\zeta)\, r(\xi,\zeta)\, d\xi\, d\zeta, \\ v(s,y) &= v(x,t) = u(s,t), \end{aligned} \tag{5.1.295}$$

then from (5.1.295), we derive

$$v_{xy}(x,y) = r_{xy}(x,y) + c(x,y)\, r(x,y),$$

which, by using (5.1.294) and the inequality $r(x,y) \leq v(x,y)$, implies

$$v_{xy}(x,y) \leq [\, a(s,t)\, b(x,y) + c(x,y)]\, v(x,y).$$

By an argument similar to that in the proof of Theorem 5.1.41, we conclude

$$v(x,y) \leq u(s,t) \exp\left(\int_x^s \int_y^t [\, a(s,t)\, b(\xi,\zeta) + c(\xi,\zeta)]\, d\xi\, d\zeta \right).$$

Substituting this bound on $v(x,y)$ in (5.1.294), we have

$$r_{xy}(x,y) \leq a(s,t)\, b(x,y)\, u(s,t) \exp\left(\int_x^s \int_y^t [\, a(s,t)\, b(\xi,\zeta) + c(\xi,\zeta)]\, d\xi\, d\zeta \right).$$

Now integrating both sides of the above inequality with respect to y from y to t, we get

$$\begin{aligned} & r_x(x,t) - r_x(x,y) \\ & \leq a(s,t)\, u(s,t) \int_y^t b(x,n)\, \exp(\int_x^s \int_n^t [\, a(s,t)\, b(\xi,\zeta) + c(\xi,\zeta)]\, d\xi\, d\zeta)\, dn. \end{aligned}$$

Integrating both sides of the above inequality with respect to x from x to s, we obtain

$$\begin{aligned} r(x,y) \leq\ & u(s,t)[1 + a(s,t)(\int_x^s \int_y^t b(m,n) \\ & \times \exp(\int_x^s \int_n^t [\, a(s,t)\, b(\xi,\zeta) + c(\xi,\zeta)]\, d\xi\, d\zeta)\, dm\, dn)]. \end{aligned} \tag{5.1.296}$$

Thus (5.1.291) follows from (5.1.292) and (5.1.296) since s and t are arbitrary in the interval I. □

The next result, due to Medveď [398], establishes the Wendroff inequality of Henry type in two independent variables. To do this, we shall need the following well-known consequence of the Jensen inequality:

$$(A_1 + A_2 + \cdots + A_n) \leq n^{r-1}(A_1^r + A_2^r + \cdots + A_n^r). \tag{5.1.297}$$

The inequality considered here is the following

$$u(x, y) \leq a(x, y) + \int_0^x \int_0^y (x-s)^{\alpha-1}(y-t)^{\beta-1} \times F(s, t)\omega(u(s, t))dsdt, \tag{5.1.298}$$

for all $(x, y) \in [0, T)^2 = [0, T) \times [0, T)$ $(0 < T \leq +\infty)$, where $\alpha > 0, \beta > 0$. Results on integral inequalities in two variables with regular kernels (i.e., with $\alpha = 1, \beta = 1, F$ continuous and $a(x, y)$ constant) are contained in the books [90, 95, 482, 621].

We need the following lemma.

Lemma 5.1.5 (Medveď [398]) *Let $\omega : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a non-negative, non-decreasing C^1-function, $a(x, y)$ be a non-negative C^2-function on $[0, T)^2$ $(0 < T \leq +\infty)$ such that on $[0, T)^2$ $(0 < T \leq +\infty)$,*

$$\frac{\partial^2 a(x, y)}{\partial x \partial y} \geq 0, \quad \frac{\partial a(x, y)}{\partial y} \geq 0, \quad (or \ \frac{\partial a(x, y)}{\partial x} \geq 0).$$

Let $k(x, y)$ be a continuous, non-negative C^2-function and $z(x, y)$ be a continuous, non-negative function on $[0, T)^2$ with for all $(x, y) \in [0, T)^2$,

$$z(x, y) \leq a(x, y) + \int_0^x \int_0^y k(s, t)\omega(z(s, t))dsdt. \tag{5.1.299}$$

Then for all $(x, y) \in [0, T_1)^2$,

$$z(x, y) \leq \Omega^{-1}\left[\Omega(a(x, y)) + \int_0^x \int_0^y k(s, t)dsdt\right], \ for\ all\ \ (x, y) \in [0, T_1)^2, \tag{5.1.300}$$

where $T_1 > 0$ is such that the argument of Ω^{-1} in the above inequality belongs to Dom (Ω^{-1})

Proof Let $V(x, y)$ be the right-hand side of (5.1.299). Then

$$\frac{\partial^2 V(x, y)}{\partial x \partial y} = \frac{\partial^2 a(x, y)}{\partial x \partial y} + k(x, y)\omega(z(x, y)), \tag{5.1.301}$$

$$\frac{\partial^2 \Omega(V(x,y))}{\partial x \partial y} = \Omega'(V(x,y))\frac{\partial^2 V(x,y)}{\partial x \partial y} + \Omega''(V(x,y))\frac{\partial V(x,y)}{\partial x}\frac{\partial V(x,y)}{\partial y}. \tag{5.1.302}$$

Since $\Omega'(V) = \frac{1}{\omega(V)}$ and $\Omega''(V) \leq 0$, we derive from (5.1.301) and (5.1.302)

$$\begin{aligned}\frac{\partial^2 \Omega(V(x,y))}{\partial x \partial y} &\leq \frac{\partial^2 a(x,y)}{\partial x \partial y}\frac{1}{\omega(V)} + k(x,y) \\ &\leq \frac{\partial^2 a(x,y)}{\partial x \partial y}\frac{1}{\omega(a(x,y))} + k(x,y).\end{aligned} \tag{5.1.303}$$

However

$$\begin{aligned}\frac{\partial^2}{\partial x \partial y}\Omega(a(x,y)) &= \frac{\partial^2}{\partial x \partial y}\int_0^{a(x,y)} \frac{d\sigma}{\omega(\sigma)} = \frac{\partial}{\partial x}[\frac{\partial a(x,y)}{\partial y}\frac{1}{\omega(a(x,y))}] \\ &= \frac{\partial^2 a(x,y)}{\partial x \partial y}\frac{1}{\omega(a(x,y))} - \omega'(a(x,y))\frac{\partial a(x,y)}{\partial x}\frac{1}{\omega(a(x,y))^2} \\ &\geq \frac{\partial^2 a(x,y)}{\partial x \partial y}\frac{1}{\omega(a(x,y))},\end{aligned}$$

i.e.,

$$\frac{\partial^2}{\partial x \partial y}\Omega(a(x,y)) \geq \frac{\partial^2 a(x,y)}{\partial x \partial y}\frac{1}{\omega(a(x,y))}. \tag{5.1.304}$$

(If $\frac{\partial a}{\partial y} \geq 0$, then we can obtain (5.1.304) by estimating $\frac{\partial^2}{\partial x \partial y}\Omega(a(x,y))$.) Thus we deduce from (5.1.303) and (5.1.304)

$$\frac{\partial^2 \Omega(V(x,y))}{\partial x \partial y} \leq \frac{\partial^2 \Omega(a(x,y))}{\partial x \partial y} + k(x,y)$$

which yields

$$\Omega(v(x,y)) \leq \Omega(a(x,y)) + \int_0^x \int_0^y k(s,t)dsdt.$$

From the above inequality, we have

$$z(x,y) \leq V(x,y) \leq \Omega^{-1}\left[\Omega(a(x,y)) + \int_0^x \int_0^y k(s,t)dsdt\right].$$

Thus the proof is complete. □

Theorem 5.1.43 (Medveď [398]) *Let $a(x, y)$ be a non-negative, C^2-function satisfying on $[0, T)^2 = [0, T) \times [0, T)$ $(0 < T \leq +\infty)$,*

$$\frac{\partial^2 a(x, y)}{\partial x \partial y} \geq 0, \ \frac{\partial a(x, y)}{\partial x} \geq 0 \ \ (or \ \frac{\partial a(x, y)}{\partial y} \geq 0)$$

$u(x, y), F(x, y)$ be continuous, non-negative functions on $[0, T)^2$ satisfying the inequality (5.1.298), where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-negative C^1-function. Then the following assertions holds:

(i) Suppose $\alpha > \frac{1}{2}, \beta > \frac{1}{2}$ and satisfies the condition (q) with $q = 2$. Then

$$u(x, y) \leq e^{x+y} \left\{ \Omega^{-1}[(2a(x, y)^2) + 2K \int_0^x \int_0^y F^2(s, t) R(s + t) ds dt] \right\}^{\frac{1}{2}} \tag{5.1.305}$$

where

$$(x, y) \in [0, T_1)^2 = [0, T_1) \times [0, T_1), \ \ K = \frac{\Gamma(2\beta - 1)\Gamma(2\alpha - 1)}{4^{\alpha+\beta-1}}$$

and Γ is the Gamma function, $\Omega(v) = \int_{v_0}^{v} \frac{dy}{w(y)}, v \geq v_0 > 0, \Omega^{-1}$ is the inverse of Ω and $T_1 > 0$ is such that the argument of Ω^{-1} in (5.1.305) belongs to Dom (Ω^{-1}) for all $(x, y) \in [0, T_1)^2$.

(ii) Suppose $\alpha = \beta = \frac{1}{z+1}$ for some real number $z \geq 1$ and ω satisfies the condition (q) with $q = z + 2$. Then for all $(x, y) \in [0, T_2)^2$,

$$u(x, y) \leq e^{x+y} \left\{ \Omega^{-1} \left[\Omega(2a(x, y)^2) + M_z \int_0^x \int_0^y F^q(s, t) R(s + t) ds dt \right] \right\}^{\frac{1}{q}}, \tag{5.1.306}$$

where

$$p = \frac{z + 2}{z + 1}, \ \ M_z = \left(\frac{\Gamma(2 - p\delta)}{p^{(1-p\delta)}} \right)^{\frac{2}{p}}, \ \ \delta = 1 - \beta = \frac{z}{z + 1},$$

and $T_2 > 0$ is such that the argument of Ω^{-1} belongs to Dom (Ω^{-1}) for all $(x, y) \in [0, T_2)^2$.

Proof (i) Using the Cauchy-Schwarz inequality, we infer from (5.1.298)

$$\begin{aligned}u(x,y) &\leq a(x,y) + \int_0^x \int_0^y (x-s)^{\alpha-1} e^s (y-t)^{\beta-1} e^t \left[e^{-(s+t)} F(s,t)\omega(u(s,t))\right] dsdt \\ &\leq a(x,y) + \left[\int_0^x \int_0^y (x-s)^{2\alpha-2} e^{2s} (y-t)^{2\beta-2} e^{2t} dsdt\right]^{\frac{1}{2}} \\ &\quad \times \left[\int_0^x \int_0^y e^{-(s+t)} F^2(s,t)\omega(u(s,t))^2 dsdt\right]^{\frac{1}{2}}. \end{aligned} \tag{5.1.307}$$

For the first integral in (5.1.307), we have

$$\begin{aligned}\int_0^x \int_0^y (x-s)^{2\alpha-2} e^{2s} (y-t)^{2\beta-2} e^{2t} dsdt &= e^{2(x+y)} \int_0^x \sigma^{2\alpha-2} e^{-2\sigma} \int_0^y \eta^{2\beta-2} e^{-2\eta} d\sigma d\eta \\ &= \frac{e^{2(x+y)}}{2^{2(\alpha+\beta)-2}} \int_0^x \sigma^{2\alpha-2} e^{-\sigma} \left(\int_0^y \eta^{2\beta-2} e^{-\xi} d\sigma\right) d\xi \\ &\leq \frac{e^{2(x+y)}}{2^{2(\alpha+\beta)-2}} \Gamma(2\beta-1)\Gamma(2\alpha-1).\end{aligned}$$

Therefore we obtain from (5.1.307),

$$u(x,y) \leq a(x,y) K^{\frac{1}{2}} \left[\int_0^x \int_0^y F^2(s,t) e^{-2(s+t)} \omega(u(s,t))^2 dsdt\right]^{\frac{1}{2}}$$

where K is as in Theorem 5.1.43. Using the inequality (5.1.298) with $n = 2, r = 2$ and applying the condition (q) with $q = 2$, we obtain

$$v(x,y) \leq \alpha(x,y) + 2K \int_0^x \int_0^y F^2(s,t) R(s+t) \omega(v(s,t)) dsdt \tag{5.1.308}$$

where

$$v(x,y) = \left(e^{-(x+y)} u^2(x,y)\right), \alpha(x,y) = 2a^2(x,y). \tag{5.1.309}$$

Applying Lemma 5.1.5 to the inequality (5.1.308), we obtain

$$v(x,y) \leq \Omega^{-1}\left[\Omega(a(x,y)) + 2K \int_0^x \int_0^y F^2(s,t) R(s+t) dtds\right]. \tag{5.1.310}$$

Using (5.1.309), we have

$$u(x,y) \leq e^{x+y}\left\{\Omega^{-1}\left[\Omega(2a(x,y))^2 + 2K \int_0^x \int_0^y F^2(s,t) R(t+s) dtds\right]\right\}^{\frac{1}{2}}. \tag{5.1.311}$$

(ii) Let $p = \frac{z+2}{z+1}, q = z + 2$. Then

$$u(x,y) \leq a(x,y) + \left[\int_0^x \int_0^y (x-s)^{-p\delta} e^{ps} (y-t)^{-p\delta} e^{pt} dsdt\right]^{\frac{1}{p}} \times \left[\int_0^x \int_0^y e^{-q(s+t)} F^q(s,t) \omega^q(u(s,t)) dtds\right]^{\frac{1}{q}}. \tag{5.1.312}$$

Noting that

$$\int_0^x \int_0^y (x-s)^{-p\delta} e^{ps} (y-t)^{-p\delta} e^{pt} dsdt = \int_0^x (x-s)^{-p\delta} e^{ps} \int_0^y (y-t)^{-p\delta} e^{-pt} dtds$$
$$\leq \frac{e^y}{p^{1-p\delta}} \Gamma(1-p\delta) \int_0^y (x-s)^{-p\delta} e^{ps} ds \leq \frac{e^{x+y}}{p^{2(1-p\delta)}} \Gamma^2(1-p\delta),$$

we deduce

$$u(x,y) \leq a(x,y) + Ke^{x+y} \left[\int_0^x \int_0^y F^q(s,t) R(t-s) \omega(e^{-q(s+t)} u^q(s,t)) dsdt\right]^{\frac{1}{q}}$$

which yields

$$\nu(x,y) \leq a(x,y) + 2K^2 \int_0^x \int_0^y F^q(s,t) R(t+s) \omega(\nu(s,t)) dsdt,$$

where

$$\alpha(x,y) = 2a^2(x,y), \quad \nu(x,y) = \left(e^{-(x+y)} u(x,y)\right)^q, \quad M_z = \left(\frac{\Gamma(1-p\delta)}{p^{1-p\delta}}\right)^{\frac{2}{p}}$$

and this gives the inequality from the assertion (ii).

If $\alpha \neq \beta, \alpha, \beta < \frac{1}{2}$, then there are some technical problems and we omit this case. □

Remark 5.1.10 If $a(x,y)$ is a constant, then the above theorem is a consequence of [388]. In this case, it suffices to assume that ω is only continuous.

Theorem 5.1.44 (Medveď [398]) *Let functions a, F be as in Theorem 5.1.43 and $u(x,y)$ be a continuous, non-negative function on $[0,T)^2$ satisfying the inequality*

$$u(x,y) \leq a(x,y) + \int_0^x \int_0^y (x-s)^{\beta-1} (y-t)^{\beta-1} s^{\gamma-1} F(s,t) u(s,t) dsdt, \tag{5.1.313}$$

where $\beta > 0, \gamma > 0$. Then the following assertions hold:

(i) If $\beta > \frac{1}{2}, \gamma > 1 - \frac{1}{2p}$*, then for all* $(x, y) \in [0, T)^2$,

$$u(x, y) \leq e^{x+y}\Phi(x, y), \tag{5.1.314}$$

where

$$\Phi(x, y) = 2^{1-\frac{1}{2q}} \exp\left[\frac{4^{q-1}}{q} K^q L^q \int_0^x \int_0^y F^{2q}(s, t)e^{q(s+t)}dsdt\right], \tag{5.1.315}$$

and K *is as Theorem 5.1.43,*

$$L = \left(\frac{\Gamma((2\gamma - 2)p + 1)}{p^{(2\gamma-2)p+1}}\right)^{\frac{2}{q}}, \quad p \geq 1, \quad q \geq 1, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

(ii) Let $\beta = \frac{1}{z+1}$ *for some real number* $z \geq 1$, $p = \frac{z+2}{z+1}, q = z + 1, \gamma > 1 - \frac{1}{kq}$, *where* $k > 1$. *Then*

$$u(x, y) \leq e^{x+y}\Psi(x, y), \tag{5.1.316}$$

where

$$\Psi(x, y) = 2^{1-\frac{1}{rq}} a(x, y) \exp\left[\frac{Q^{rq}}{nq} \int_0^x \int_0^y e^{r(s+t)}F^{rq}(s, t)dsdt\right],$$

and $\gamma > 1$ *is such that* $1/k + 1/r = 1$, $Q = M_z P$, M_z *is as in Theorem 5.1.43,* $P = [\Gamma(sq(\gamma - 1) + 1)]^{2/k}$ *and* $\alpha = -z/(z + 1) = \beta - 1$.

Proof (i) From (5.1.313) it follows

$$\begin{aligned}
u(x, y) &\leq a(x, y) + \left[\int_0^x \int_0^y (x - s)^{2\gamma-2}e^{2s}(y - t)^{2\beta-2}e^{2t}dsdt\right]^{1/2} \\
&\quad \times \left[\int_0^x \int_0^y s^{2\gamma-2}t^{2\gamma-2}F^2(s, t)(e^{-(s+t)}u(s, t))^2 dsdt\right]^{1/2} \\
&\leq a(x, y) + e^{x+y}K^{1/2} \\
&\quad \times \left[\int_0^x \int_0^y s^{2\gamma-2}t^{2\gamma-2}F^2(s, t)(e^{-(s+t)}u(s, t))^2 dsdt\right]^{1/2},
\end{aligned}$$

where K is in Theorem 5.1.43, we derive

$$v(x, y) \leq c(x, y) + 2K \int_0^x \int_0^y s^{2\gamma-2}t^{2\gamma-2}F^2(s, t)v(s, t)dsdt, \tag{5.1.317}$$

where

$$v(x,y) = \left(e^{-(x+y)}u(x,y)\right)^2, \quad c(x,y) = 2a^2(x,y). \tag{5.1.318}$$

From (5.1.317), it follows

$$v(x,y) \le c(x,y) + 2K\left[\int_0^x\int_0^y s^{(2\gamma-2)p}t^{(2\gamma-2)p}e^{-p(s+t)}dsdt\right]^{1/p} \times\left[\int_0^x\int_0^y F^{2q}(s,t)e^{q(s+t)}v^q(s,t)dsdt\right]^{1/q} \tag{5.1.319}$$

where p, q are as in theorem. For the first integral in (5.1.319), we have

$$\begin{aligned}&\int_0^x\int_0^y s^{(2\gamma-2)p}t^{(2\gamma-2)p}e^{-p(s+t)}dsdt\\ &= \frac{1}{(p^{(2\gamma-2)p+1})^2}\int_0^{px}\sigma^{(2\gamma-2)p}e^{-\sigma}\left(\int_0^{py}\tau^{(2\gamma-2)p}e^{-\tau}d\tau\right)d\sigma\\ &\le \left(\frac{\Gamma((2\gamma-2)p+1)}{p^{(2\gamma-2)p+1}}\right)^2\end{aligned}$$

and whence from (5.1.319)

$$v(x,y) \le c(x,y) + 2KL\int_0^x\int_0^y F^{2q}(s,t)e^{q(s+t)}v^q(s,t)dsdt, \tag{5.1.320}$$

where L is defined in theorem. This yields

$$v^q(x,y) \le 2^{q-1}\left[c^q(x,y) + 2^qK^qL^q\int_0^x\int_0^y F^{2q}(s,t)e^{q(s+t)}v^q(s,t)dsdt\right]. \tag{5.1.321}$$

We note that from the assumptions it follows that

$$\frac{\partial c(x,y)}{\partial x\partial y} \ge 0, \quad \frac{\partial c(x,y)}{\partial x} \ge 0, \quad (or\ \frac{\partial c(x,y)}{\partial y} \ge 0).$$

Thus from Lemma 5.1.5 and (5.1.321), we derive

$$v^q(x,y) \le 2^{q-1}c^q(x,y)\exp\left[\frac{4^q}{2}K^qL^q\int_0^x\int_0^y F^{2q}(s,t)e^{q(s+t)}dsdt\right]$$

which, along with (5.1.318), yields (5.1.314).

(ii) From the inequality (5.1.313), we obtain

$$u(x,y) \le a(x,y) + \left[\int_0^x \int_0^y (x-s)^{-p\alpha}(y-t)^{-p\alpha} e^{p(s+t)} dsdt\right]^{1/p}$$
$$\times \left[\int_0^x \int_0^y s^{q(\gamma-1)} t^{q(\gamma-1)} e^{-q(s+t)} F^q(s,t) u^q(s,t) dsdt\right]^{1/q}$$
$$\le a(x,y) + e^{x+y}\left(\frac{\Gamma(1-\alpha p)}{p^{1-\alpha p}}\right)^{2/p}$$
$$\times \left[\int_0^x \int_0^y s^{kq(\gamma-1)} t^{kq(\gamma-1)} e^{-(s+t)} dsdt\right]^{1/k}$$
$$\times \left[\int_0^x \int_0^y e^{r(s+t)} F^{rq}(s,t) \left(e^{-(s+t)}u(s,t)\right)^{rq} dsdt\right]^{1/rq}$$
$$\le a(x,y) + e^{x+y} Q \left[\int_0^x \int_0^y e^{r(s+t)} F^{rq}(s,t) \left(e^{-(s+t)}u(s,t)\right)^{rq} dsdt\right]^{1/rq}$$

where $Q = M_z P$, M_z is defined in Theorem 5.1.43, P is as in theorem and r, k are as in the assertion (ii). The above inequality yields

$$v(x,y) \le 2^{qr-1}\left[a^{rq}(x,y) + Q^{rq}\int_0^x \int_0^y e^{r(s+t)} F^{rq}(s,t) v(s,t) dsdt\right]$$

where

$$v(x,y) = \left(e^{-(x+y)}u(x,y)\right)^{rq}. \tag{5.1.322}$$

Therefore

$$v(x,y) \le 2^{qr-1} a^{rq}(x,y) \exp\left[Q^{rq}\int_0^x \int_0^y e^{r(s+t)} F^{rq}(s,t) dsdt\right]$$

and using (5.1.322), we can obtain (5.1.316). □

We shall prove a theorem corresponding to an analogue of Ou-Yang-Pachpatte inequality (see, e.g., [396, 493]).

Theorem 5.1.45 (Medveď [398]) *Let $T > 0$, F and ω be as in Theorem 5.1.43 and a be a positive constant. Let $u(x,y)$ be a continuous, non-negative function on $[0,T)^2$ satisfying the inequality for all $(x,y) \in [0,T)^2$,*

$$u^2(x,y) \le a + \int_0^x \int_0^y (x-s)^{\alpha-l}(y-t)^{\beta-1} F(s,t)\omega(u(s,t)) dsdt. \tag{5.1.323}$$

Then the following assertions hold:

(i) Suppose $\alpha > 1/2, \beta > 1/2$, *and* ω *satisfies the condition(q) with* $q = 2$. *Then for all* $(x, y) \in [0, T)^2$,

$$u(x, y) \le e^{x+y}\Phi(x, y), \tag{5.1.324}$$

where for all $(x, y) \in [0, T)^2$,

$$\Phi(x, y) = \left[\Lambda^{-1}\left(\Lambda(2a^2) + 2K\int_0^x\int_0^y F^2(s, t)R(s + t)dsdt\right)\right]^{1/4},$$

and K is the number from Theorem 5.1.43 and $\Lambda(\nu) = \int_{\nu_0}^{\nu} d\sigma/\omega(\sqrt{\sigma}),\ \nu \ge \nu_0 > 0,\ T_1 > 0$ *is such that the argument of* Λ^{-1} *belongs to Dom* (Λ^{-1}) *for all* $(x, y) \in [0, T_1)^2$.

(ii) Suppose $\alpha = \beta = 1/(z + 1)$ *for some real numbers* $z \ge 1$ *and let* $p = (z + 2)/(z + 1), q = z + 2$. *Assume that* ω *satisfies the condition (q) with* $q = z + 2$. *Then for all* $(x, y) \in [0, T_2)^2$,

$$u(x, y) \le e^{x+y}\Psi(x, y), \tag{5.1.325}$$

where

$$\Psi(x, y) = \left[\Lambda^{-1}\left(\Lambda(2^{q-1}a^q)\right) + 2^{q-1}M_z^q\int_0^x\int_0^y F^q(s, t)R(s + t)dsdt\right]^{1/2q}$$

for all $(x, y) \in [0, T_2), T_2 > 0$ *is such that the argument of* Λ^{-1} *in the above inequality belongs to Dom* (Λ^{-1}) *for all* $(x, y) \in [0, T_2)$, M_z *is as in Theorem 5.1.43.*

Proof Let us prove (ii). Using the Cauchy-Schwarz inequality and inequality (5.1.308), we obtain

$$\begin{aligned}
u^2(x, y) &\le a + \int_0^x\int_0^y (x - s)^{\alpha-1}(y - t)^{\beta-1}e^{s+t}F(s, t)\omega(u(s, t))dsdt \\
&\le a + \left(\int_0^x\int_0^y (x - s)^{2\alpha-2}(y - t)^{2\beta-2}e^{2(s+t)}dsdt\right)^{1/2} \\
&\quad \times \left(\int_0^x\int_0^y F^2(s, t)R(s + t)\omega(e^{-2(s+t)}u(s, t)^2)dsdt\right)^{1/2} \\
&\le a + Ke^{-(x+y)}\left(\int_0^x\int_0^y F^2(s, t)R(s + t)\omega(e^{-2(s+t)}u(s, t)^2)dsdt\right)^{1/2}
\end{aligned}$$

where K is as in Theorem 5.1.43. Applying the inequality (5.1.297) similarly as in the proof of Theorem 5.1.43, we obtain

$$e^{-(x+y)}u^2(x,y) \le 2a^2 + 2K\int_0^x\int_0^y F^2(s,t)R(s+t)\omega(e^{-(s+t)}u(s,t))dsdt,$$

where K is an in Theorem 5.1.43. This yields

$$v^2(x,y) \le c + 2K\int_0^x\int_0^y F^2(s,t)R(s+t)\omega(v(s,t))dsdt, \tag{5.1.326}$$

where

$$v(x,y) = (e^{-(x+y)}u(x,y))^2, \quad c = 2a^2. \tag{5.1.327}$$

Let $V(x,y)$ be the right-hand side of (5.1.326). Then

$$v(x,y) \le \sqrt{V(x,y)}, \qquad \omega(v(x,t)) \le \omega(\sqrt{V(x,y)}. \tag{5.1.328}$$

We have

$$\frac{\partial^2 V(x,y)}{\partial x\partial y} = 2KF^2(x,y)R(x+y)\omega(v(x,y)) \tag{5.1.329}$$

and

$$\begin{aligned}
&\frac{\partial^2}{\partial x\partial y}\int_0^{V(x,y)}\frac{dt}{\omega(\sqrt{t})} = \frac{\partial}{\partial x}\Big(\frac{\partial V(x,y)/\partial y}{\omega(\sqrt{V(x,y)})}\Big)\\
&= \frac{\partial^2 V(x,y)}{\partial x\partial y}\frac{1}{\omega(\sqrt{V(x,y)})} - \frac{\partial V(x,y)}{\partial y}\frac{\partial V(x,y)}{\partial x}\frac{\omega'(\sqrt{V(x,y)})}{2\sqrt{V(x,y)}\omega(\sqrt{V(x,y)})^2}\\
&\le \frac{\partial^2 V(x,y)}{\partial x\partial y}\frac{1}{\omega(\sqrt{V(x,y)})},
\end{aligned}$$

i.e.,

$$\frac{\partial^2}{\partial x\partial y}\Lambda(V(x,y)) \le \frac{\partial^2 V(x,y)}{\partial x\partial y}\frac{1}{\omega(\sqrt{V(x,y)})}, \tag{5.1.330}$$

which, together with (5.1.329), gives us

$$\frac{\partial^2}{\partial x\partial y}\Lambda(V(x,y)) \le 2K\int_0^x\int_0^y F^2(s,t)R(s+t)dsdt,$$

and using (5.1.327) and (5.1.328), we obtain (5.1.324).

Now let us prove (ii). Following the proof of the assertion (ii) of Theorem 5.1.43, we can show that

$$w^2(x,y) \leq \alpha + 2K^2 \int_0^x \int_0^y F^q(s,t)R(s+t)\omega(w(s,t))dsdt \tag{5.1.331}$$

where

$$\alpha = 2a^2, \quad w(x,y) = \left(e^{-(x+y)}u(x,y)\right)^q .$$

Applying the same procedure to (5.1.331) as that used in the proof of the assertion (ii) as well as that one from the proof of (ii) of Theorem 5.1.43, we can prove (5.1.325). □

5.2 Linear Two-Dimensional Continuous Integral Inequalities of Volterra Type

In this section, we shall introduce the so-called inequality of the Volterra-Fredholm type which can be applied to study the boundedness, stability and uniqueness of the solutions of some integral equations and their systems.

Hacia [249] studied some special cases of two-dimensional inequalities of the Volterra type

$$u(x,t) \leq f(x,t) + \int_0^t \int_a^b k(x,t,y,s)u(y,s)dyds, \tag{5.2.1}$$

and presented some generations of the results on the generalizations of the Bellmon-Gronwall inequality in two independent variables in [42, 95, 152].

Using the theory of Volterra-Fredholm equations (see, [247]), the following result on integral inequalities can be obtained.

Theorem 5.2.1 (Hacia [247]) *Let f be a continuous function in $D = \{(x,t) : a \leq x \leq b, t \geq 0\}$ and k be non-negative and continuous in $\Omega = \{(x,y,s,t) : a \leq x, y \leq b, 0 \leq s \leq t < +\infty\}$. If the continuous function u satisfies inequality (5.2.1), then*

$$u(x,t) \leq f(x,t) + \int_0^t \int_a^b r(x,t,y,s)u(y,s)dyds \tag{5.2.2}$$

where the resolvent kernel k is of the form

$$r(x,y,s,t) = \sum_{n=0}^{+\infty} k_n(x,t,y,s) \tag{5.2.3}$$

and the iterated kernel k_n is defined by the following formula

$$\begin{cases} k_n(x,t,y,s) = \int_s^t \int_a^b k(x,t,p,q)k_{n-1}(p,q,y,s)dpdq, \quad n = 1,2,3,\dots \\ k_0(x,t,y,s) = k(x,t,y,s). \end{cases} \tag{5.2.4}$$

Proof For a continuous and non-negative function g in D, from inequality (5.2.1), we get the Volterra-Fredholm integral equation

$$u(x,t) = f(x,t) - g(x,t) + \int_0^t \int_a^b k(x,t,y,s)u(y,s)dyds.$$

Using the resolvent method, we get

$$u(x,t) = f(x,t) - g(x,t) + \int_0^t \int_a^b r(x,t,y,s)\big[f(y,s) - g(y,s)\big]dyds.$$

Since $g(x,t) \geq 0$, we obtain (5.2.1). □

Next, let us consider a special case of inequality (5.2.1) with $k(x,t,y,s) = A(x,t)B(y,s)$.

Theorem 5.2.2 (Hacia [247]) *Let A, B, f, u be a continuous in D. If $A \cdot B$ is non-negative in Ω and u satisfies*

$$u(x,t) \leq f(x,t) + A(x,t)\int_0^t \int_a^b B(y,s)u(y,s)dsdt, \tag{5.2.5}$$

then

$$u(x,t) \leq f(x,t) + A(x,t)\int_0^t \int_a^b B(y,s)\exp\left[\int_s^t \int_a^b A(z,\tau)B(z,\tau)dzd\tau\right]f(y,s)dsdt. \tag{5.2.6}$$

Proof In this case,

$$k_0(x,t,y,s) = k(x,t,y,s) = A(x,t)B(y,s).$$

By virtue of (5.2.4), we get

$$k_1(x,t,y,s) = \int_s^t \int_a^b A(x,t)B(z,\tau)A(z,\tau)B(y,s)dzd\tau = A(x,t)B(y,s) \cdot L(t),$$

where

$$L(t) = \int_s^t M(\tau)d\tau,\ M(\tau) = \int_a^b A(z,\tau)B(z,\tau)dz$$

and

$$L(s) = 0,\ L'(t) = M(t).$$

Similarly, we have

$$\begin{aligned} k_2(x,t,y,s) &= \int_s^t \int_a^b A(x,t)B(z,\tau)A(z,\tau)B(y,s)L(\tau)d\tau dz \\ &= A(x,t)B(y,s)\int_x^t M(\tau)L(\tau)d\tau \\ &= A(x,t)B(y,s)\int_s^t L'(t)L(\tau)d\tau \\ &= A(x,t)B(y,s)\int_s^t \frac{d}{d\tau}\Big(\frac{L^2(\tau)}{2!}\Big)d\tau \\ &= A(x,t)B(y,s)\frac{L^2(t)}{2!}. \end{aligned}$$

By induction, we obtain

$$k_n(x,t,y,s) = A(x,t)B(y,s)\frac{L^n(t)}{n!}.$$

Next, from (5.2.3) it follows that

$$\begin{aligned} r(x,t,y,s) &= A(x,t)B(y,s)\sum_{n=0}^{+\infty}\frac{L^n(t)}{n!} \\ &= A(x,t)B(y,s)\exp\big[L(t)\big] \\ &= A(x,t)B(y,s)\exp\left(\int_s^t\int_a^b A(z,\tau)B(z,\tau)dzd\tau\right). \end{aligned}$$

Using Theorem 5.2.1, the proof is thus complete. □

Lemma 5.2.1 (Hacia [247]) *If h is continuous in D, then*

$$1 + \int_0^t \int_a^b h(y,s)\exp\left(\int_s^t\int_a^b h(z,\tau)dzd\tau\right)dyds = \exp\left(\int_0^t\int_a^b h(y,s)dyds\right).$$

Proof If we introduce the notation

$$\int_a^b h(y,s)dy = H(s),\ \int_0^t H(s) = \chi(t),$$

then

$$\chi'(t) = h(t),\ \chi(0) = 0.$$

Thus a direct computation gives us

$$\begin{aligned}
&1 + \int_0^t \int_a^b h(y,s)\exp\left(\int_s^t \int_a^b h(z,\tau)dzd\tau\right)dyds\\
&= 1 + \int_0^t H(s)\exp\left(\int_s^t H(\tau)d\tau\right)ds = 1 + \int_0^t \chi'(s)\exp\big[\chi(t) - \chi(s)\big]ds\\
&= 1 + \exp\chi(t)\int_0^t \chi'(s)\exp\big[-\chi(s)\big]ds = 1 - \exp\chi(t)\exp\big[-\chi(s)\big]\Big|_0^t\\
&= 1 - \exp\chi(t)\exp\big[-\chi(t)\big] + \exp\chi(t) = \exp\chi(t)\\
&= \exp\left(\int_0^t H(s)ds\right) = \exp\left(\int_0^t \int_a^b h(y,s)dyds\right).
\end{aligned}$$

The proof is now complete. □

Corollary 5.2.1 *If the assumptions of Theorem 5.2.2 hold, then*

$$u(x,y) \le F(t)\left[1 + A(x,t)\int_0^t \int_a^b B(y,s)\exp\left(\int_s^t \int_a^b A(z,\tau)B(z,\tau)dzd\tau\right)dyds\right], \tag{5.2.7}$$

where

$$F(t) = \sup\Big\{f(x,s) : a \le x \le b, 0 \le s \le t\Big\}.$$

Remark 5.2.1 If $A(x,t) = 1$, then we get an analogue of the Gronwall-Bellman inequality

$$u(x,t) \le F(t)\exp\left(\int_0^t \int_a^b B(y,s)dyds\right). \tag{5.2.8}$$

In fact, to achieve the above inequality, it suffices to employ Lemma 5.2.1.

Corollary 5.2.2 *If $f(x,t) = C$, a constant, or f is bounded in D, (i.e., there exists a constant $C > 0$ such that $|f(x,t)| \leq C$), then the inequality*

$$u(x,t) \leq f(x,t) + \int_0^t \int_a^b B(y,s)u(y,s)dyds \tag{5.2.9}$$

implies

$$u(x,t) \leq C \exp\left(\int_0^t \int_a^b B(y,s)dyds\right). \tag{5.2.10}$$

Theorem 5.2.3 (Hacia [247]) *Let the assumptions of Theorem 5.2.2 be satisfied. If u satisfies inequality (5.2.5), then the following inequality holds,*

$$u(x,t) \leq H(x,t) \exp\left(\int_0^t \int_a^b M(y,s)B(y,s)dyds\right) \tag{5.2.11}$$

where $H(x,t) = max\{A(x,t), f(x,t)\} \neq 0$.

Proof In fact, inequality (5.2.5) leads to

$$u(x,t) \leq H(x,t)\left[1 + \int_0^t \int_a^b B(y,s)u(y,s)dyds\right]$$

or

$$\frac{u(x,t)}{H(x,t)} \leq 1 + \int_0^t \int_a^b H(y,s)B(y,s)\frac{u(y,s)}{H(y,s)}dyds.$$

Applying Corollary 5.2.2 with $C = 1$, we get

$$\frac{u(x,t)}{H(x,t)} \leq \exp\left(\int_0^t \int_a^b H(y,s)B(y,s)dyds\right)$$

which implies (5.2.11). □

Corollary 5.2.3 (Hacia [247]) *If the assumptions of Theorem 5.2.3 are satisfied, then inequality (5.2.11) leads to*

$$u(x,t) \leq A(x,t) \exp\left(\int_0^t \int_a^b A(y,s)B(y,s)dyds\right),$$

as

$$f(x,t) \leq A(x,t) \neq 0, \tag{5.2.12}$$

or

$$u(x,t) \leq f(x,t) \exp\left(\int_0^t \int_a^b f(y,s)B(y,s)dyds\right),$$

as

$$0 \leq f(x,t) < A(x,t). \tag{5.2.13}$$

Theorem 5.2.4 (Hacia [247]) *Suppose that the assumptions of Theorem 5.2.2 are fulfilled. If* $A(x,t) \neq 0$, *then inequality (5.2.5) implies*

$$u(x,t) \leq \Phi(t)A(x,t) \exp\left(\int_0^t \int_a^b A(y,s)B(y,s)dyds\right), \tag{5.2.14}$$

where $\Phi(t) = sup\{\frac{f(x,s)}{A(x,s)} : a \leq x \leq b, 0 \leq s \leq t\}$.

Proof Obviously, inequality (5.2.5) can be written in the form

$$\frac{u(x,t)}{A(x,t)} \leq \frac{f(x,t)}{A(x,t)} + \int_0^t \int_a^b A(y,s)B(y,s)\frac{u(y,s)}{A(y,s)}dyds.$$

By virtue of Remark 5.2.1, we get

$$\frac{u(x,t)}{A(x,t)} \leq \Phi(t) \exp\left(\int_0^t \int_a^b A(y,s)B(y,s)dyds\right),$$

which concludes the proof. □

Theorem 5.2.5 (Hacia [247]) *Let* f *and* k *are continuous functions in* D *and* Ω, *respectively. If* k *is non-negative and satisfies in* Ω *the condition*

$$k(x,t,y,s) \leq K(y,s) \tag{5.2.15}$$

and a continuous function u *satisfies inequality (5.2.2), then*

$$u(x,t) \leq F(t) \exp\left(\int_0^t \int_a^b K(y,s)dyds\right). \tag{5.2.16}$$

Proof Estimate (5.2.16) easily follows by applying Remark 5.2.1 to the inequality

$$u(x,t) \leq f(x,t) + \int_0^t \int_a^b K(y,s)u(y,s)dyds.$$

Thus the proof is complete. □

Theorem 5.2.6 (Hacia [247]) *Suppose that the assumptions of Theorem 5.2.5 are satisfied and the condition (5.2.15) is replaced by*

$$k(x,t,y,s) \le N(x,t). \tag{5.2.17}$$

Then

$$u(x,t) \le N^*(t)N(x,t)\exp\left(\int_0^t \int_a^b N(y,s)dyds\right) \tag{5.2.18}$$

where

$$N^*(t) = \sup\left\{\frac{f(x,s)}{N(x,s)} : a \le x \le b, 0 \le s \le t\right\}.$$

Proof From (5.2.17) and (5.2.1) it follows that

$$u(x,t) \le f(x,t) + N(x,t)\int_0^t \int_a^b u(y,s)dyds,$$

i.e.,

$$\frac{u(x,t)}{N(x,t)} \le \frac{f(x,t)}{N(x,t)} + \int_0^t \int_a^b N(y,s)\frac{u(y,s)}{N(y,s)}dyds.$$

Using Remark 5.2.1, we get

$$\frac{u(x,t)}{N(x,t)} \le N^*(t)\exp\left(\int_0^t \int_a^b N(y,s)dyds\right),$$

which finishes the proof. □

Remark 5.2.2 If, in (5.2.1), $k(x,t,y,s) \le A(x,t)B(y,s), A(x,t) = 0$, then we get inequality (5.2.5), which leads to (5.2.14).

Remark 5.2.3 The above results are true for the Volterra-Fredholm inequality

$$u(x,t) \le f(x,t) + \int_0^t \int_G k(x,t,y,s)u(y,s)dyds, \tag{5.2.19}$$

where G is a certain compact subset of $\mathbb{R}^K$.

Remark 5.2.4 Obviously, the results in Theorems 5.2.1–5.2.6 can extended to the class L^2.

5.3 Linear Two-Dimensional Continuous Retarded Integral Inequalities

In this section, we introduce some two-dimensional linear continuous retarded integral inequalities.

In what follows, $\mathbb{R}_1 = [1, +\infty)$, $I = [t_0, T)$, $J_1 = [x_0, X)$, and $J_2 = [y_0, Y)$ are the given subsets of $\mathbb{R}$; $\triangle = J_1 \times J_2$. The first-order partial derivatives of a function $z(x, y)$ for $x, y \in \mathbb{R}$ with respect to x and y are denoted by $D_1z(x, y)$, $D_2z(x, y)$ and $D_1D_2z(x, y)$ (or z_{xy}) , respectively.

The next result is due to Pachpatte [501].

Theorem 5.3.1 (Pachpatte [501]) *Let $a, b \in C(\triangle, \mathbb{R}_+)$ and $\alpha \in C^1(J_1, J_1), \beta \in C^1(J_2, J_2)$ be non-decreasing with $\alpha(x) \le x$ on J_1, $\beta(y) \le y$ on J_2. Let k, c, p be as in Theorem 1.2.30.*

If $u \in C(\triangle, \mathbb{R}_+)$ and for all $(x, y) \in \triangle$,

$$u(x, y) \le k + \int_{x_0}^{x}\int_{y_0}^{y} a(s, t)u(s, t)dtds + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s, t)u(s, t)dtds, \quad (5.3.1)$$

then for all $(x, y) \in \triangle$,

$$u(x, y) \le k \exp(A(x, y) + B(x, y)), \quad (5.3.2)$$

where for all $(x, y) \in \triangle$,

$$A(x, y) = \int_{x_0}^{x}\int_{y_0}^{y} a(s, t)dtds, \quad B(x, y) = \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s, t)dtds. \quad (5.3.3)$$

Proof First, from the hypotheses, we can see that $\alpha'(t) \ge 0$ for all $t \in I, \alpha'(x) \ge 0$ for all $x \in J_1$, $\beta'(y) \ge 0$ for all $y \in J_2$.

Let $k > 0$ and define a function $z(x, y)$ by the right-hand side of (5.3.1). Then $z(x, y) > 0, z(x_0, y) = z(x, y_0) = k$, and $u(x, y) \le z(x, y)$, and hence

$$\begin{aligned}
D_1z(x, y) &= \int_{y_0}^{y} a(x, t)u(x, t)dt + \Big(\int_{\beta(y_0)}^{\beta(y)} b(\alpha(x), t)u(\alpha(x), t)dt\Big)\alpha'(x) \\
&\le \int_{y_0}^{y} a(x, t)z(x, t)dt + \Big(\int_{\beta(y_0)}^{\beta(y)} b(\alpha(x), t)z(\alpha(x), t)dt\Big)\alpha'(x) \\
&\le z(x, y)\int_{y_0}^{y} a(x, t)dt + z(\alpha(x), \beta(y))\Big(\int_{\beta(y_0)}^{\beta(y)} b(\alpha(x), t)dt\Big)\alpha'(x) \\
&\le z(x, y)[\int_{y_0}^{y} a(x, t)dt + \Big(\int_{\beta(y_0)}^{\beta(y)} b(\alpha(x), t)dt\Big)\alpha'(x)],
\end{aligned}$$

which gives us

$$\frac{D_1 z(x,y)}{z(x,y)} \le \int_{y_0}^{y} a(x,t)dt + \Big(\int_{\beta(y_0)}^{\beta(y)} b(\alpha(x),t)dt \Big) \alpha'(x). \tag{5.3.4}$$

Keeping y fixed in (5.3.4), setting $x = \sigma$, and integrating it with respect to σ from x_0 to $x, x \in J_1$, and making the change of variable, we conclude

$$u(x,y) \le k \exp\Big(A(x,y) + B(x,y) \Big). \tag{5.3.5}$$

Using (5.3.5) in $u(x,y) \le z(x,y)$, we can get (5.3.2). Note that the proof can also be carried out by differentiation of $z(x,y)$ with respect to y. □

It is well-known that the integral inequalities which furnish explicit bounds on unknown functions has become a rich source of inspiration in the development of the theory of differential and integral equations. A detailed account related to such inequalities can be found in [42, 364, 495, 498, 501, 504] and the references given therein. However, in certain situations the bounds provided by such inequalities available in the literature are inadequate and we need bounds on some new integral inequalities in order to achieve a diversity of desired goals. In this section, we present some basic integral inequalities in two independent variables which can be used more conveniently in specific applications.

Theorem 5.3.2 (Pachpatte [507]) *Let $u, a, b_i \in C(\triangle, \mathbb{R}_+)$ and $\alpha_i \in C^1(J_1, J_1)$, $\beta_i \in C^1(J_2, J_2)$ be non-decreasing with $\alpha_i(x) \le x$ on J_1, $\beta_i(y) \le y$ on J_2 for $i = 1, \ldots, n$ and $k \ge 0$ be a constant.*

(A1) If for all $x \in J_1$, $y \in J_2$,

$$u(x,y) \le k + \int_{x_0}^{x} a(s,y)u(s,y)ds + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)u(s,t)dtds, \tag{5.3.6}$$

then for all $x \in J_1$, $y \in J_2$,

$$u(x,y) \le kq(x,y) \exp\Big(\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)u(s,t)dtds \Big), \tag{5.3.7}$$

where for all $x \in J_1$, $y \in J_2$,

$$q(x,y) = \exp\Big(\int_{x_0}^{x} a(\xi,y)d\xi \Big). \tag{5.3.8}$$

(A2) *If for all* $x \in J_1$, $y \in J_2$,

$$u(x,y) \le k + \int_{y_0}^{y} a(x,t)u(x,t)dt + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)u(s,t)dtds, \quad (5.3.9)$$

then for all $x \in J_1, y \in J_2$,

$$u(x,y) \le k\gamma(x,y) \exp\Big(\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)\gamma(s,t)dtds\Big), \quad (5.3.10)$$

where for all $x \in J_1$, $y \in J_2$,

$$\gamma(x,y) = \exp\Big(\int_{y_0}^{y} a(x,\eta)d\eta\Big). \quad (5.3.11)$$

Proof We only give the details of the proof of (A1). The proof of (A2) can be done similarly.

(A1) Define a function $z(x,y)$ by

$$z(x,y) = k + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)u(s,t)dtds. \quad (5.3.12)$$

Then (5.3.6) can be rewritten as

$$u(x,y) \le z(x,y) + \int_{x_0}^{x} a(s,y)u(s,y)ds. \quad (5.3.13)$$

It is easy to check that $z(x,y)$ is a non-negative, continuous and non-decreasing function for all $x \in J_1, y \in J_2$. Fixing $y \in J_2$ in (5.3.13) and using Theorem 1.1.4 to (5.3.13), we get, for all $x \in J_1, y \in J_2$,

$$u(x,y) \le q(x,y)z(x,y), \quad (5.3.14)$$

where $q(x,y)$ is defined by (5.3.8). From (5.3.12) and (5.3.14) it follows

$$z(x,y) \le k + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)q(s,t)z(s,t)dtds. \quad (5.3.15)$$

Let $k > 0$ and define a function $v(x,y)$ by the right-hand side of (5.3.15). Then it is easy to check that

$$v(x,y) > 0,\ v(x_0,y) = v(x,y_0) = k,\ z(x,y) \le v(x,y)$$

and

$$
\begin{aligned}
D_1 v(x,y) &= \sum_{i=1}^{n} \Big(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\alpha_i(x),t) q(\alpha_i(x),t) z(\alpha_i(x),t) dt \Big) \alpha_i'(x) \\
&\le \sum_{i=1}^{n} \Big(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\alpha_i(x),t) q(\alpha_i(x),t) v(\alpha_i(x),t) dt \Big) \alpha_i'(x) \\
&\le v(x,y) \sum_{i=1}^{n} \Big(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\alpha_i(x),t) q(\alpha_i(x),t) \Big) \alpha_i'(x)
\end{aligned}
$$

i.e.,

$$
\frac{D_1 v(x,y)}{v(x,y)} \le \sum_{i=1}^{n} \Big(\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(\alpha_i(x),t) q(\alpha_i(x),t) dt \Big) \alpha_i'(x). \tag{5.3.16}
$$

Keeping y fixed in (5.3.16) , setting $x = \sigma$ and integrating it with respect to σ from x_0 to $x, x \in J_1$, and making the change of variables, we conclude for all $x \in J_1, y \in J_2$,

$$
v(x,y) \le k \exp \Big(\sum_{i=1}^{n} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t) q(s,t) dt ds \Big). \tag{5.3.17}
$$

Using (5.3.17) in $z(x,y) \le v(x,y)$, we get

$$
z(x,y) \le k \exp \Big(\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t) q(s,t) dt ds \Big). \tag{5.3.18}
$$

Using (5.3.17) in (5.3.14), we can get the required inequality in (5.3.7). If $k \ge 0$, we carry out the above procedure with $k + \epsilon$ instead of k, where $\epsilon > 0$ is an arbitrary small constant, and subsequently pass the limit $\epsilon \to 0$ to obtain (5.3.7). □

The inequalities in the following theorems can be used in the qualitative analysis of certain partial integro-differential equations involving several retarded arguments.

Theorem 5.3.3 (Pachpatte [507]) *Let $u, a, b_i, \alpha_i, \beta_i, k$ be as in Theorem 5.3.2. Let $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ be non-decreasing and sub-multiplicative function with $g(u) > 0$ for all $u > 0$.*

(C1) If $c \in C(\Delta, \mathbb{R}_+)$ and for all $x \in J_1, y \in J_2$,

$$
\begin{aligned}
u(x,y) \le k &+ \int_{x_0}^{x} a(s,y) \Big(u(s,y) + \int_{x_0}^{s} c(\sigma,y) u(\sigma,y) d\sigma \Big) ds \\
&+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t) u(s,t) dt ds,
\end{aligned} \tag{5.3.19}
$$

then for all $x \in J_1, y \in J_2$,

$$u(x,y) \le kp(x,y)\exp\Big(\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)g(r(s,t))dtds\Big), \tag{5.3.20}$$

where for all $x \in J_1, y \in J_2$,

$$p(x,y) = 1 + \int_{x_0}^{x} a(\xi,y)\exp\Big(\int_{x_0}^{\xi}[a(\sigma,y)+c(\sigma,y)]d\xi\Big). \tag{5.3.21}$$

(C2) *If* $c \in C(\Delta,\mathbb{R}_+)$ *and for all* $x \in J_1, y \in J_2$,

$$u(x,y) \le k + \int_{y_0}^{y} a(x,t)\Big(u(x,t) + \int_{y_0}^{t} c(x,\tau)u(x,\tau)d\tau\Big)dt \tag{5.3.22}$$

$$+\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)u(s,t)dtds,$$

then for all $x \in J_1, y \in J_2$,

$$u(x,y) \le kw(x,y)\exp\Big(\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)w(s,t)dtds\Big) \tag{5.3.23}$$

where for all $x \in J_1, y \in J_2$,

$$w(x,y) = 1 + \int_{y_0}^{y} a(x,\eta)\exp\Big(\int_{y_0}^{\eta}[a(x,\tau)+c(x,\tau)]d\tau\Big)d\eta. \tag{5.3.24}$$

Proof We only give the details of the proof of (C1).

(C1) Define a function $z(x,y)$ by (5.3.12) . Then (5.3.19) can be restated as

$$u(x,y) \le z(x,y) + \int_{x_0}^{x} a(x,y)\Big(u(x,y) + \int_{x_0}^{s} c(\sigma,y)u(\sigma,y)d\sigma\Big)ds. \tag{5.3.25}$$

Clearly, $z(x,y)$ is non-negative, continuous and non-decreasing function for all $x \in J_1, y \in J_2$. Fixing $y \in J_2$ in (5.3.25) and applying Theorem 1.7.4 given in [495] to (5.3.25) yields

$$u(x,y) \le p(x,y)z(x,y),$$

where $p(x,y)$ and $z(x,y)$ are defined by (5.3.21) and (5.3.12), respectively. Now following the proof of (A1) with suitable changes, we get the desired inequality in (5.3.20). □

Theorem 5.3.4 (Pachpatte [507]) *Let $u, a, b_i, \alpha_i, \beta_i, k, g(u)$ be as in Theorem 5.3.2 and g be as in Theorem 5.3.3.*

If $c \in C(\Delta, \mathbb{R}_+)$ and for all $x \in J_1, y \in J_2$,

$$u(x,y) \le k + \int_{x_0}^{x} a(s,y)\Big(u(s,y) + \int_{x_0}^{s} c(\sigma,y)u(\sigma,y)d\sigma\Big)ds + \sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)g(r(s,t))dtds, \tag{5.3.26}$$

then for $x_0 \le x \le x_3, y_0 \le y \le y_3$; $x, x_3 \in J_1$, $y, y_3 \in J_2$,

$$u(x,y) \le p(x,y)G^{-1}\Big(G(k) + \sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)g(p(s,t))dtds\Big) \tag{5.3.27}$$

where $p(x,y)$ is given by (5.3.21), G, G^{-1} are as in part (B1) in Theorem 5.3.3 and $x_3 \in J_1$, $y_3 \in J_2$ are chosen so that

$$G(k) + \sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} b_i(s,t)g(p(s,t))dtds \in Dom\ (G^{-1})$$

for all $x \in [x_0, x_3]$ and $y \in [y_0, y_3]$.

Theorem 5.3.5 (Pachpatte [504]) *Let $u(x,y), a(x,y) \in C(\Delta, \mathbb{R}_+), b(x,y,s,t) \in C(\Delta^2, \mathbb{R}_+)$, for $x_0 \le s \le a(x) \le X$, $y_0 \le t \le y \le Y$, $\alpha(x) \in C^1(J_1, J_1)$, $\beta(y) \in C^1(J_2, J_2)$ be non-decreasing with $\alpha(x) \le x$ on J_1, $\beta(y) \le y$ on J_2 and $k \ge 0$ be a constant. If for all $(x,y) \in \Delta$,*

$$u(x,y) \le k + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}\Big[a(s,t)u(s,t) + \int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} b(s,t,\sigma,\eta)u(\sigma,\eta)d\eta d\sigma\Big]dtds, \tag{5.3.28}$$

then for all $(x,y) \in \Delta$,

$$u(x,y) \le k\exp(A(x,y)), \tag{5.3.29}$$

where for all $(x,y) \in \Delta$,

$$A(x,y) = \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}\Big[a(s,t) + \int_{\alpha(x_0)}^{s}\int_{\beta(y_0)}^{t} b(s,t,\sigma,\eta)d\eta d\sigma\Big]dtds. \tag{5.3.30}$$

Proof Let $k > 0$ and define a function $z(x, y)$ by the right-hand side of (5.3.28). Then $z(x, y) > 0, z(x_0, y) = z(x, y_0) = k, u(x, y) \leq z(x, y)$ and

$$\begin{aligned} D_1 z(x, y) &= \Big[\int_{\beta(y_0)}^{\beta(y)} \Big[a(\alpha(x), t) u(\alpha(x), t) \\ &\quad + \int_{\alpha(y_0)}^{\alpha(y)} \int_{\beta(y_0)}^{t} b(\alpha(x), t, \sigma, \eta) u(\sigma, \eta) d\eta d\sigma \Big] dt \Big] \alpha'(x) \\ &\leq \Big[\int_{\beta(y_0)}^{\beta(y)} \Big[a(\alpha(x), t) z(\alpha(x), t) \\ &\quad + \int_{\alpha(y_0)}^{\alpha(y)} \int_{\beta(y_0)}^{t} b(\alpha(x), t, \sigma, \eta) z(\sigma, \eta) d\eta d\sigma \Big] dt \Big] \alpha'(x). \end{aligned} \tag{5.3.31}$$

From (5.3.31) it follows that

$$\frac{D_1 z(x, y)}{z(x, y)} \leq \int_{\beta(y_0)}^{\beta(y)} \left[a(\alpha(x), t) + \int_{\alpha(y_0)}^{\alpha(y)} \int_{\beta(y_0)}^{t} b(\alpha(x), t, \sigma, \eta) d\eta d\sigma \right] dt\, \alpha'(x) dx. \tag{5.3.32}$$

Keeping y fixed in (5.3.32), setting $x = \xi$ and integrating it with respect to ξ from x_0 to x and making the change of variables, we get

$$z(x, y) \leq k \exp(A(x, y)). \tag{5.3.33}$$

Using (5.3.33) in $u(x, y) \leq z(x, y)$, we get the required inequality in (5.3.29). The case $k \geq 0$ follows as mentioned in the proof of (A_1) of Theorem 5.3.2. □

In the next result, due to Pachatte [506], an explicit bound on a new retarded integral inequality in two independent variables is established.

A detailed account on such inequalities and some of their applications can be found in [42, 364, 495, 501, 503]. In [506], Pachpatte has established the following useful integral inequality.

Lemma 5.3.1 (Pachpatte [506]) *Let $u(t) \in C(I, \mathbb{R}_+), a(t, s), b(t, s) \in C(D, \mathbb{R}_+)$ and $a(t, s), b(t, s)$ are non-decreasing in t for each $s \in I$, where $I = [\alpha, \beta]$, $D = \{(t, s) \in I^2 : \alpha \leq s \leq t \leq \beta\}$ and suppose that for all $t \in I$,*

$$u(t) \leq k + \int_{\alpha}^{t} a(t, s) u(s) ds + \int_{\alpha}^{\beta} b(t, s) u(s) ds, \tag{5.3.34}$$

where $k \geq 0$ *is a constant. If* $p(t) = \int_\alpha^\beta b(t,s)\exp\left(\int_\alpha^s a(s,\sigma)d_\sigma\right)ds < 1$ *for all* $t \in I$, *then for all* $t \in I$,

$$u(t) \leq \frac{k}{1-p(t)}\exp\left(\int_\alpha^t a(t,s)ds\right). \tag{5.3.35}$$

Note that, a version of the above inequality when $a(t,s) = a(s), b(t,s) = b(s)$ was first given in [42]. In [505], a useful general retarded version of the above inequality was given.

Next result is to establish a general two independent variable retarded version of the above inequality which can be used as a tool to study the behavior of solutions of a general retarded Volterra-Fredholm integral equation in two independent variables.

Let $E = \{(x,y,s,t) \in \Delta^2 : x_0 \leq s \leq x \leq X, y_0 \leq t \leq y \leq Y\}$.

Theorem 5.3.6 (Pachpatte [506]) *Let* $u(x,y) \in C(\Delta, \mathbb{R}_+), a(x,y,s,t), b(x,y,s,t) \in C(E, \mathbb{R}_+)$ *and* $a(x,y,s,t)$, $b(x,y,s,t)$ *be non-decreasing in x and y for each* $s \in J_1, t \in J_2, \alpha \in C^1(J_1,J_1), \beta \in C^1(J_2,J_2)$ *be non-decreasing with* $\alpha(x) \leq x$ *on* J_1, $\beta(y) \leq y$ *on* J_2 *and suppose that for all* $x \in J_1, y \in J_2$,

$$u(x,y) \leq c + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)u(s,t)dtds + \int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)} b(x,y,s,t)u(s,t)dtds \tag{5.3.36}$$

where $c \geq 0$ *is a constant. If for all* $x \in J_1, y \in J_2$,

$$p(x,y) = \int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)} b(x,y,s,t)\exp\left(\int_{\alpha(x_0)}^{\alpha(s)}\int_{\beta(y_0)}^{\beta(t)} a(s,t,\sigma,\tau)d\tau d\sigma\right)dtds < 1, \tag{5.3.37}$$

then for all $x \in J_1, y \in J_2$,

$$u(x,y) \leq \frac{c}{1-p(x,y)}\exp\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)dtds\right). \tag{5.3.38}$$

Proof Fix any arbitrary $(x,y) \in \Delta$. Then for all $x_0 \leq x \leq M, y_0 \leq y \leq N$, we have

$$u(x,y) \leq c + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(X,Y,s,t)u(s,t)dtds + \int_{\alpha(x_0)}^{\alpha(X)}\int_{\beta(y_0)}^{\beta(Y)} b(M,N,s,t)u(s,t)dtds. \tag{5.3.39}$$

Let

$$k = c + \int_{\alpha(x_0)}^{\alpha(X)} \int_{\beta(y_0)}^{\beta(Y)} b(X, Y, s, t)u(s, t)dtds, \tag{5.3.40}$$

then (5.3.36) can be restated as

$$u(x, y) \le k + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(M, N, s, t)u(s, t)dtds, \tag{5.3.41}$$

for all $x_0 \le x \le M, y_0 \le y \le N$. Now a suitable application of the inequality (5.3.36) to (5.3.41) yields

$$u(x, y) \le k \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(M, N, s, t)dtds\right) \tag{5.3.42}$$

for all $x_0 \le x \le M, y_0 \le y \le N$. Since $(M, N) \in \Delta$ is arbitrary, from (5.3.42) and (5.3.40) with M and N replaced by x and y, we have

$$u(x, y) \le k \exp\Big(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x, y, s, t)dtds\Big), \tag{5.3.43}$$

where

$$k = c + \int_{\alpha(x_0)}^{\alpha(X)} \int_{\beta(y_0)}^{\beta(Y)} b(x, y, s, t)u(s, t)dtds \tag{5.3.44}$$

for all $x \in J_1, y \in J_2$.

Using (5.3.43) on the right-hand side of (5.3.44) and in view of (5.3.37), we have

$$k \le \frac{c}{1 - p(x, y)}. \tag{5.3.45}$$

Using (5.3.45) in (5.3.43), we get the desired inequality in (5.3.38). The proof is thus complete. □

Taking $b(x, y, s, t) = 0$ in Theorem 5.3.6, we get the following useful inequality.

Corollary 5.3.1 (Pachpatte [506]) *Let $u(x, y), a(x, y, s, t), \alpha(x), \beta(y)$ and c be as in Theorem 5.3.6. If for all $x \in J_1, y \in J_2$,*

$$u(x, y) \le c + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x, y, s, t)u(s, t)dtds, \tag{5.3.46}$$

then for all $x \in J_1, y \in J_2$,

$$u(x,y) \le c \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)dtds\right). \tag{5.3.47}$$

The following corollaries of Theorem 5.3.6 and Corollary 5.3.1 are also obtained readily.

Corollary 5.3.2 (Pachpatte [506]) *Let* $u(x,y), a(x,y,s,t), b(x,y,s,t)$ *and* c *be as in Theorem 5.3.6 and suppose that for all* $x \in J_1,\ y \in J_2$,

$$\begin{aligned} u(x,y) \le c &+ \int_{x_0}^{x} \int_{y_0}^{y} a(x,y,s,t)u(s,t)dtds \\ &+ \int_{x_0}^{M} \int_{y_0}^{N} b(x,y,s,t)u(s,t)dtds. \end{aligned} \tag{5.3.48}$$

If for all $x \in J_1,\ y \in J_2$,

$$q(x,y) = \int_{x_0}^{M} \int_{y_0}^{N} b(x,y,s,t) \exp\left(\int_{x_0}^{s} \int_{y_0}^{t} a(s,t,\sigma,\tau)d\tau d\sigma\right) dtds < 1, \tag{5.3.49}$$

then for all $x \in J_1,\ y \in J_2$,

$$u(x,y) \le \frac{c}{1 - q(x,y)} \exp\left(\int_{x_0}^{x} \int_{y_0}^{y} a(x,y,s,t)dtds\right). \tag{5.3.50}$$

Corollary 5.3.3 (Pachpatte [506]) *Let* $u(x,y), a(x,y,s,t)$ *and* c *be as in Corollary 5.3.1. If for all* $x \in J_1$, $y \in J_2$,

$$u(x,y) \le c + \int_{x_0}^{x} \int_{y_0}^{y} a(x,y,s,t)u(s,t)dtds, \tag{5.3.51}$$

then for all $x \in J_1, y \in J_2$,

$$u(x,y) \le c \exp\left(\int_{x_0}^{x} \int_{y_0}^{y} a(x,y,s,t)dtds\right). \tag{5.3.52}$$

The proofs of Corollaries 5.3.2 and 5.3.3 follow by taking $\alpha(x) = x, \beta(y) = y$ in Theorem 5.3.6 and Corollary 5.3.1.

5.4 Linear Multi-Dimensional Continuous Integral Inequalities

5.4.1 Linear Multi-Dimensional Continuous Integral Inequalities and Their Generalizations

In this section, we shall introduce some multi-dimensional linear continuous integral inequalities.

First, we give some notations. If $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in \mathbb{R}^n$, we write $x \leq y$ ($x < y$) if and only if $x_i \leq y_i$ ($x_i < y_i$), $i = 1, \cdots, n$. If $x < y$, then $[x,\ y]$ denotes the $n-$dimensional interval $\{z \in \mathbb{R}^n : x \leq z \leq y\}$. We also adopt the notation

$$\int_x^y f(s)ds = \int_{x_1}^{y_1} \cdots \int_{x_n}^{y_n} f(s_1, \cdots, s_n)ds_1 \cdots ds_n$$
$$= \int_{x_1}^{y_1} \left(\int_{x_2}^{y_2} \cdots \left(\int_{x_n}^{y_n} f(s_1, \cdots, s_n)ds_n \right) \cdots ds_2 \right) ds_1,$$

$x = (x_1, x^1)$, $x^1 = (x_2, \cdots, x_n)$, $dx^1 = dx_2 \cdots dx_n$. If $D \subset \mathbb{R}^n$ and $f : D \to \mathbb{R}$, we say that $f(x)$ is a non-decreasing function in D if $x, y \in D$ and $x \leq y$ imply $f(x) \leq f(y)$.

Theorem 5.4.1 (Zahariev-Bainov [683]) *Let $\alpha, \beta \in \mathbb{R}^n, \alpha < \beta$. Let $u(x), b(x)$ be non-negative continuous functions for all $x \in [\alpha,\ \beta]$ satisfying the inequality for all $x \in [\alpha,\ \beta]$,*

$$u(x) \leq a + \int_\alpha^x b(s)u(s)ds, \tag{5.4.1}$$

where $a \geq 0$ is a constant. Then for all $x \in [\alpha,\ \beta]$,

$$u(x) \leq a \exp\left(\int_\alpha^x b(s)ds \right). \tag{5.4.2}$$

Proof Obviously, (5.4.1) implies

$$u(x) \leq a + \int_{\alpha_1}^{x_1} \left(\int_{\alpha^1}^{x^1} b(s_1, s^1)u(s_1, s^1) \right) ds_1 \equiv v(x_1, x^1). \tag{5.4.3}$$

For any fixed $x^1 \in [\alpha^1,\ \beta^1]$, the function $w(x_1) = v(x_1, x^1)$ satisfies the relations

$$\begin{cases} w(\alpha_1) = a, & (5.4.4) \\ w'(x_1) = \displaystyle\int_{\alpha^1}^{x^1} b(x_1, s^1)u(x_1, s^1)ds^1 \leq \int_{\alpha^1}^{x^1} b(x_1, s^1)ds^1 w(x_1) & (5.4.5) \end{cases}$$

since $v(x_1, x^1)$ is non-decreasing in $[\alpha,\ \beta]$ and $u(x_1, s^1) \leq v(x_1, s^1) \leq v(x, x^1) = w(x_1)$. Lemma 1.1.1 and (5.4.5) imply

$$w(x_1) \leq a \exp\left(\int_{\alpha_1}^{x_1}\left(\int_{\alpha^1}^{x^1} b(s_1, s^1)ds^1\right)ds_1\right)$$

which, together with (5.4.3), implies (5.4.1). □

Corollary 5.4.1 (Zahariev-Bainov [683]) *If $a(x)$ is a non-decreasing function in $[\alpha,\ \beta] \subset \mathbb{R}^n$ and satisfies for all $x \in [\alpha,\ \beta]$,*

$$u(x) \leq a(x) + \int_{\alpha}^{x} b(s)u(s)ds, \tag{5.4.6}$$

then for all $x \in [\alpha,\ \beta]$,

$$u(x) \leq a(x) \exp\left(\int_{\alpha}^{x} b(s)ds\right). \tag{5.4.7}$$

Theorem 5.4.2 (The Gronwall-Bellman Inequality [459]) *If u, η, ϕ, ψ are non-negative continuous functions $(\mathbb{R}^n \to \mathbb{R}_+)$ and if*

$$u(x) \leq \eta(x) + \phi(x)\int_{0}^{x} \psi(s)u(s)ds, \tag{5.4.8}$$

then

$$u(x) \leq \eta(x) + \phi(x)\int_{0}^{x} \psi(s)\eta(s)\exp\left(\int_{s}^{x} \psi(t)\phi(t)dt\right)ds. \tag{5.4.9}$$

Proof Define a function $z(x)$ by

$$z(x) = \int_{0}^{x} \psi(s)u(s)ds,$$

then $z(0) = 0, u(x) \leq \eta(x) + \phi(x)z(x)$ and

$$z'(x) = \psi(x)u(x) \leq \psi(x)\eta(x) + \psi(x)\phi(x)z(x).$$

Multiplying above inequality by the integrating factor $\exp\left(-\int_0^x \psi(s)\phi(s)ds\right)$, we have

$$\frac{d}{dt}\left[z(x)\exp\left(-\int_{0}^{x} \psi(s)\phi(s)ds\right)\right] \leq \psi(x)\eta(x)\exp\left(-\int_{0}^{x} \psi(s)\phi(s)ds\right).$$

Setting $x = s$ in this above formula and integrating it with respect to s from 0 to x, we get

$$z(x)\exp\left(-\int_0^x \psi(s)\phi(s)ds\right) \le \int_0^x \psi(s)\eta(s)\exp\left(-\int_0^x \psi(s)\phi(s)ds\right)ds.$$

Using the bound on $z(x)$ from the above formula in $u(x) \le \eta(x) + \phi(x)z(x)$. We get the required inequality in (5.4.9). ☐

In the same manner, we can readily prove the following three theorems.

Theorem 5.4.3 (Zahariev-Bainov [683]) *Let $\alpha, \beta \in \mathbb{R}^n, \alpha < \beta$. If $u(x), b(x)$ are non-negative continuous functions for all $x \in [\alpha,\ \beta]$ satisfying the inequality for all $x \in [\alpha,\ \beta]$,*

$$u(x) \le a + \int_x^\beta b(s)u(s)ds, \tag{5.4.10}$$

then for all $x \in [\alpha,\ \beta]$,

$$u(x) \le a\exp\left(\int_x^\beta b(s)ds\right). \tag{5.4.11}$$

Theorem 5.4.4 (Zahariev-Bainov [683]) *Let $\alpha, \beta \in \mathbb{R}^n, \alpha < \beta$. If $u(x), b(x)$ are non-negative continuous functions for all $x \in [\alpha,\ \beta]$ satisfying the inequality for all $\alpha \le x \le \tau \le \beta$,*

$$u(x) \le u(\tau) + \int_x^\tau b(s)u(s)ds. \tag{5.4.12}$$

Then for all $x \in [\alpha,\ \beta]$,

$$u(x) \ge u(\alpha)\exp\left(-\int_\alpha^x b(s)ds\right). \tag{5.4.13}$$

Theorem 5.4.5 (Hristova-Bainov [291]) *Let $\alpha, \beta \in \mathbb{R}^n, \alpha < \beta$. If $u(x), b(x), k(s,t)$ are non-negative continuous functions for $\alpha \le \tau \le s \le \beta$ satisfying the inequality for all $x \in [\alpha,\ \beta]$,*

$$u(x) \le a + \int_\alpha^x \left[b(s)u(s) + \int_\alpha^s k(s,t)u(\tau)d\tau\right]ds, \tag{5.4.14}$$

where $a \ge 0$ is a constant. Then for all $x \in [\alpha,\ \beta]$,

$$u(x) \le a\exp\left(\int_\alpha^x \left[b(s)ds + \int_\alpha^s k(s,\tau)d\tau\right]ds\right). \tag{5.4.15}$$

Theorem 5.4.6 (Hristova-Bainov [291]) *Let $\alpha, \beta \in \mathbb{R}^n, \alpha < \beta$. If $u(x)$, $a(x)$, $b(x)$, $f(x)$, $g(x)$ are non-negative continuous functions for all $x \in [\alpha, \beta]$ with $a(x)$ non-decreasing in $[\alpha, \beta]$. If the inequality holds for all $x \in [\alpha, \beta]$,*

$$u(x) \leq a(x) + \int_{\alpha}^{x} f(s) \left[u(s) + \int_{\alpha}^{s} g(\tau)u(\tau)d\tau \right] ds, \tag{5.4.16}$$

then for all $x \in [\alpha, \beta]$, we have

$$u(x) \leq a(x) + \int_{\alpha}^{x} a(s)f(s) \exp\left(\int_{\alpha}^{s} (f(\tau) + g(\tau))d\tau \right) ds. \tag{5.4.17}$$

Proof Set $r(s) \equiv u(s) + \int_{\alpha}^{s} g(\tau)u(\tau)d\tau$. Then (5.4.16) takes the form

$$u(x) \leq a(x) + \int_{\alpha}^{x} f(s)r(s)ds. \tag{5.4.18}$$

Noting that $u(s) \leq r(s)$, we obtain

$$\begin{aligned} r(x) &= u(x) + \int_{\alpha}^{x} g(s)u(s)ds \\ &\leq a(x) + \int_{\alpha}^{x} f(s)r(s)ds + \int_{\alpha}^{x} g(s)r(s)ds. \end{aligned} \tag{5.4.19}$$

By Corollary 5.4.1, we conclude for all $x \in [\alpha, \beta]$,

$$r(x) \leq a(x) \exp\left(\int_{\alpha}^{x} \big(f(\tau) + g(\tau)\big)d\tau \right),$$

which, along with (5.4.18), implies (5.4.16). □

Corollary 5.4.2 (Hristova-Bainov [291]) *If $a(x)$ is non-decreasing in $[\alpha, \beta]$, then (5.4.17) implies for all $x \in [\alpha, \beta]$,*

$$u(x) \leq a(x) \left[1 + \int_{\alpha}^{x} f(s) \exp\left(\int_{\alpha}^{s} [f(\tau) + g(\tau)]d\tau \right) ds \right]. \tag{5.4.20}$$

As Bainov and Simeonov [42] pointed out, the proofs of Theorems 5.4.5–5.4.6 are based on the method of "preliminary single differentiation". The following question arises: Is it possible to find better estimates than those given in these theorems, if the right-hand sides of the inequalities are first differentiated with respect to all n variables. This method was used by many authors (see, e.g., Corduneanu [154], Pachpatte [483], Yeh [667], Shih [587],

Yang [659], Young [680]). In these works, the crucial inequalities are as follows

$$\begin{cases} u(x) \le a + \int_{\alpha}^{x} b(s)u(s)ds \equiv v(x), \quad x \in [\alpha,\ \beta], & (5.4.21) \\ u(x) \le a + \int_{x}^{\beta} b(s)u(s)ds \equiv v(x), \quad x \in [\alpha,\ \beta], & (5.4.22) \\ u(x) \le a + \int_{\alpha}^{x} b(s)g(u(s))ds \equiv v(x), \quad x \in [\alpha,\ \beta], & (5.4.23) \\ u(x) \le a + \int_{x}^{\beta} b(s)g(u(s))ds \equiv v(x), \quad x \in [\alpha,\ \beta]. & (5.4.24) \end{cases}$$

After differentiation of the right-hand sides of $v(x)$ with respect to all variables $x_1, \cdots, x_n$, these inequalities reduce to the differential inequalities

$$\begin{cases} D_1 \cdots D_n v(x) \le\ b(x)v(x), & (5.4.25) \\ (-1)^n D_1 \cdots D_n v(x) \le\ b(x)v(x), & (5.4.26) \\ D_1 \cdots D_n v(x) \le\ b(x)g(v(x)), & (5.4.27) \\ (-1)^n D_1 \cdots D_n v(x) \le\ b(x)g(v(x)), & (5.4.28) \end{cases}$$

respectively, where $x \in [\alpha,\ \beta]$ and $D_i = \partial/\partial x_i, \quad i = 1, 2, \cdots, n$. Now we begin with estimate of function $v(x)$ defined by, respectively, (5.4.21)–(5.4.24) using the inequalities (5.4.25)–(5.4.28). To demonstrate this estimate procedure, following Young [680] we consider the differential inequality (5.4.27) for the function $v(x)$ defined by (5.4.23). Thus it follows from (5.4.27) that

$$\frac{D_1 \cdots D_n v(x)}{v(x)} \le b(x).$$

Since

$$D_n \left[\frac{D_1 \cdots D_{n-1} v(x)}{v(x)} \right] = \frac{D_1 \cdots D_n v(x)}{v(x)} - \frac{D_n v(x) D_1 \cdots D_{n-1} v(x)}{v^2(x)}$$

and $D_n v(x) = v'(x) D_n v(x) \ge 0, D_1 \cdots D_{n-1} v(x) \ge 0$, the above inequality implies

$$D_n \left[\frac{D_1 \cdots D_{n-1} v(x)}{v(x)} \right] \le b(x).$$

Consequently, integrating with respect to x_n from α_n to x_n, noting that fact that $D_1 \cdots D_{n-1} v(x) = 0$ for $x_n = \alpha_n$, we get

$$\frac{D_1 \cdots D_{n-1} v(x)}{g(v(x))} \le \int_{\alpha_n}^{x_n} b(x_1, \cdots, x_{n-1}, s_n) ds_n.$$

Repeating this process, we find after $n-1$ steps,

$$\frac{D_1 v(x)}{g(v(x))} \leq \int_{\alpha_2}^{x_2} \cdots \left(\int_{\alpha_n}^{x_n} b(x_1, s_2, \cdots, s_n) ds_n \right) \cdots ds_2.$$

For $G(u) = \int_{u_0}^{u} dz/g(z), u \geq u_0 > 0$, we find $D_1 G(v(x)) = D_1 v(x)/g(v(x))$, so that

$$D_1 G(v(x)) \leq \int_{\alpha^1}^{x^1} b(x_1, s^1) ds^1.$$

Therefore, integrating with respect to x_1 from α_1 to x_1 yields

$$G(v(x_1, \cdots, x_n)) - G(v(\alpha_1, x_2, \cdots, x_n)) \leq \int_{\alpha}^{x} b(s) ds$$

and since $v(\alpha_1, x_2, \cdots, x_n) = a$,

$$u(x) \leq v(x) \leq G^{-1} \left[G(a) + \int_{\alpha}^{x} b(s) ds \right]. \tag{5.4.29}$$

Comparing this estimate with the proof of Theorem 12.10 in Bainov and Simeonov [42] readily concluded that

(1) both ways of estimate lead to the same results- estimate (12.22) in Bainov and Simeonov [42] and (5.4.29);
(2) the above estimate procedure is more involved, and can only be applied under additional restrictions, e.g., the existence of a non-negative derivative $g'(u)$ is required.

Consequently, the method of single differentiation is to be preferred to the estimate procedure above. However, a sharp estimate for the function $u(x)$ using the inequalities (5.4.21)–(5.4.24) cannot be obtained by either method. Finding such an estimate is related to the investigation of a corresponding comparison integral equation. For example, for $n = 2$, the comparison integral equation for (5.4.24) is

$$u(x, y) = a + \int_{x_0}^{x} \int_{y_0}^{y} b(s, t) u(s, t) ds dt.$$

This equation is closely related to the partial differential equation

$$v_{xy}(x, y) = b(x, y) v(x, y)$$

and this relationship is reflected in the following theorem which is the scalar form of Theorem 5.1.11.

Theorem 5.4.7 (Snow [603]) *Let $u(x, y), a(x, y), b(x, y)$ be continuous functions in a domain D, with $b \geq 0$ in D. Let $P_0(x_0, y_0), P(x, y)$ be two points in D such that $(x-x_0)(y-y_0) > 0$, and let R be the rectangular region with P and P_0 as two of its opposite vertices. Let $v(s, y; x, y)$ be the solution of the characteristic initial value problem*

$$L[v] = v_{st} - b(s,t)v = 0, \quad v(x, t; x, y) = v(s, y; x, y) = 1, \tag{5.4.30}$$

and let D^+ be a connected sub-domain of D, containing P, on which $v > 0$. If $R \subset D^+$ and u satisfies

$$u(x, y) \leq a(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} b(s,t)u(s,t)dsdt, \tag{5.4.31}$$

then

$$u(x, y) \leq a(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} a(s,t)b(s,t)v(s, y; x, y)dsdt. \tag{5.4.32}$$

This theorem is a special case of the result of Young [677], which is based on the next two lemmas.

The first is a variant of the scalar form of Theorem 5.1.10.

Lemma 5.4.1 (Snow [603]) *Let $b(s)$ be a continuous function in $D \subset \mathbb{R}^n$. Then the characteristic initial value problem*

$$\begin{cases} (-1)^n v_s(s; x) - b(s)v(s; x) = 0, \quad in \ \ D, & (5.4.33) \\ v(s; x) = 1 \ \ on \ \ s_i = x_i, \quad i = 1, 2, \cdots, n, & (5.4.34) \end{cases}$$

has a unique solution $v(s; x)$ for s near x satisfying $\Pi_{i=1}^{n}(x_i - s_i) \geq 0$. This solution is continuous; if $b(s)$ is non-negative, so is $v(s; x)$.

Proof The function $v(s; x)$ is the Riemann function relative to the point x. Problem (5.4.33)–(5.4.34) is equivalent to the integral equation

$$v(s; x) = 1 + \int_{s}^{x} b(t)v(t; x)dt. \tag{5.4.35}$$

The existence, uniqueness, and possible non-negativity of $v(s; x)$ follows by successive approximation arguments, as given in the proof of Theorem 5.1.10. Since $v(s; x)$ is continuous, and $v = 1$ on $s_i = x_i, i = 1, 2, \cdots, n$, there is a domain D^+, containing x, on which $v \geq 0$ even if $b(s)$ is not non-negative. □

Lemma 5.4.2 (Snow [603]) *Let $b(x), f(x)$ be continuous functions in $D \subset \mathbb{R}^n$. Let $v(s; x)$ be the solution of problem (5.4.33)–(5.4.34), and let D^+ be a connected sub-domain of D, containing x, on which $v \geq 0$ for all $s \in D^+$. Let $[\alpha, x] \subset D, \alpha \leq x$,*

and let

$$L[w] = w_x(x) - b(x)w(x) \leq f(x), \tag{5.4.36}$$

where w vanishes together with all its mixed derivatives up to order $n-1$ on $x_i = \alpha_i,\ i = 1, 2, \cdots, n$. Then

$$w(x) \leq \int_{\alpha}^{x} f(t)v(t;x)dt. \tag{5.4.37}$$

Proof If ϕ is an n times continuously differentiable function in D, then

$$\phi Lw - wM\phi = \sum_{i=1}^{n}(-1)^{i-1}D_i[(D_0D_1\cdots D_{i-1}\phi)(D_{i+1}\cdots D_nD_{n+1})w], \tag{5.4.38}$$

where $M\phi = (-1)^n\phi_x(x) - b(x)\phi(x)$ with $D_0 \equiv D_{n+1} = I$, the identity operator. Integrating (5.4.38) over $[\alpha, x]$, with t as an integration variable, and noting that w vanishes together with all its mixed derivatives up to order $n-1$ on $t_i = \alpha_i,\ i = 1, \cdots, n$, this gives us

$$\int_{\alpha}^{x}(\phi Lw - wM\phi)dt = \sum_{i=1}^{n}(-1)^{i-1}\int_{t_i=x_i}(D_1\cdots D_{i-1})(D_{i+1}\cdots D_nw)dt^i \tag{5.4.39}$$

where $dt^i = dt_1\cdots dt_{i-1}dt_{i+1}\cdots dt_n$. Now we set ϕ equal to the function v satisfying problem (5.4.33)–(5.4.34). Since $v = 1$ on $t_i = x_i,\ i = 1, \cdots, n$, it follows that $D_1\cdots D_{i-1}v(t;x) = 0$ on $t_i = x_i$ for $i = 2, \cdots, n$. Thus (5.4.39) becomes

$$\begin{aligned}\int_{\alpha}^{x} v(t;x)L[w](t)dt &= \int_{t_1=x_1} v(t;x)D_2\cdots D_nw(t)dt^1 \\ &= \int_{\alpha_2}^{x_2}\cdots\int_{\alpha_n}^{x_n} D_2\cdots D_nw(x_1, t_2, \cdots, t_n)dt_2\cdots dt_n = w(x).\end{aligned} \tag{5.4.40}$$

By Lemma 5.4.1, there is a domain D^+, containing x, on which $v \geq 0$. Multiplying (5.4.36) throughout by v and using (5.4.40), we can obtain the required (5.4.37). □

Remark 5.4.1 Lemma 5.4.2 still holds if "$\leq$" is replaced by "$\geq$" in (5.4.36) and (5.4.37), or even if the inequalities in (5.4.36) and (5.4.37) are replaced by equalities. In this sense, (5.4.37) is the best estimate for a function $w(x)$ satisfying (5.4.36).

The next result is a generalization of Theorem 5.4.7, which is an analogous of those in Snow [603, 604].

Theorem 5.4.8 (Yeh [669]) *Suppose that $u(x)$, $D_1 \dots D_n u(x)$, $a(x)$ and $b(x)$ are real-valued non-negative continuous functions defined on Q. Let $v(s;x)$ be a solution of the characteristic initial value problems*

$$(-1)^n \frac{\partial^n v(s;x)}{\partial s_1 \dots \partial s_n} - [1 + b(s))]v(s;x) = 0 \qquad in \quad Q,$$
$$v(s;x) = 1 \qquad on\ s_i = x_i,\ i = 1, \dots, n,$$

and let D^+ be a connected sub-domain of Q which contains x such that $v \geq 0$ for all $s \in D^+$. If $D \subset D^+$ and

$$D_1 \dots D_n u(x) \leq a(x) + \int_{x^0}^{x} b(s) + D_1 \dots D_n u(s)]ds],$$

then

$$u(x) \leq h(x) + \int_{x^0}^{x} \{a(s) + \int_{x^0}^{s} b(t)[a(t) + h(t) + \int_{x^0}^{t} v(m;s)(b(m)(a(m) + h(m)) + a(m))dm]dt\}ds, \quad (5.4.41)$$

where

$$h(x) = \sum u(x_1^0, x_2, \dots, x_n) - \sum u(x_1^0, x_2^0, x_3, \dots, x_n) + \dots + (-1)^{i-1} \sum u(x_1^0, \dots, x_i^0, x_{i+1}, \dots, x_n) + \dots + (-1)^{n-1} u(x_1^0, \dots, x_n^0) \geq 0. \quad (5.4.42)$$

Here

$$\sum u(x_1^0, x_2, \dots, x_n) = u(x_1^0, x_2, \dots, x_n) + u(x_1, x_2^0, x_3, \dots, x_n) + \dots + u(x_1, \dots, x_{n-1}, x_n^0),$$
$$\sum u(x_1^0, x_2^0, \dots, x_n) = u(x_1^0, x_2^0, \dots, x_n) + u(x_1^0, x_2, x_3^0, x_4, \dots, x_n) + \dots + u(x_1, \dots, x_{n-2}, x_{n-1}^0, x_n^0),$$
$$\vdots$$
$$\sum u(x_1^0, \dots, x_{n-1}^0, x_n) = u(x_1^0, x_2^0, \dots, x_{n-2}^0, x_{n-1}, x_n^0) + \dots + u(x_1, x_2^0, \dots, x_n^0).$$

Proof Let

$$A(s) = \int_{x^0}^{x} b(s)[u(s) + D_1 \cdots D_n u(s)]ds.$$

Then

$$\begin{aligned} A(x) &= 0 \qquad on\ x_i = x_i^0,\ i = 1, \ldots, n, \\ D_1 \ldots D_n A(x) &= b(x)[u(x) + D_1 \ldots D_n u(x)] \end{aligned} \tag{5.4.43}$$

and

$$D_1 \ldots D_n u(x) \le a(x) + A(x). \tag{5.4.44}$$

Integrating both sides of (5.4.44) from x^0 to x, we obtain

$$u(x) \le h(x) + \int_{x^0}^{x} [a(s) + A(s)]ds, \tag{5.4.45}$$

where $h(s)$ is the function as defined in (5.4.42). It follows from (5.4.43)–(5.4.45) that

$$D_1 \ldots D_n A(x) \le b(x)[h(x) + a(x) + A(x) + \int_{x^0}^{x} (a(s) + A(s))ds]. \tag{5.4.46}$$

Let

$$B(x) = A(x) + \int_{x^0}^{x} (a(s) + A(s))ds.$$

Then

$$\begin{aligned} B(x) &= A(x) \qquad \text{on } \mathrm{x_i} = \mathrm{x_i^0},\ \mathrm{i} = 1, \ldots, \mathrm{n}, \\ A(x) &\le B(x), \\ D_1 \ldots D_n B(x) &= D_1 \ldots D_n A(x) + a(x) + A(x) \end{aligned}$$

and

$$D_1 \ldots D_n A(x) \le b(x)[h(x) + a(x) + B(x)].$$

Thus

$$D_1 \ldots D_n B(x) - [1 + b(x)]B(x) \le b(x)[a(x) + h(x)] + a(x).$$

As in the proof of Theorem 5.4.7, we obtain

$$B(x) \le \int_{x^0}^{x} v(s;x)[b(s)(a(s)+h(s))+a(s)],$$

which, along with (5.4.46), implies

$$D_1 \dots D_n A(x) \le b(x)[a(x)+h(x)+\int_{x^0}^{x} v(s;x)[b(s)(a(s)+h(s))+a(s)]ds]. \tag{5.4.47}$$

Since $A(x)=0$ on $x_i=x_i^0$ for $i=1,\dots,n$, it follows from (5.4.47) that

$$A(x) \le \int_{x^0}^{x} b(s)(a(s)+h(s)+\int_{x^0}^{s} v(t;s)[b(t)(a(t)+h(t))+a(t)]dt)ds.$$

Substituting this estimate for $A(x)$ in (5.4.44) and integrating both sides from x^0 to x, we obtain the desired bound in (5.4.41). □

Let $a,\ b \in \mathbb{R}^n,\ b>a$. We shall introduce the following notations:

$$\begin{cases} B(a,b) = & I_1(a,b)\times I_2(a,b)\times\cdots\times I_n(a,b), \quad B(0,b)=B^b, \\ B_k(a,b) = & I_1(a,b)\times\cdots\times I_{k-1}(a,b)\times I_{k+1}(a,b)\times\cdots\times I_n(a,b), \end{cases}$$

where for all $1\le k\le n$,

$$I_k(a,b)=[a_k,\ b_k].$$

The next result is a variant of Theorem 5.4.1 with the same proof.

Theorem 5.4.9 (Zahariev-Bainov [682]) *Let the functions $u(x),\ f(x):\mathbb{R}^n\to\mathbb{R}_+$ be non-negative and continuous for all $x\in B(a,b)$, and satisfy the inequality for all $x\in B(a,b)$,*

$$u(x)\le u_0+\int_{B(a,x)} f(s)u(s)ds,$$

where $u_0\ge 0$ is a constant. Then for all $x\in B(a,b)$, there holds that

$$u(x)\le u_0\exp\Big(\int_{B(a,x)} f(s)ds\Big). \tag{5.4.48}$$

Theorem 5.4.10 (Hristova-Bainov [293]) *Let the following conditions hold*

(1) The functions $u(x),\ f(x):\mathbb{R}^n\to\mathbb{R}_+$ are continuous and non-negative for all $x\in B(a,b)$ where $a,\ b,\ b>a$ are fixed points in $\mathbb{R}^n$.

(2) The function $g(x) : \mathbb{R}^n \to \mathbb{R}_0 \equiv (0, +\infty)$ is continuous, positive and non-decreasing for all $x \in B(a, b)$.
(3) The inequality satisfies that for all $x \in B(a, b)$,

$$u(x) \leq g(x) + \int_{B(a,x)} f(s)u(s)ds. \tag{5.4.49}$$

Then for all $x \in B(a, b)$, there holds that

$$u(x) \leq g(x) \exp\Big(\int_{B(a,x)} f(s)ds \Big). \tag{5.4.50}$$

Proof From the condition (2) of Theorem 5.4.10 and (5.4.42) it follows that

$$\begin{aligned} \frac{u(x)}{g(x)} &\leq 1 + \int_{B(a,x)} \frac{f(s)u(s)}{g(x)} ds \\ &\leq 1 + \int_{B(a,x)} f(s) \frac{u(s)}{g(s)} ds. \end{aligned}$$

Applying Theorem 5.4.9 with $u_0 \equiv 1$ to the function $\frac{u(x)}{g(x)} : \mathbb{R}^n \to \mathbb{R}_+$, we obtain

$$\frac{u(x)}{g(x)} \leq \exp\left(\int_{B(a,x)} f(s)ds \right)$$

which implies (5.4.50). □

Theorem 5.4.11 (The Gronwall-Bellman Inequality [145]) *If h, f, g, u are non-negative continuous functions ($\mathbb{R}^n \to \mathbb{R}_+$), and if the following inequality holds,*

$$u(x) \leq h(x) + \int_0^x f(s)u(s) + \int_0^x f(s)\{\int_0^s g(t)u(t)dt\}ds, \tag{5.4.51}$$

then there are continuous non-negative functions η, ϕ, ψ (not depending on u), such that

$$u(x) \leq \eta(x) + \phi(x) \int_0^x \psi(s)\eta(s) \exp\{\int_s^x \psi(t)\phi(t)dt\}ds. \tag{5.4.52}$$

Proof Let $z(x) = u(x) + \int_0^x g(s)u(s)ds$, then

$$\begin{aligned} u(x) \leq z(x) &\leq h(x) + \int_0^x f(s)\left\{u(s) + \int_0^s g(t)u(t)dt\right\} ds + \int_0^x g(s)u(s)ds \\ &\leq h(x) + \int_0^x [f(s) + g(s)]z(s)ds. \end{aligned}$$

Then the conclusion follows from Theorem 5.4.2 on putting $\nu = h, \phi = 1$, and $\psi = f + g$ in (5.4.52). □

Note that Theorem 5.4.11 was extended by Pachpatte [477] to the 2-dimensional case, again by using the Riemann functions, with

$$u'(x) = a + \int_0^x f(s)\{u(s) + u'(s)\}ds + \int_0^x f(s)\{\int_0^s h(t)u'(t)dt\}ds \tag{5.4.53}$$

replaced by

$$u_{xy}(x,y) \le a(x,y) + \int_0^x \int_0^y b(s,t)\Big(u(s,t) + u_{st}(s,t)\Big)dsdt + \int_0^x \int_0^y c(s,t)\left(\int_0^s \int_0^t p(\xi,\eta)[u(\xi,\eta) + u_{\xi\eta}(\xi,\eta)]d\xi d\eta\right)dsdt.$$

It was then shown that u satisfies a complicated inequality involving a Riemann function.

To minimize notational complexities, we introduce a generalized version of Pachpatte's theorem for two-dimensions, but it should be clear that the method of proof is valid for n-dimensions.

Let $D_1 = \frac{\partial}{\partial x}, \quad D_2 = \frac{\partial}{\partial y}$,

$$P(x,y) = 1 + x + y + xy, \quad P(D_1 + D_2) = 1 + D_1 + D_2 + D_1D_2 = P(D),$$

so that

$$P(D)u = P(D_1, D_2)u = 1 + u_1 + u_2 + u_{12},$$

where $u_i = D_i u$, etc.

Theorem 5.4.12 (Conlan-Wang [145]) *Let*

$$u_{xy} = u_{12} \le a + b\int_0^x \int_0^y fP(D)udsdt + k\int_0^x \int_0^y g\Big(\int_0^s \int_0^t hP(D)ud\xi d\eta\Big)dsdt \tag{5.4.54}$$

where all functions and their relevant derivatives are continuous, non-negative for all $x \ge 0, y \ge 0$. Then the conclusion of Theorem 5.4.11 holds for u.

Proof Let

$$R(x,y) = b\int_0^x \int_0^y fzdsdt + k\int_0^x \int_0^y g\Big\{\int_0^s \int_0^t hzd\xi d\eta\Big\}dsdt$$

where $z = P(D)u$. Then from (5.4.54) it follows

$$\begin{aligned} u_{12} &\le a + R, \\ u_1(x,y) - u_1(x,0) &= \int_0^y u_{12}(x,t)dt \le \int_0^y a(x,t)dt + \int_0^y R(x,t)dt, \\ u_2(x,y) - u_2(0,y) &\le \int_0^x a(s,y)ds + \int_0^x R(s,y)ds, \\ u(x,y) - u(0,y) - u(x,0) + u(0,0) &\le \int_0^x \int_0^y adsdt + \int_0^x \int_0^y R(s,t)dsdt. \end{aligned}$$

Now

$$\begin{aligned} \int_0^x Rds = & \int_0^x b(s,y)\Big(\int_0^s \int_0^y fzd\xi dt\Big)ds \\ & + \int_0^x k(s,y)\Big(\int_0^s \int_0^y g[\int_0^\xi \int_0^t hzd\lambda d\eta]d\xi dt\Big)ds \\ \le & (\int_0^x bds)\Big\{\int_0^x \int_0^y fzd\xi dt\Big\} \\ & \times(\int_0^x kds) \cdot \Big\{\int_0^x \int_0^y g\Big(\int_0^s \int_0^t hzd\lambda d\eta\Big)d\xi dt\Big\}, \end{aligned}$$

and similarly for $\int_0^y R(s,t)dt$ and $\int_0^x \int_0^y R(s,t)dsdt$.

Thus setting

$$\begin{cases} \Phi(x,y) = u_x(x,0) + u_y(0,y) + u(x,0) + u(0,y) - u(0,0) + (1 + \int_0^x + \int_0^y + \int_0^x \int_0^y)a, \\ B(x,y) = (1 + \int_0^x + \int_0^y + \int_0^x \int_0^y)b, \end{cases}$$

and $K(x,y) = (1 + \int_0^x + \int_0^y + \int_0^x \int_0^y)k$, we have

$$u(x,y) \le z(x,y) \le \Phi(x,y) + B(x,y) \int_0^x \int_0^y fzdsdt$$

which is a special case of (5.4.54) with u replaced by $P(D_1,D_2)u = z, a = \Phi$, $f_1 = B, f_2 = f_1, g_1 = K, g_2 = g, g_3 = h, h_1 = 0$. □

Let $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ be arbitrary points from $\mathbb{R}^n$. We shall say that the point x follows the point y ($x > y$) if $x_i > y_i$ for $i = 1, \cdots, n$. The

following denotations will be used:

$$\begin{cases} B(y,x) = [y_1,x_1] \times [y_2,x_2] \times \cdots \times [y_n,x_n], \\ B_k(y,x) = [y_1,x_1] \times \cdots \times [y_{k-1},x_{k-1}] \times [y_{k+1},x_{k+1}] \times \cdots \times [y_n,x_n], \\ ds' = ds_1 ds_2 \cdots ds_n, \quad d\tau' = d\tau_1 d\tau_2 \cdots d\tau_n, \\ ds = ds_1 \cdots ds_{k-1} ds_{k+1} \cdots ds_n, \qquad d\tau = d\tau_1 \cdots d\tau_{k-1} d\tau_{k+1} \cdots d\tau_n. \end{cases}$$

Theorem 5.4.13 (Zahariev-Bainov [683]) *Let* $x_0 = (x_1^0, x_2^0 \cdots, x_n^0)$, $x = (x_1, x_2 \cdots, x_n)$, $x > x_0$ *be arbitrary points from* $\mathbb{R}^n$ *and let the following conditions be satisfied:*

(1) The function $u(y)$, $f(y) : \mathbb{R}^n \to \mathbb{R}_+$ *are continuous and non-negative.*
(2) There exists a positive integer k, $1 \le k \le n$*, such that the inequality holds for every point* $y \in B_k(x_0, x)$, $y > x_0$,

$$u(y) \le u(x_0) + \int_{x_k^0}^{y_k} \left[\int_{B_k(x_0,x)} f(s)u(s)ds \right] ds_k. \tag{5.4.55}$$

Then there holds that

$$u(x) \le u(x_0) \exp\left(\int_{B_k(x_0,x)} f(s)ds \right). \tag{5.4.56}$$

Proof Let $u(x_0) \neq 0$ and let $y > x_0$ be an arbitrary point from $B_k(x_0, x)$. From (5.4.48) it follows that

$$\frac{u(y)}{u(x_0) + \int_{x_k^0}^{y_k} [\int_{B_k(x_0,x)} f(s)u(s)ds] ds_k} \le 1. \tag{5.4.57}$$

After multiplying both sides of inequality (5.4.57) by $f(y)$ and integrating with respect to $B_k(x_0, x)$, we obtain

$$\int_{x_k^0}^{y_k} \frac{\int_{B_k(x_0,x)} f(\tau)u(\tau)d\tau'}{u(x_0) + \int_{x_k^0}^{\tau_k} [\int_{B_k(x_0,x)} f(s)u(s)ds] ds_k} d\tau_k \le \int_{B_k(x_0,x)} f(s)ds$$

which implies

$$\ln \left[u(x_0) + \int_{B_k(x_0,x)} f(s)u(s)ds \right] - \ln |u(x_0)| \le \int_{B_k(x_0,x)} f(s)ds. \tag{5.4.58}$$

Combining (5.4.55) and (5.4.58), we obtain

$$u(x) \le u(x_0) \exp\left(\int_{B_k(x_0,x)} f(s)ds\right).$$

Let $u(x_0) = 0$. Then from (5.4.55), it follows that for every positive number ϵ and for every $y \in B_k(x_0, x),\ y > x_0$,

$$u(y) \le \epsilon + \int_{B_k(x_0,x)} f(s)u(s)ds.$$

Hence we conclude

$$u(x) \le \epsilon \exp\left(\int_{B_k(x_0,x)} f(s)ds\right). \tag{5.4.59}$$

Since $u(x) \ge 0$ and $\epsilon > 0$ is an arbitrary number independent of x and x_0, then from (5.4.59) it follows that $u(x) = 0$. Thus the proof is complete. □

Theorem 5.4.14 (Zahariev-Bainov [683]) *Let* $x_0 = (x_1^0, x_2^0 \cdots, x_n^0)$, $x = (x_1, x_2 \cdots, x_n)$, $x > x_0$ *be arbitrary points from* $\mathbb{R}^n$ *and let the following conditions hold*

(1) The function $u(y),\ f(y) : \mathbb{R}^n \to \mathbb{R}_+$ *are continuous and non-negative.*
(2) There exists a positive integer $k,\ 1 \le k \le n$, *such that the inequality holds, for every point* $y \in B(x_0, x),\ y < x$,

$$u(y) \le u(x_0) + \int_{y_k}^{x_k} \left[\int_{B_k(x_0,x)} f(s)u(s)ds\right] ds_k. \tag{5.4.60}$$

Then there holds that

$$u(x) \ge u(x_0) \exp\left(-\int_{B_k(x_0,x)} f(s)ds\right). \tag{5.4.61}$$

Proof Let $u(x_0) \ne 0$ and let $y < x$ be an arbitrary point from $B(x_0, x)$. From (5.4.60) it follows that

$$\frac{-u(y)}{u(x_0) + \int_{y_k}^{x_k}[\int_{B_k(x_0,x)} f(s)u(s)ds']ds_k} \ge -1. \tag{5.4.62}$$

Multiplying both sides of inequality (5.4.62) by $f(y)$ and integrating with respect to $B_k(x_0, x)$, we obtain

$$-\int_{x_k^0}^{y_k} \frac{\int_{B_k(x_0,x)} f(\tau)u(\tau)d\tau'}{u(x) + \int_{\tau_k}^{x_k}[\int_{B_k(x_0,x)} f(s)u(s)ds']ds_k} d\tau_k \ge -\int_{B(x_0,x)} f(s)ds. \tag{5.4.63}$$

From (5.4.63) it follows that

$$\ln|u(x)| - \ln\left[u(x) + \int_{B_k(x_0,x)} f(s)u(s)ds\right] \geq -\int_{B(x_0,x)} f(s)ds,$$

which yields

$$u(x) \geq u(x_0)\exp\left(\int_{B(x_0,x)} f(s)ds\right).$$

In case $u(x) = 0$, we may proceed as in Theorem 5.4.13 when $u(x_0) = 0$. Thus the proof is complete. □

Remark 5.4.2 Theorems 5.4.13 and 5.4.14 would also hold if in the inequalities (5.4.55) and (5.4.60), $u(x_0)$ and $u(x)$ are replaced by an arbitrary non-negative constant.

To introduce the following result, we need the following notations. Let $U(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ be an open ball in $\mathbb{R}^n$. Let $x, y \in \mathbb{R}^n$, and let $Q(x, y)$ be sets satisfying the following conditions (H1)–(H6):

(H1): $Q(x, y)$ is a compact measurable subset of $\mathbb{R}^n$;
(H2): Measure $[Q(x, y)\backslash Q(x, z)] \cup [Q(x, z)\backslash Q(x, y)] \to 0$ as $y \to x$;
(H3): $Q(x, y) \subset U(x, |x - y|)$;
(H4): if $z \in Q(x, y)$, then $Q(x, z) \subseteq Q(x, y)$;
(H5): $Q(x, y) = Q(x, z)$ if and only if $y = z$;
(H6): if $z \in Q(x, y)$, then $\{w \in Q(x, y) : z \in Q(x, w)\} = Q(x, y)$.

Note that the $n-$dimensional intervals

$$[x, y] = \{z \in \mathbb{R}^n : z_i = \lambda_i x_i + (1 - \lambda_i)y_i, 0 \leq \lambda_i \leq 1, i = 1, \cdots, n\} = \{z \in \mathbb{R}^n : x \leq z \leq y\}$$

satisfy the above conditions (H1)–(H6).

Theorem 5.4.15 (Simeonov-Bainov [592]) *Let $Q(x, y)$ be sets satisfying the conditions (H1)–(H4). Let $u, a : Q(x, y) \to \mathbb{R}, G : Q(\alpha, \beta) \times Q(\alpha, \beta) \to \mathbb{R}_+$ be continuous non-negative functions. If for all $x \in Q(\alpha, \beta)$, there holds that*

$$u(x) \leq a(x) + \int_{Q(\alpha,x)} G(x, s)u(s)ds, \tag{5.4.64}$$

then

(1) $u(x) \leq v(x)$, for all $x \in Q(\alpha, \beta)$, where $v(x)$ is a solution of the equation

$$v(x) = a(x) + \int_{Q(\alpha,x)} G(x, s)v(s)ds. \tag{5.4.65}$$

(2) *The solution $v(x)$ of (5.4.65) is unique, and can be represented as the sum of the series*

$$v(x) = v_0(x) + v_1(x) + \cdots + v_m(x) + \cdots \tag{5.4.66}$$

with $v_0(x) \equiv a(x)$,

$$v_m(x) = \int_{Q(\alpha,x)} G(x,s)v_{m-1}(s)ds, \quad m = 1, 2, \cdots .$$

Proof Define on the space C of continuous functions $w : Q(\alpha, \beta) \to \mathbb{R}$ with the norm $\|w\| = \max_{z \in Q(\alpha,\beta)} |w(z)|$ the operator T by

$$Tw(x) = \int_{Q(\alpha,x)} G(x,s)w(s)ds.$$

The conditions (H1)–(H2) imply that $T(C) \subset C$. The non-negativity of G and (5.4.64) successively imply that

$$\begin{aligned} u &\le a + T(a + Tu) = a + Ta + T^2u \le a + Ta + T^2a + T^3u \\ &\le a + Ta + T^2a + \cdots + T^m a + \cdots \end{aligned}$$

or

$$u(x) \le v(x)$$

where

$$v(x) = v_0(x) + v_1(x) + \cdots + v_m(x) + \cdots ,$$

$$v_0(x) \equiv a(x), \;\; v_m(x) = T^m a(x) = \int_{Q(\alpha,x)} G(x,s)v_{m-1}(s)ds.$$

By induction with respect to m, we can derive from the conditions (H1), (H3) and (H4) that

$$\|T^m w\| \le \frac{Q^m \|w\|}{m!}, \quad w \in C, \; m \in \mathbb{N}, \tag{5.4.67}$$

where $Q = |\beta - \alpha|^n / n! \cdot \sigma M$, where $M = \max_{Q(\alpha,\beta)\times Q(\alpha,\beta)} |G(x,s)|$ and $\sigma = \int_{S(1)} ds$ is the measure (area) of the unit sphere $S(1) = \{x \in \mathbb{R}^n : |x| = 1\}$.

In particular, (5.4.67) holds for $w = a$. Weierstrass's theorem and (5.4.67) imply that the series (5.4.66) is uniformly convergent in $Q(\alpha, \beta)$. A straightforward verification shows that its sum $v(x)$ satisfies (5.4.65). Since, moreover, (5.4.67)

implies that some power of T is a contraction, $v(x)$ is the unique solution of equation (5.4.65). □

Lemma 5.4.3 (Simeonov-Bainov [592]) *Let $Q(x, y)$ be sets satisfying the conditions (H1), and (H4)–(H5), and let $b : Q(\alpha, \beta) \to \mathbb{R}_+$ be a non-negative continuous function. Then functions*

$$v_0(x) \equiv 1,\ v_m(x) = \int_{Q(\alpha,x)} b(s)v_{m-1}(s)ds,\ x \in Q(\alpha, \beta),\ m = 1, \cdots, \tag{5.4.68}$$

satisfy

$$0 \le v_m(x) \le \frac{1}{m!}\left(\int_{Q(\alpha,x)} b(s)ds\right)^m,\quad x \in Q(\alpha, \beta),\ m = 1, \cdots. \tag{5.4.69}$$

Proof Let $\nu = (1, \cdots, m)$, and let μ be a permutation of ν, i.e., $\mu = (i_1, \cdots, i_m)$. We can define the following $m!$ sets:

$$P_m(\alpha, x; \mu) = \Big\{(s_1, \cdots, s_m) \in \mathbb{R}^{mn} : z_{i_1} \in Q(\alpha, x), z_{i_2} \in Q(\alpha, z_{i_1}), \cdots, z_{i_m} \in Q(\alpha, Z_{i_{m-1}})\Big\}.$$

In particular, when $\mu = \nu$, we set

$$P_m(\alpha, x) = P_m(\alpha, x, \nu) = \Big\{(s_1, \cdots, s_m) \in \mathbb{R}^{mn} : z_1 \in Q(\alpha, x), z_2 \in Q(\alpha, z_1), \cdots, z_m \in Q(\alpha, Z_{m-1})\Big\}.$$

Now introducing the notation

$$D_m(x) = \Big\{(s_1, \cdots, s_m) \in \mathbb{R}^{mn} : s_1 \in Q(\alpha, x), \cdots, s_m \in Q(\alpha, x)\Big\},$$

the function $v_m(x)$ takes the form

$$v_m(x) = \int \cdots \int_{P_m(\alpha,x)} b(s_1) \cdots b(s_m)ds_1 \cdots ds_m. \tag{5.4.70}$$

We divide the proof into several steps:

Step 1. The equality holds:

$$\int \cdots \int_{P_m(\alpha,x)} b(s_1) \cdots b(s_m)ds_1 \cdots ds_m = \int \cdots \int_{P_m(\alpha,x;\mu)} b(s_1) \cdots b(s_m)ds_1 \cdots ds_m \tag{5.4.71}$$

since each integral in (5.4.71) is equal to the integral

$$\int \cdots \int_{P_m(\alpha,x;\mu)} b(s_{i_1}) \cdots b(s_{i_m})ds_{i_1} \cdots ds_{i_m}.$$

Step 2. The definition of $P_m(\alpha, x; \mu)$ and $D_m(x)$ imply that

$$\bigcup_{\mu} P_m(\alpha, x; \mu) \subset D_m(x), \tag{5.4.72}$$

where the union is over all permutations μ.

Step 3. Let μ and λ be two distinct permutations, in which the numbers k and j form an inversion.

Let $(s_1, \cdots, s_k, \cdots, s_j, \cdots, s_m) \in P_m(\alpha, x; \mu) \cap P_m(\alpha, x; \lambda)$. The definition of the sets $P_m(\alpha, x; \mu)$ implies that $s_k \in Q(\alpha, s_j)$ and $s_j \in Q(\alpha, s_k)$, and (H4) implies that $Q(\alpha, s_k) \subset Q(\alpha, s_j)$ and $Q(\alpha, s_j) \subset Q(\alpha, s_k)$, and hence $Q(\alpha, s_k) = Q(\alpha, s_j)$. Hence by (H5), $s_j = s_k$, so that the set $P_m(\alpha, x; \mu) \cap P_m(\alpha, x; \lambda)$ is contained in the subset $L = \{(s_1, \cdots, s_m) \in \mathbb{R}^{mn} : s_k = s_j\}$ of $\mathbb{R}^{mn}$. Since $dimL < mn$, we have

$$\int \cdots \int_{P_m(\alpha,x;\mu)\cap P_m(\alpha,x;\lambda)} b(s_1) \cdots b(s_m) ds_1 \cdots ds_m = 0. \tag{5.4.73}$$

Step 4. The following equality holds:

$$\begin{aligned} &\int \cdots \int_{\cup_\mu P_m(\alpha,x;\mu)} b(s_1) \cdots b(s_m) ds_1 \cdots ds_m \\ &= \sum_\mu \int \cdots \int_{P_m(\alpha,x;\mu)} b(s_1) \cdots b(s_m) ds_1 \cdots ds_m + J, \end{aligned} \tag{5.4.74}$$

where in J we have collected the integrals over all possible intersections of sets of the type $P_m(\alpha, x; \mu)$, $P_m(\alpha, x; \lambda)$ with distinct μ, λ. But (5.4.73) implies that all such integrals vanish, i.e., $J = 0$. Hence, it follows from (5.4.74), (5.4.72) and the non-negativity of b that

$$\begin{aligned} &\sum_\mu \int \cdots \int_{P_m(\alpha,x;\mu)} b(s_1) \cdots b(s_m) ds_1 \cdots ds_m \\ &= \int \cdots \int_{D_m(x)} b(s_1) \cdots b(s_m) ds_1 \cdots ds_m = \left(\int_{Q(\alpha,x)} b(s) ds \right)^m, \end{aligned}$$

and (5.4.71) and (5.4.70) imply (5.4.69). The inequalities $v_m(x) \geq 0, m = 0, 1, \cdots$, are obvious. □

Corollary 5.4.3 (Simeonov-Bainov [592]) *If $b(x) \geq 0$ is continuous for all $x \in [\alpha, \beta], \alpha < \beta$, the solution $v(x)$ of problem (5.4.40)–(5.4.41) satisfies for all $\alpha \leq s \leq x \leq \beta$,*

$$0 \leq v(s, x) \leq \exp\left(\int_s^x b(\tau) d\tau \right). \tag{5.4.75}$$

Proof Indeed, Theorem 5.4.15 with $G(x, s) = b(s), a(x) = 1$, and $Q(s, x) = [s, x]$ implies the representation

$$v(s; x) = \sum_{m=0}^{+\infty} v_m(s; x).$$

By Lemma 5.4.3 with $Q(s, x) = [s, x]$, we obtain

$$0 \le v(s; x) = \sum_{m=0}^{+\infty} v_m(s; x) \le \sum_{m=0}^{+\infty} \frac{1}{m!} \left(\int_s^x b(\tau) d\tau \right)^m = \exp \left(\int_s^x b(\tau) d\tau \right).$$

□

Corollary 5.4.4 (Simeonov-Bainov [592]) *If $\alpha, \beta \in \mathbb{R}^n, \alpha < \beta$, and $u(x), a(x)$, $b(x), q(x)$ are non-negative continuous functions in $[\alpha, \beta]$, if the inequality holds for all $x \in [\alpha, \beta]$,*

$$u(x) \le a(x) + q(x) \int_\alpha^x b(s) u(s) ds, \tag{5.4.76}$$

then for all $x \in [\alpha, \beta]$,

$$u(x) \le a(x) + q(x) \int_\alpha^x a(s) b(s) \exp \left(\int_s^x q(\tau) b(\tau) d\tau \right) ds. \tag{5.4.77}$$

In particular, if $q(x) = 1$ and $a(x)$ is non-decreasing in $[\alpha, \beta]$, then for all $x \in [\alpha, \beta]$,

$$u(x) \le a(x) \left[1 + \int_\alpha^x b(s) \exp \left(\int_s^x b(\tau) d\tau \right) ds \right]. \tag{5.4.78}$$

Proof Indeed, (5.4.78) follows from Theorem 5.4.15 and Corollary 5.4.3. Moreover, (5.4.77) is better than estimate

$$u(x) \le a(x) \exp \left(\int_\alpha^x b(s) ds \right). \tag{5.4.79}$$

□

Remark 5.4.3 It follows from the proof of Lemma 5.4.3 that (5.4.69) holds if condition (H1) is replaced by the condition $\int_{Q(\alpha,\beta)} b(s) ds < +\infty$.

To obtain estimates similar to (5.4.77)–(5.4.79) with $[\alpha, x]$ replaced by $Q(x, \alpha)$, we need the next lemma.

Lemma 5.4.4 (Simeonov-Bainov [592]) *Let $Q(x, y)$ be sets satisfying (H4)–(H6), and let $a, b, q : Q(\alpha, \beta) \to \mathbb{R}$ be non-negative continuous functions with*

$$\int_{Q(\alpha,\beta)} q(s)b(s)ds < +\infty, \quad \int_{Q(\alpha,\beta)} a(s)b(s)ds < +\infty. \tag{5.4.80}$$

If $v_0(x) = a(x)$, then the functions

$$v_m(x) = q(x) \int_{Q(\alpha,x)} b(s)v_{m-1}(s)ds, \quad m = 1, \cdots,$$

satisfy

$$0 \le v_{m+1}(x) \le \frac{q(x)}{m!} \int_{Q(\alpha,x)} a(s)b(s) \left(\int_{Q(\alpha,x)} q(\tau)b(\tau)d\tau \right)^m ds, \tag{5.4.81}$$

with $m = 0, 1, \cdots$.

Proof For function $v_{m+1}(x), m = 0, 1, \cdots$, we have

$$\begin{aligned}
v_{m+1}(x) &= \int_{Q(\alpha,x)} q(x)b(s_1) \\
&\quad \times \left(\int_{Q(\alpha,s_1)} q(s_1)b(s_2) \left(\int_{Q(\alpha,s_2)} \cdots \left(\int_{Q(\alpha,s_m)} q(s_m)b(s)a(s)ds \right) ds_m \right) \cdots \right) ds_1 \\
&= \int \cdots \int_{P_{m+1}(\alpha,x)} q(x)q(s_1)b(s_1) \cdots q(s_m)b(s_m)a(s)b(s)ds_1 \cdots ds_m ds \\
&= q(x) \int_{Q(\alpha,x)} a(s)b(s) \left(\int \cdots \int_{Q(s)} q(s_1)b(s_1) \cdots q(s_m)b(s_m)ds_1 \cdots ds_m \right) ds,
\end{aligned} \tag{5.4.82}$$

where $Q(s) = \{(s_1, \cdots, s_m) \in \mathbb{R}^{mn}, (s_1, \cdots, s_m, s) \in P_{m+1}(\alpha, x)\}$. For fixed $s \in Q(\alpha, x)$, we get by using (H4)–(H6),

$$\begin{aligned}
Q(s) &= \{(s_1, \cdots, s_m) \in \mathbb{R}^{mn} : s_1 \in Q(\alpha, x), s_2 \in Q(\alpha, s_1), \cdots, s_m \in Q(\alpha, s_{m-1}), \\
&\qquad s \in Q(\alpha, s_1), \cdots, s \in Q(\alpha, s_m)\} \\
&= \{(s_1, \cdots, s_m) \in \mathbb{R}^{mn} : s_1 \in Q(s, x), s_2 \in Q(\alpha, s_1), \cdots, s_m \in Q(\alpha, s_{m-1})\} \\
&= P_m(s, x).
\end{aligned} \tag{5.4.83}$$

By Lemma 5.4.3, we obtain

$$\int \cdots \int_{P_m(s,x)} q(s_1)b(s_1) \cdots q(s_m)b(s_m)ds_1 \cdots ds_m \le \frac{1}{m!} \left(\int_{Q(s,x)} q(\tau)b(\tau)d\tau \right)^m$$

which yields (5.4.81). □

Theorem 5.4.16 (Simeonov-Bainov [592]) *Let $Q(x,y)$ be sets satisfying conditions (H4)–(H6), and let $u,a,b,q : Q(\alpha,\beta) \to \mathbb{R}_+$ be non-negative continuous functions with*

$$\int_{Q(\alpha,\beta)} u(s)b(s)ds < +\infty,\ \int_{Q(\alpha,\beta)} q(s)b(s)ds < +\infty,\ \int_{Q(\alpha,\beta)} a(s)b(s)ds < +\infty. \tag{5.4.84}$$

If the inequality holds for all $x \in Q(\alpha,\beta)$,

$$u(x) \le a(x) + q(x)\int_{Q(\alpha,x)} b(s)u(s)ds, \tag{5.4.85}$$

then for all $x \in Q(\alpha,\beta)$,

$$u(x) \le a(x) + q(x)\int_{Q(\alpha,x)} a(s)b(s)\exp\left(\int_{Q(s,x)} q(\tau)b(\tau)d\tau\right)ds. \tag{5.4.86}$$

Proof Since $b \ge 0, q \ge 0$, as in Theorem 5.4.15 with $G(s,x) = q(x)b(s)$, (5.4.85) implies

$$u(x) \le \sum_{j=0}^{m} v_j(x) + R_{m+1}(x),\quad x \in Q(\alpha,\beta), \tag{5.4.87}$$

where $v_0(x) = a(x)$,

$$\begin{cases} v_j(x) = q(x)\displaystyle\int_{Q(\alpha,x)} b(s)v_{j-1}(s)ds,\quad x \in Q(\alpha,\beta), j = 1,2,\cdots, & (5.4.88)\\ R_{m+1}(x) = \displaystyle\int_{Q(\alpha,x)} q(x)b(s_1)\Big(\int_{Q(\alpha,s_1)} q(s_1)b(s_2)\cdots & \\ \qquad\qquad \cdots(\displaystyle\int_{Q(\alpha,s_m)} q(s_m)b(s)u(s)ds)\cdots\Big)ds_1. & (5.4.89)\end{cases}$$

By Lemma 5.4.4, we get for all $x \in Q(\alpha,\beta)$,

$$\begin{cases} 0 \le v_j(x) = q(x)\displaystyle\int_{Q(\alpha,x)} a(s)b(s)\frac{1}{(j-1)!}\left(\int_{Q(s,x)} q(\tau)b(\tau)d\tau\right)^{j-1} ds, & (5.4.90)\\ 0 \le R_{m+1}(x) \le q(x)\displaystyle\int_{Q(\alpha,x)} u(s)b(s)\frac{1}{m!}\left(\int_{Q(s,x)} q(\tau)b(\tau)d\tau\right)^{m}, & (5.4.91)\end{cases}$$

$j = 1, \cdots, m = 0, 1, \cdots$. It thus follows from (5.4.84) and (5.4.91) that $R_{m+1}(x) \to 0$ as $m \to +\infty$. Hence, by (5.4.87), we conclude

$$u(x) \leq \sum_{j=0}^{+\infty} v_j(x)$$

which, together with (5.4.90), implies (5.4.86). □

Corollary 5.4.5 (Simeonov-Bainov [592]) *Let $u, a, b, q : \mathbb{R}_+^2 \to \mathbb{R}_+$ be nonnegative continuous functions satisfying for all $x \geq 0, \quad y \geq 0$,*

$$u(x, y) \leq a(x, y) + q(x, y) \int_x^{+\infty} \int_y^{+\infty} b(s, t)u(s, t)dsdt, \tag{5.4.92}$$

where

$$\int_0^{+\infty} \int_0^{+\infty} u(s, t)b(s, t)dsdt < +\infty, \quad \int_0^{+\infty} \int_0^{+\infty} q(s, t)b(s, t)dsdt < +\infty,$$

$$\int_0^{+\infty} \int_0^{+\infty} a(s, t)b(s, t)dsdt < +\infty.$$

Then for all $x \geq 0, y \geq 0$,

$$u(x, y) \leq a(x, y) + q(x, y) \int_x^{+\infty} \int_y^{+\infty} a(s, t)b(s, t) \exp\left(\int_x^s \int_y^t q(\sigma, \tau)b(\sigma, \tau)d\sigma d\tau \right) dsdt. \tag{5.4.93}$$

In particular, if $a(x, y) \equiv a \geq 0$, and $q(x, y) \equiv 1$, then

$$u(x, y) \leq a \left[1 + q(x, y) \int_x^{+\infty} \int_y^{+\infty} b(s, t) \exp\left(\int_x^s \int_y^t b(\sigma, \tau)d\sigma d\tau \right) dsdt \right] \tag{5.4.94}$$

which is better than the estimate

$$u(x, y) \leq a \exp\left(\int_x^{+\infty} \int_y^{+\infty} b(s, t)dsdt \right). \tag{5.4.95}$$

Proof The conclusion follows from Theorem 5.4.16 in case $n = 2$, and $Q[(s, t), (x, y)] = \left\{ (\sigma, \tau) \in \mathbb{R}_+^2 : x \leq \sigma \leq s, y \leq \tau \leq t \right\}, \alpha = (+\infty, +\infty), \beta = (0, 0)$. □

The next result is related to some new integrodifferential inequalities of the Gronwall and Wendroff type in several independent variables which generalize some existing results in the literature.

An important generalization of this inequality is by Wendroff [47].

However, Wendroff's inequality had not received the attention it deserved until or so where many generalizations of it have been established by, e.g., Snow [603, 604], Ghoshal and Masood [227, 228], Pachpatte [445, 446], Bondge and Pachpatte [90], and many others.

To state the next result, we need to introduce some notations. Let $n \geq 2$ be a fixed integer. Let $\mathbb{R}_0 = (0, +\infty)$. For each $i = 1, \ldots, n$, the differential operator $\partial/\partial x_i$ will be abbreviated by ∂_i. As usual, partial derivatives of a function $f : \mathbb{R}_+^n \to \mathbb{R}_+$ will be denoted by f_i, f_j, etc, and for the sake of simplicity, f will stand for $f_{1\ldots n}$. If X, Y are subsets of some Euclidean spaces, $C^k(X, Y)$ will denote the set of all functions of X into Y with continuous $k-$th order derivatives. The collection of all functions $f \in C^1(\mathbb{R}_+^n, \mathbb{R}_0)$ such that $f_{i_1\ldots i_n}$ is continuous for any permutation $i_1, \ldots, i_n$ of $1, \ldots, n$ will be denoted by $\mathcal{D}$. Let $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ be any functions, $i = 1, \ldots, n$. Then

$$\begin{cases} \displaystyle\sum_j^* g_i(x_i) := \sum_{i=1,\, i\neq j}^n g_i(x_i) + g_j(0), \\ \displaystyle\sum_{j,k}^* g_i(x_i) := \sum_{i=1,\, i\neq j,\, k}^n g_i(x_i) + g_j(0) + g_k(0). \end{cases}$$

We first observe the following important lemma.

Lemma 5.4.5 (Cheung [132]) *Suppose $f \in C^1(\mathbb{R}_+^n, \mathbb{R}_0)$ is given by*

$$f(x) = \sum_{i=1}^n g_i(x_i) + \int_0^x h(s)ds \tag{5.4.96}$$

with

$$f'(x) \leq k(x)f(x) \tag{5.4.97}$$

for some functions $g_i \in C^1(\mathbb{R}_+^n, \mathbb{R}_0)$, $i = 1, \ldots, n$, and $h, k \in C^1(\mathbb{R}_+^n, \mathbb{R}_+)$. If f is non-decreasing in each variable and f is continuous, then for any $x \in \mathbb{R}_+^n$,

$$f(x) \leq \frac{(\sum_1^* g_i(x_i))(\sum_2^* g_i(x_i))}{\sum_{1,2}^* g_i(x_i)} \exp\left(\int_0^x k(s)ds\right). \tag{5.4.98}$$

In particular, if $f \in \mathcal{D}$, then for any $x \in \mathbb{R}_+^n$,

$$y'(x) \leq \min_{1\leq j<k\leq n} \frac{(\sum_j^* g_i(x_i))(\sum_k^* g_i(x_i))}{\sum_{j,k}^* g_i(x_i)} \exp\left(\int_0^x k(s)ds\right). \tag{5.4.99}$$

Proof Note first that by assumption, $f_i \geq 0$ for all i. Furthermore, since $h \geq 0$, we have

$$f_1, f_{12}, f_{123}, \ldots, f_{1,\ldots,n-1}, f \geq 0.$$

Hence it follows from (5.4.90) that for each $s \in \mathbb{R}^n_+$,

$$\partial_n \left(\frac{f_{1,\ldots,n-1}}{f}\right)(s) = \frac{\dot{f}}{f}(s) - \frac{f_{1\ldots n-1}}{f^2}(s) \leq \frac{\dot{f}}{f}(s) \leq k(s).$$

By integrating this inequality with respect to s_n from 0 to x_n, we have

$$\frac{f_{1,\ldots,n-1}}{f}(s_1, \ldots, s_{n-1}, x_n) - \frac{f_{1,\ldots,n-1}}{f}(s_1, \ldots, s_{n-1}, 0) \leq \int_0^{x_n} k(s) ds_n.$$

If $n - 1 \geq 2$, we have, by (5.4.96),

$$f_{1,\ldots,n-1}(s) = \int_0^{s_n} h(s_1, \ldots, s_{n-1}, t_n) dt_n$$

and so $f_{1,\ldots,n-1}(s_1, \ldots, s_{n-1}, 0) = 0$. Hence

$$\frac{f_{1,\ldots,n-1}}{f}(s_1, \ldots, s_{n-1}, x_n) \leq \int_0^{x_n} k(s) ds_n.$$

Repeating the above process, we get

$$\begin{aligned}
&\partial_{n-1} \left(\frac{f_{1,\ldots,n-2}}{f}\right)(s_1, \ldots, s_{n-1}, x_n) \\
&= \frac{f_{1,\ldots,n-1}}{f}(s_1, \ldots, s_{n-1}, x_n) - \frac{f_{1,\ldots,n-2} f_{n-1}}{f^2}(s_1, \ldots, s_{n-1}, x_n) \\
&\leq \frac{f_{1,\ldots,n-1}}{f}(s_1, \ldots, s_{n-1}, x_n) \leq \int_0^{x_n} k(s) ds_n.
\end{aligned}$$

Integrating with respect to s_{n-1} from 0 to x_{n-1}, we get

$$\frac{f_{1,\ldots,n-2}}{f}(s_1, \ldots, s_{n-2}, x_{n-1}, x_n) - \frac{f_{1,\ldots,n-2}}{f}(s_1, \ldots, s_{n-2}, 0, x_n) \leq \int_0^{x_{n-1}} \int_0^{x_n} k(s) ds_n ds_{n-1}.$$

Again, if $n - 2 \geq 2$, $f_{1,\ldots,n-2}(s_1, \ldots, s_{n-2}, 0, x_n) = 0$, then

$$\frac{f_{1,\ldots,n-2}}{f}(s_1, \ldots, s_{n-2}, 0, x_n) \leq \int_0^{x_{n-1}} \int_0^{x_n} k(s) ds_n ds_{n-1}.$$

By an induction, we know

$$\begin{aligned}\partial_2\left(\frac{f_1}{f}\right)(s_1,s_2,x_3,\ldots,x_n) &\le \frac{f_{12}}{f}(s_1,s_2,x_3,\ldots,x_n)\\ &\le \int_0^{x_3}\cdots\int_0^{x_n} k(s)ds_n\cdots ds_3.\end{aligned}$$

Integrating with respect to s_2 from 0 to x_2, we obtain

$$\frac{f_1}{f}(s_1,x_2,\ldots,x_n)-\frac{f_1}{f}(s_1,0,x_3,\ldots,x_n)\le\int_0^{x_2}\cdots\int_0^{x_n}k(s)ds_m\cdots ds_2.$$

Integrating again with respect to s_1 from 0 to x_1, we conclude

$$\ln f(x)-\ln f(0,x_2,\ldots,x_n)-\ln f(x_1,0,x_3,\ldots,x_n)+\ln f(0,0,x_3,\ldots,x_n)\le\int_0^x k(s)ds,$$

which yields (5.4.98) and (5.4.99) follows from similar arguments by replacing the subscripts 1, 2 by all possible j, k's satisfying $1\le j<k\le n$. □

The following inequalities are Wendroff type inequalities which generalize the results of Pachpatte [445, 446] to the case of many independent variables.

Theorem 5.4.17 (Cheung [132]) *Let $y, w \in C(\mathbb{R}_+^n,\mathbb{R}_+)$ be such that $y'\in C(\mathbb{R}_+^n,\mathbb{R}_+)$ and $y(x)=0$ if $x_i=0$ for some i. If for any $x\in\mathbb{R}_+^n$,*

$$y'(x)\le\sum_{i=1}^n g_i(x_i)+\int_0^x w(s)(y(s)+y'(s))ds, \tag{5.4.100}$$

where $g_i\in C^1(\mathbb{R}_+,\mathbb{R}_0)$ satisfies $g_i'\ge 0,\ i=1,\ldots,n$, then for any $x\in\mathbb{R}_+^n$,

$$y'(x)\le\sum_{i=1}^n g_i(x_i)+\min_{1\le j<k\le n}\int_0^x w(s)\frac{(\sum_j^* g_i(x_i))(\sum_k^* g_i(x_i))}{\sum_{j,k}^* g_i(x_i)}\exp\Big(\int_0^x(1+w(t))dtds\Big). \tag{5.4.101}$$

Proof Let $u(x)$ be the right-hand side of (5.4.100). Then $0\le y'(t)\le u(x)$ and so by the initial conditions of $y(x)$, we have for all $x\in\mathbb{R}_+^n$,

$$y(x)\le\int_0^x u(s)ds.$$

Let

$$f(x)=u(x)+\int_0^x u(s)ds,$$

then $u \le f$ and by the definition of $u(x)$, we get

$$0 \le u'(x) = w(x)(y(x) + \dot{y}(x)) \le w(x)f(x). \tag{5.4.102}$$

Noting now that $f \in \mathcal{D}$,

$$f'(x) = u'(x) + u(x) \le (1 + w(x))f(x),$$

and for any $x \in \mathbb{R}^n_+$,

$$f(x) = \sum_{i=1}^{n} g_i(x_i) + \int_0^x [u(s) + w(s)(y(s) + y'(s))]ds,$$

thus from Lemma 5.4.5 and (5.4.102) it follows that for any $x \in \mathbb{R}^n_+$ and $1 \le j < k \le n$,

$$0 \le u'(x) \le w(x)\frac{(\sum_j^* g_i(x_i))(\sum_k^* g_i(x_i))}{\sum_{j,k}^* g_i(x_i)} \exp\left[\int_0^x (1 + w(s))ds\right]. \tag{5.4.103}$$

Hence by the definition of $u(x)$, we obtain

$$\begin{aligned}
\int_0^x u'(s)ds &= \int_0^{x_1} \cdots \int_0^{x_{n-1}} [u_{1,\dots,n-1}(s_1, \dots, s_{n-1}, x_n) \\
&\quad -u_{1,\dots,n-1}(s_1, \dots, s_{n-1}, 0)]ds_{n-1} \cdots ds_1 \\
&= \int_0^{x_1} \cdots \int_0^{x_{n-1}} u_{1,\dots,n-1}(s_1, \dots, s_{n-1}, x_n)ds_{n-1} \cdots ds_1 \\
&\quad \cdots \\
&= \int_0^{x_1} \int_0^{x_2} u_{12}(s_1, s_2, x_3, \dots, x_n)ds_2 ds_1 \\
&= \int_0^{x_1} [u_1(s_1, x_2, \dots, x_n) - u_1(s_1, 0, x_3, \dots, x_n)]ds_1 \\
&= u(x) - u(0, x_2, \dots, x_n) - u(x_1, 0, x_3, \dots, x_n) + u(0, 0, x_3, \dots, x_n) \\
&= u(x) - \sum_1^* g_i(x_i) - \sum_2^* g_i(x_i) + \sum_{1,2}^* g_i(x_i) \\
&= u(x) - \sum_{i=1}^{n} g_i(x_i).
\end{aligned} \tag{5.4.104}$$

Inserting (5.4.103) and (5.4.104) into $y'(t) \le u(x)$, (5.4.101) follows immediately. $\square$

Corollary 5.4.6 (Cheung [132]) *Under the assumptions of Theorem 5.4.16, if* $\sum_{i=1}^{n} g_i(x_i) \equiv \alpha > 0$, *then for any* $x \in \mathbb{R}_+^n$,

$$y'(x) \le \alpha \left[1 + \int_0^x w(s) \exp\left(\int_0^s (1 + w(t))dt\right)\right].$$

Proof It easily follows from Theorem 5.4.17. □

Theorem 5.4.18 (Cheung [132]) *Let* $y, w \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ *be such that* $y' \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ *and* $y(x) = 0$ *if* $x_i = 0$ *for some i. If for any* $x \in \mathbb{R}_+^n$,

$$y'(x) \le \sum_{i=1}^{n} g_i(x_i) + M\left[y(x) + \int_0^x w(s)(y(s) + y'(s))\right] ds \tag{5.4.105}$$

where $g_i \in C^1(\mathbb{R}_+, \mathbb{R}_0)$ *satisfies* $g_i' \ge 0$, $i = 1, \ldots, n$, *and* $M \ge 0$ *is a constant, then for any* $x \in \mathbb{R}_+^n$,

$$y'(x) \le \min_{1 \le j < k \le n} \frac{(\sum_j^* g_i(x_i))(\sum_k^* g_i(x_i))}{\sum_{j,k}^* g_i(x_i)} \exp\left[\int_0^x (M + w(s) + Mw(s))ds\right]. \tag{5.4.106}$$

Proof Let $u(x)$ be the right-hand side of (5.4.105). Then $u \in \mathcal{D}$ and for any $x \in \mathbb{R}_+^n$,

$$\begin{aligned} u'(x) &= M[y'(x) + w(x)(y(x) + y'(x))] \\ &= [M + w(x) + Mw(x)]u(x). \end{aligned}$$

Furthermore, by the initial conditions of $y(x)$, we have for any $x \in \mathbb{R}_+^n$,

$$u(x) = \sum_{i=1}^{n} g_i(x_i) + \int_0^x M[y'(s) + w(s)(y(s) + y'(s))]ds.$$

Hence inequality (5.4.106) follows from Lemma 5.4.5. □

Corollary 5.4.7 (Cheung [132]) *Under the assumptions of Theorem 5.4.18, if* $\sum_{i=1}^{n} g_i(x_i) \equiv \alpha > 0$, *then for any* $x \in \mathbb{R}_+^n$,

$$y'(x) \le \alpha \exp\left[\int_0^x (M + w(s) + Mw(s))ds\right].$$

Proof It follows immediately from Theorem 5.4.18. □

Theorem 5.4.19 (Cheung [132]) *Under the assumptions in Theorem 5.4.18, if, instead of (5.4.105), for any* $x \in \mathbb{R}_+^n$,

$$y'(x) \leq \sum_{i=1}^{n} g_i(x_i) + M\left[y(x) + \int_0^x w(s)y'(s)\right] ds, \tag{5.4.107}$$

then for any $x \in \mathbb{R}_+^n$,

$$y'(x) \leq \min_{1\leq j<k\leq n} \frac{(\sum_j^* g_i(x_i))(\sum_k^* g_i(x_i))}{\sum_{j,k}^* g_i(x_i)} \exp\left[\int_0^x M(1+w(s))ds\right]. \tag{5.4.108}$$

Proof It is analogous to the proof of Theorem 5.4.18. In fact, if we set $u(x)$ to be the right-hand side of (5.4.107), then $u \in \mathcal{D}$ and for any $x \in \mathbb{R}_+^n$,

$$\begin{aligned} u'(x) &= M[y'(x) + w(x)y'(x)] \\ &= M(1+w(x))u(x). \end{aligned}$$

Furthermore, by the initial conditions of $y(x)$, we have for any $x \in \mathbb{R}_+^n$,

$$u(x) = \sum_{i=1}^{n} g_i(x_i) + \int_0^x M(1+w(s))y'(s)ds.$$

Thus inequality (5.4.108) now follows from Lemma 5.4.5 immediately. □

Remark 5.4.4 These theorems have generalized the results in [445, 446] to the case of many independent variables. The significance of these theorems is that they give genuine upper bound estimates for y' (that is, upper bounds for y' which do not involve the function y' itself) and hence for y, after integration with respect to x.

Now we shall establish the following generalization of some existing Gronwall-Bellman inequalities in [236, 443, 444].

Theorem 5.4.20 (Cheung [132]) *If* $y, p, q \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ *satisfy for any* $x \in \mathbb{R}_+^n$,

$$y(x) \leq \sum_{i=1}^{n} g_i(x_i) + \int_0^x p(s)y(s)ds + \int_0^x p(s)\left[\int_0^s q(t)y(t)dt\right] ds \tag{5.4.109}$$

where $g_i \in C^1(\mathbb{R}_+, \mathbb{R}_0)$ *satisfies* $g_i' \geq 0$ *for* $i = 1, \ldots, n$, *then*

$$y(x) \leq \sum_{i=1}^{n} g_i(x_i) + \int_0^x p(s)R(s)ds, \tag{5.4.110}$$

where

$$R(s) = \min_{1 \le j < k \le n} \frac{(\sum_j^* g_i(x_i))(\sum_k^* g_i(x_i))}{\sum_{j,k}^* g_i(x_i)} \exp\left[\int_0^s (p(t) + q(t))dt\right].$$

Proof Let $u(x)$ be the right-hand side of (5.4.109). Then

$$\begin{aligned} 0 \le u'(x) &= p(x)y(x) + p(x)\int_0^x q(t)y(t)dt \\ &\le p(x)[u(x) + \int_0^x q(t)u(t)dt]. \end{aligned}$$

Let

$$f(x) = u(x) + \int_0^x q(t)u(t)dt.$$

Then $u \le f$ and

$$u'(x) \le p(x)f(x). \tag{5.4.111}$$

We know that $f \in \mathcal{D}$,

$$f'(x) = u'(x) + q(x)u(x) \le (p(x) + q(x))f(x),$$

and for any $x \in \mathbb{R}_+^n$,

$$f(x) = \sum_{i=1}^n g_i(x_i) + \int_0^x [q(s)u(s) + p(s)y(s) + p(s)\int_0^s q(t)y(t)dt]ds.$$

By Lemma 5.4.5 and (5.4.111), we have for any $x \in \mathbb{R}_+^n$ and $1 \le j < k \le n$,

$$u(x) \le p(x)\frac{(\sum_j^* g_i(x_i))(\sum_k^* g_i(x_i))}{\sum_{j,k}^* g_i(x_i)} \exp\left[\int_0^x (p(s) + q(s))ds\right].$$

Hence for any $x \in \mathbb{R}_+^n$,

$$u'(x) \le p(x)R(x). \tag{5.4.112}$$

Now by exactly the same arguments as those in the derivation of (5.4.104), we conclude

$$\int_0^x u'(s)ds = u(x) - \sum_{i=1}^n g_i(x_i). \tag{5.4.113}$$

Therefore, inserting (5.4.112) and (5.4.113) into $y(x) \leq u(x)$, we can prove (5.4.110). □

Corollary 5.4.8 *Under the hypotheses of Theorem 5.4.20, if* $\sum_{i=1}^{n} g_i(x_i) \equiv \alpha > 0$, *then for any* $x \in \mathbb{R}_+^n$,

$$y(x) \leq \alpha \left[1 + \int_0^x p(s) \exp\left(\int_0^s (p(t) + q(t))dt\right) ds\right].$$

Proof It follows immediately from Theorem 5.4.20. □

All the functions which appear in the following inequalities in Theorem 5.4.21 are assumed to be real-valued of n variables which are non-negative and continuous. All integrals are assumed to exist on their domains of definitions.

We assume that $I = [a, b]$ in any bounded open set in the dimensional Euclidean space $\mathbb{R}^n$ and that our integrals are on $\mathbb{R}^n$ ($n \geq 1$), where $a = (a_1, a_2, ..., a_n)$, $b = (b_1, b_2, ..., b_n) \in \mathbb{R}_+^n$. Let $C(I, \mathbb{R}_+)$ denote the class of continuous functions from I to $\mathbb{R}_+$.

The following theorem deals with n-independent variable versions of the inequalities established in Pachpatte [502].

Theorem 5.4.21 (Pachpatte [502]) *Let* $u(x)$, $f(x)$, $a(x)$ *be in* $C(\mathbb{R}, \mathbb{R}_+)$ *and let* $K(x, t), D_i k(x, t)$ *be in* $C(\mathbb{R} \times \mathbb{R}, \mathbb{R}_+)$ *for all* $i = 1, 2, \cdots, n$, *and let c be a non-negative constant.*

(1) If for all $x \in I$ *and* $a \leq \tau \leq s \leq b$,

$$u(x) \leq c + \int_a^x f(s) \left[u(s) + \int_a^s k(s, \tau)u(\tau)d\tau\right] ds, \tag{5.4.114}$$

then

$$u(x) \leq c \left[1 + \int_a^x f(t) \exp(\int_a^t (k(b, s) + f(s))ds)dt\right]. \tag{5.4.115}$$

(2) If for all $x \in I$ *and* $a \leq \tau \leq s \leq b$,

$$u(x) \leq a(x) + \int_a^x f(s) \left[u(s) + \int_a^s k(s, \tau)u(\tau)d\tau\right] ds, \tag{5.4.116}$$

then

$$u(x) \leq a(x) + e(x) \left[1 + \int_a^x f(t) \exp(\int_a^t (k(b, s) + f(s))ds)dt\right], \tag{5.4.117}$$

where

$$e(x) = \int_a^x f(s)\left[a(s) + \int_a^s k(s,\tau)a(\tau)d\tau\right]ds. \tag{5.4.118}$$

Proof (1) The inequality (5.4.114) implies

$$u(x) \le c + \int_a^x f(s)\left[u(s) + \int_a^s k(s,\tau)u(\tau)d\tau\right]ds.$$

Define

$$z(x) = c + \int_a^x f(s)\left[u(s) + \int_a^s k(s,\tau)u(\tau)d\tau\right]ds.$$

Then $z(a_1, x_2, ..., x_n) = c,\ u(x) \le z(x)$ and

$$\begin{aligned} Dz(x) &= f(x)\left[u(x) + \int_a^s k(b,s)u(s)ds\right] \\ &\le f(x)\left[z(x) + \int_{x_0}^x k(b,s)z(s)ds\right]. \end{aligned}$$

Define

$$v(x) = z(x) + \int_a^x k(b,s)z(s)ds,$$

then $z(a_1, x_2, \cdots, x_n) = v(a_1, x_2, \cdots, x_n) = c, Dz(x) \le f(x)v(x)$ and $z(x) \le v(x)$, and

$$Dv(x) = Dz(x) + k(b,x)z(x) \le (f(x) + k(b,x))v(x). \tag{5.4.119}$$

Clearly, $v(x)$ is positive for all $x \in I$, hence it follows from (5.4.119)

$$\frac{v(x)Dv(x)}{v^2(x)} \le f(x) + k(b,x),$$

i.e.,

$$\frac{v(x)Dv(x)}{v^2(x)} \le f(x) + k(b,x) + \frac{(D_n v(x))(D_1 D_2 \cdots D_{n-1} v(x))}{v^2(x)},$$

whence

$$D_n\left(\frac{D_1 D_2 \cdots D_{n-1} v(x)}{v(x)}\right) \le f(x) + k(b,x).$$

Integrating with respect to x_n from a_n to x_n, we get

$$\frac{D_1D_2...D_{n-1}v(x)}{v(x)} \leq \int_{a_n}^{x_n} [f(x_1, \cdots, x_{n-1}, t_n) + k(b, x_1, \cdots, x_{n-1}, t_n)]dt_n$$

whence

$$\frac{v(x)D_1D_2\cdots D_{n-1}v(x)}{v^2(x)} \leq \int_{a_n}^{x_n} [f(x_1, \cdots, x_{n-1}, t_n) + k(b, x_1, \cdots, x_{n-1}, t_n)]dt_n \\ + \frac{(D_{n-1}v(x))(D_1D_2\cdots D_{n-2}v(x))}{v^2(x)},$$

i.e.,

$$D_{n-1}\left(\frac{D_1D_2\cdots D_{n-2}v(x)}{v(x)}\right) \leq \int_{a_n}^{x_n} \Big[f(x_1, \cdots, x_{n-1}, t_n) + k(b, x_1, \cdots, x_{n-1}, t_n)\Big]dt_n.$$

Integrating with respect to x_{n-1} from a_{n-1} to x_{n-1}, we obtain

$$\frac{D_1D_2\cdots D_{n-2}v(x)}{v(x)} \leq \int_{a_{n-1}}^{x_{n-1}} \int_{a_n}^{x_n} \Big[f(x_1, \cdots, x_{n-2}, t_{n-1}t_n) \\ + k(b, x_1, \cdots, x_{n-2}, t_{n-1}t_n)\Big]dt_n dt_{n-1}.$$

Continuing this process, we can obtain

$$\frac{D_1v(x)}{v(x)} \leq \int_{a_2}^{x_2} \cdots \int_{a_n}^{x_n} \Big[f(x_1, t_2, t_3, \cdots, t_n) + k(b, x_1, t_2, t_3, \cdots, t_n)\Big]dt_n \cdots dt_2.$$

Integrating with respect to x_1 from a_1 to x_1, we arrive at

$$\log \frac{v(x)}{v(a_1, x_2, \cdots, x_n)} \leq \int_a^x [f(t) + k(b, t)]dt$$

which implies

$$v(x) \leq c \exp\left(\int_a^x [f(t) + k(b, t)]dt\right). \tag{5.4.120}$$

Inserting (5.4.120) into $Dz(x) \leq f(x)v(x)$, we obtain

$$Dz(x) \leq cf(x) \exp\left(\int_a^x [f(t) + k(b, t)]dt\right). \tag{5.4.121}$$

Integrating (5.4.121) with respect to the x_n component from a_n to x_n, then with respect to a_{n-1} to x_{n-1}, and continuing until finally a_1 to x_1, and noting that $z(a_1, x_2, \cdots, x_n) = c$, we conclude

$$z(x) \le c\left[1 + \int_a^x f(t)\exp\left(\int_a^t [f(s) + k(b,s)]ds\right)dt\right],$$

which completes the proof of the first part.

(2) Define

$$z(x) = \int_a^x f(s)\left[u(s) + \int_a^s k(s,\tau)u(\tau)d\tau\right]ds. \tag{5.4.122}$$

Then from (5.4.116), $u(x) \le a(x) + z(x)$ and using this in (5.4.122), we get

$$\begin{aligned} z(x) &\le \int_a^x f(s)\left[a(s) + z(s) + \int_a^s k(s,\tau)[a(\tau) + z(\tau)]d\tau\right]ds, \\ &\le e(x) + \int_a^x f(s)\left[z(s) + \int_a^s k(s,\tau)z(\tau)d\tau\right]ds, \end{aligned} \tag{5.4.123}$$

where $e(x)$ is defined by (5.4.118).

Clearly, $e(x)$ is positive, continuous and non-decreasing for all $x \in I$. From (5.4.123) it follows that

$$\frac{z(x)}{e(x)} \le 1 + \int_a^x f(s)\left[z(s)e(s) + \int_a^s k(s,\tau)z(\tau)e(\tau)d\tau\right]ds.$$

Now applying the inequality in part (1), we can obtain

$$z(x) \le e(x)\left[1 + \int_a^x f(t)\exp\left(\int_a^t (f(s) + k(b,s))ds\right)dt\right]. \tag{5.4.124}$$

The desired inequality (5.4.117) follows from (5.4.124) and the fact that $u(x) \le a(x) + z(x)$. □

Next we establish n-independent-variable generalizations, due to Yeh [668], of the integral inequalities established by Bellman [75], Bihari [82] and Dhongade and Deo [182] for $n = 1$.

Let $J = [1, +\infty)$. First, Bellman's inequality [75] and the Dhongade-Deo inequality [182] can be unified and embodied in the following Theorem 5.4.22.

Theorem 5.4.22 (Yeh [668]) *Suppose that*

(a) $w(x),\ h(x) \in C(\mathbb{R}_+^n, \mathbb{R}_+)$,
(b) $f(x) \in C(\mathbb{R}_+^n, \mathbb{R}_0)$ *and non-decreasing in x,*
(c) $g(x) \in C(\mathbb{R}_+^n, J)$.

If for all $x \in \mathbb{R}^n_+$,

$$w(x) \le f(x) + g(x) \int_0^x h(s)w(s)\,ds, \tag{5.4.125}$$

then for all $x \in \mathbb{R}^n_+$,

$$w(x) \le f(x)g(x) \exp\left(\int_0^x h(s)g(s)ds\right). \tag{5.4.126}$$

Proof Since $f(x)$ is non-decreasing and $g(x) \ge 1$, we have for all $x \in \mathbb{R}^n_+$,

$$\begin{aligned} \frac{w(x)}{f(x)} &\le 1 + g(x) \int_0^x \frac{h(s)w(s)}{f(s)}\,ds \\ &\le g(x)\left[1 + \int_0^x \frac{h(s)w(s)}{f(s)}ds\right]. \end{aligned} \tag{5.4.127}$$

Define

$$r(x) = 1 + \int_0^x \frac{h(s)w(s)}{f(s)}ds. \tag{5.4.128}$$

Then

$$\begin{aligned} &r(x) = 1 \quad \text{on} \quad x_i = 0, \quad i = 1, \cdots, n; \\ &D_1 \cdots D_n r(x) = \frac{h(x)w(x)}{f(x)} \end{aligned}$$

and

$$\frac{w(x)}{f(x)} \le g(x)r(x). \tag{5.4.129}$$

Thus

$$\frac{D_1 \cdots D_n r(x)}{r(x)} \le h(x)g(x),$$

which implies

$$\frac{r(x)D_1 \cdots D_n r(x)}{r^2(x)} \le h(x)g(x) + \frac{D_n r(x)D_1 \cdots D_{n-1} r(x)}{r^2(x)}.$$

Thus

$$D_n\left(\frac{D_1\cdots D_{n-1}r(x)}{r(x)}\right)\le h(x)g(x).$$

Integrating both sides of the above inequality with respect to the component x_n of x from 0 to x_n, we obtain

$$\begin{aligned}\frac{D_1\cdots D_{n-1}r(x)}{r(x)}-\frac{D_1\cdots D_{n-1}r(x_1,\cdots,x_{n-1},0)}{r(x_1,\cdots,x_{n-1},0)}\\ \le\int_0^{x_n}h(x_1,\cdots,x_{n-1},t_n)g(x_1,\cdots,x_{n-1},t_n)\,dt_n\end{aligned}$$

which implies

$$D_{n-1}\left(\frac{D_1\cdots D_{n-2}r(x)}{r(x)}\right)\le\int_0^{x_n}h(x_1,\cdots,x_{n-1},t_n)g(x_1,\cdots,x_{n-1},t_n)\,dt_n. \tag{5.4.130}$$

Integrating both sides of (5.4.130) with respect to the component x_{n-1} of x from 0 to x_{n-1}, we get

$$\begin{aligned}\frac{D_1\cdots D_{n-2}r(x)}{r(x)}\le\int_0^{x_{n-1}}\int_0^{x_n}h(x_1,\ldots,x_{n-2},t_{n-1},t_n)\\ \times g(x_1,\ldots,x_{n-2},t_{n-1},t_n)\,dt_ndt_{n-1}.\end{aligned}$$

Continuing in this way, we obtain

$$\begin{aligned}D_2\left(\frac{D_1r(x)}{r(x)}\right)\le\int_0^{x_3}\cdots\int_0^{x_n}h(x_1,x_2,t_3,\ldots,t_n)\\ \times g(x_1,x_2,t_3,\ldots,t_n)\,dt_n\cdots dt_3.\end{aligned} \tag{5.4.131}$$

Integrating both sides of (5.4.131) with respect to the component x_2 of x from 0 to x_2, we have

$$\frac{D_1r(x)}{r(x)}\le\int_0^{x_2}\cdots\int_0^{x_n}h(x_1,t_2,\ldots,t_n)g(x_1,t_2,\ldots,t_n)\,dt_n\cdots dt_2.$$

Integrating both sides of the above inequality with respect to the component x_1 of x from 0 to x_1, we deduce for all $x\in\mathbb{R}^n_+$,

$$\log r(x)\le\int_0^x h(t)g(t)\,dt,$$

i.e.,

$$r(x) \leq \exp\left(\int_0^x h(t)g(t)\,dt\right),$$

which, together with (5.4.122), implies (5.4.126). □

Remark 5.4.5 If $n = 1$ and $g(x) = 1$, then Theorem 5.4.22 reduces to Bellman's Lemma [75], or Theorem 1.1.2.

Remark 5.4.6 If $n = 1$, then Theorem 5.4.22 reduces to Lemma 3.1 of Dhongade and Deo [182].

As an application of Theorem 5.4.22, we establish the following n-independent-variable generalization of the Dhongade-Deo inequality [182] (see, Theorem 1.2.9).

Theorem 5.4.23 (Yeh [668]) *Let*

(a) the functions $w(x)$ and $f(x)$ be defined as in Theorem 5.4.22,
(b) $g_i(x) \in C(\mathbb{R}_+^n, J)$ for $i = 1, 2, \cdots, m$,
(c) $h_i(x) \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ for $i = 1, 2, \cdots, m$.

If for all $x \in \mathbb{R}_+^n$,

$$w(x) \leq f(x) + \sum_{i=1}^{m} g_i(x) \int_0^x h_i(s)w(s)\,ds, \tag{5.4.132}$$

then for all $x \in \mathbb{R}_+^n$,

$$w(x) \leq E^m f(x), \tag{5.4.133}$$

where E^k is defined inductively as follows:

$$\begin{cases} E^0 f(x) = f(x), \\ E^k f(x) = f(x)E^{k-1}g_k(x)\exp\left(\displaystyle\int_0^x h_k(s)E^{k-1}g_k(s)\,ds\right), \ k = 1, 2, \cdots, m. \end{cases} \tag{5.4.134}$$

Proof The proof can be done by finite induction. Note that Theorem 5.4.22 reduces to Theorem 5.4.21 for $m = 1$ and hence is true. Now assume that (5.4.133) is true for given k, $k = 2, 3, \ldots, m-1$. Thus

$$\begin{aligned} w(x) &\leq E^k f(x) \\ &\leq E^k f(x)\left[1 + g_{k+1}(x)\int_0^x \frac{h_{k+1}(s)w(s)}{f(s)}\,ds\right]. \end{aligned} \tag{5.4.135}$$

It follows from (5.4.134) that $E^k f(x)/f(x) \geq 1$. From (5.4.135) and condition (b), we deduce for all $x \in \mathbb{R}^n_+$,

$$\frac{w(x)}{f(x)} \leq \frac{g_{k+1}(x)E^k f(x)}{f(x)}\left[1 + \int_0^x \frac{h_{k+1}(s)w(s)}{f(s)}\,ds\right]. \tag{5.4.136}$$

This inequality is of the form (5.4.127). Hence, as in Theorem 5.4.22, (5.4.136) takes form

$$w(x) \leq g_{k+1}E^k f(x)\exp\left(\int_0^x \frac{h_{k+1}(s)g_{k+1}(s)E^k f(s)}{f(s)}\,ds\right).$$

which, along with (5.4.134), implies

$$w(x) \leq f(x)E^k g_{k+1}(x)\exp\left(\int_0^x h_{k+1}(s)E^k g_{k+1}(s)\,ds\right) = E^{k+1}f(x).$$

This proves that (5.4.133) holds for $m = k+1$, and the proof is thus complete. □

Corollary 5.4.9 (Yeh [668]) *If $g_i(x) = 1$ for all $x \in \mathbb{R}^n_+$ and $i = 1, 2, \cdots, m$, in Theorem 5.4.23, then*

$$w(x) \leq E^m f(x),$$

where E^k is defined inductively as follows:

$$\begin{cases} E^0(x) = f(x), \\ E^k f(x) = E^{k-1}f(x)\exp\displaystyle\int_0^x E^{k-1}h_k(s)\,ds, \quad \textit{for all } x \in \mathbb{R}^n_+. \end{cases} \tag{5.4.137}$$

Proof The proof is similar to that of Theorem 5.4.24, so we omit here the details. □

Remark 5.4.7 For $n = 1$, Corollary 5.4.9 reduces to Corollary 1 of Dhongade and Deo [182], which is a linear generalization of Bellman's inequality [75] (i.e., Theorem 1.1.2) for m terms.

As an application of Theorem 5.4.23, we may consider the following example.

Example 5.4.1 Consider the following inequality for all $x \in \mathbb{R}^2_+$:

$$w(x_1, x_2) \leq x_1^2 + x_2^3 + \int_0^{x_1}\int_0^{x_2} w(s_1, s_2)\,ds_s ds_1$$
$$+ e^{x_1 x_2}\int_0^{x_1}\int_0^{x^2} e^{s_1 - 2s_1 s_2} w(s_1, s_2)\,ds_2 ds_1.$$

Here

$$\begin{cases} f(x_1, x_2) = x_1^2 + x_2^3, \quad g_1(x_1, x_2) = 1, \quad g_2(x_1, x_2) = e^{x_1 x_2}, \\ h_1(s_1, s_2) = 1, \qquad h_2(s_1, s_2) = e^{s_1 - 2s_1 s_2}. \end{cases}$$

Thus it follows from Theorem 5.4.23 that

$$w(x) \le E^2 f(x) = (x_1^2 + x_2^3) \exp(2x_1 x_2) \exp(x_2 e^{x_1} - x_2).$$

Now let S denote an open bounded set in $\mathbb{R}^n$. For $x = (x_1, \ldots, x_n)$ and $x^0 = (x_1^0, \ldots, x_n^0) \in S$, we shall first establish the following theorem, which is due to Yeh and Shin [673].

Theorem 5.4.24 (Yeh-Shin [673]) *Let $w(x)$, $f(x)$ and $g(x)$ be real-valued, non-negative and continuous functions on S, and $n(x)$ be a positive, non-decreasing continuous function on S. Suppose that the inequality holds for all x in S with $x \ge x^0$,*

$$w(x) \le n(x) + \int_{x^0}^{x} f(s) w(s)\, ds + \int_{x^0}^{x} \left(\int_{x^0}^{s} g(t) w(t)\, dt \right) ds. \tag{5.4.138}$$

Then for all x in S with $x \ge x^0$,

$$w(x) \le n(x) \left\{ 1 + \int_{x^0}^{x} f(s) \exp\left(\int_{x^0}^{s} (f(t) + g(t))\, dt \right) ds \right\}. \tag{5.4.139}$$

Proof Since $n(x)$ is positive and non-decreasing, we derive from (5.4.138),

$$\frac{w(x)}{n(x)} \le 1 + \int_{x^0}^{x} f(s) \frac{w(s)}{n(s)}\, ds + \int_{x^0}^{x} f(s) \left(\int_{x^0}^{s} g(t) \frac{w(t)}{n(t)}\, dt \right) ds. \tag{5.4.140}$$

Setting $w(x)/n(x) = p(x)$, we have

$$p(x) \le 1 + \int_{x^0}^{x} f(s) p(s)\, ds + \int_{x^0}^{x} f(s) \left(\int_{x^0}^{x} g(t) p(t)\, dt \right) ds. \tag{5.4.141}$$

Let

$$u(x) = 1 + \int_{x^0}^{x} f(s) p(s)\, ds + \int_{x^0}^{x} f(s) \left(\int_{x^0}^{s} g(t) p(t)\, dt \right) ds.$$

Then

$$p(x) \le u(x), \quad u(x) = 1 \qquad \text{on} \qquad x_1 = x_i^0, \qquad i = 1, 2, \cdots, n, \tag{5.4.142}$$

and

$$D_1 \cdots D_n u(x) = f(x) \left(p(x) + \int_{x^0}^{x} g(t)p(t)\,dt \right). \tag{5.4.143}$$

It follows from (5.4.142) and (5.4.143) that

$$D_1 \cdots D_n u(x) \le f(x) \left(u(x) + \int_{x^0}^{x} g(t)u(t)\,dt \right).$$

Let

$$v(x) = u(x) + \int_{x^0}^{x} g(t)u(t)\,dt.$$

Then

$$\begin{cases} u(x) \le v(x), \\ v(x) = u(x) \quad \text{on} \quad x_i = x_i^0, \quad i = 1, \cdots, n, \end{cases} \tag{5.4.144}$$

and

$$D_1 \cdots D_n v(x) = D_1 \cdots D_n u(x) + g(x)u(x).$$

Hence

$$D_1 \cdots D_n v(x) \le \Big(f(x) + g(x) \Big) v(x),$$

which implies

$$\frac{v(x) D_1 \cdots D_n v(x)}{v^2(x)} \le f(x) + g(x),$$

whence

$$\frac{v(x) D_1 \cdots D_n v(x)}{v^2(x)} \le f(x) + g(x) + \frac{(D_n v(x))(D_1 \cdots D_n v(x))}{v^2(x)},$$

i.e.,

$$D_n \left(\frac{D_1 \cdots D_{n-1} v(x)}{v(x)} \right) \le f(x) + g(x). \tag{5.4.145}$$

Integrating both sides of (5.4.145) with respect to the component x_n of x from x_n^0 to x_n, we get

$$\frac{D_1 \cdots D_{n-1} v(x)}{v(x)} \le \int_{x_n^0}^{x_n} \Big(f(x_1, \ldots, x_{n-1}, t_n) + g(x_1, \ldots, x_{n-1}, t_n) \Big)\, dt_n.$$

Therefore

$$\frac{v(x)(D_1 \cdots D_{n-1} v(x))}{v^2(x)} \le \int_{x_n^0}^{x_n} \Big(f(x_1, \ldots, x_{n-1}, t_n) + g(x_1, \ldots, x_{n-1}, t_n) \Big)\, dt_n + \frac{(D_{n-1} v(x))(D_1 \cdots D_{n-2} v(x))}{v^2(x)},$$

i.e.,

$$D_{n-1} \left(\frac{D_1 \cdots D_{n-2} v(x)}{v(x)} \right) \le \int_{x_n^0}^{x_n} \left(f(x_1, \cdots, x_{n-1}, t_n) + g(x_1, \ldots, x_{n-1}, t_n) \right)\, dt_n. \tag{5.4.146}$$

Integrating both sides of (5.4.146) with respect to the component x_{n-1} of x from x_{n-1}^0 to x_{n-1}, we obtain

$$\frac{D_1 \cdots D_{n-2} v(x)}{v(x)} \le \int_{x_{n-1}^0}^{x_{n-1}} \int_{x_n^0}^{x^n} (f(x_1, \cdots, x_{n-1}, t_{n-1}, t_n) + g(x_1, \cdots, x_{n-1}, t_{n-1}, t_n))\, dt_n\, dt_{n-1}.$$

Continuing in this way, we derive

$$\frac{D_1 D_2 v(x)}{v(x)} \le \int_{x_3^0}^{x_3} \cdots \int_{x_n^0}^{x_n} (f(x_1, x_2, t_3, \cdots, t_n) + g(x_1, x_2, t_3, \cdots, t_n))\, dt_n \cdots dt_3. \tag{5.4.147}$$

Thus from (5.4.147) it follows

$$D_2 \left(\frac{D_1 v(x)}{v(x)} \right) \le \int_{x_3^0}^{x_3} \cdots \int_{x_n^0}^{x_n} \Big(f(x_1, x_2, t_3, \cdots, t_n) + g(x_1, x_2, t_3, \cdots, t_n) \Big)\, dt_n \cdots dt_3. \tag{5.4.148}$$

Integrating both sides of (5.4.148) with respect to the component x_2 of x from x_2^0 to x_2, we have

$$\frac{D_1 v(x)}{v(x)} \le \int_{x_2^0}^{x_2} \cdots \int_{x_n^0}^{x_n} \Big(f(x_1, t_2, \cdots, t_n) + g(x_1, t_2, \cdots, t_n) \Big)\, dt_n \cdots dt_2.$$

Integrating both sides of the above inequality with respect to the component x_1 of x from x_1^0 to x_1, we conclude

$$\log \frac{v(x)}{v(x_1^0, x_2, \cdots, x_n)} \leq \int_{x^0}^{x} \Big(f(t) + g(t)\Big)\, dt,$$

which implies

$$v(x) \leq \exp\left(\int_{x^0}^{x} (f(t) + g(t))\, dt\right).$$

Hence

$$D_1 \cdots D_n u(x) \leq f(x)v(x) \leq f(x) \exp\left(\int_{x^0}^{x} (f(t) + g(t))\, dt\right).$$

Integrating first with respect to the component x_n of x from x_n^0 to x_n, then with respect to the component x_{n-1} of x from x_{n-1}^0 to x_{n-1}, and continuing in this way and finally integrating with respect to the component x_1 of x from x_1^0 to x_1, we finally obtain

$$u(x) \leq 1 + \int_{x^0}^{x} f(s) \exp\left(\int_{x^0}^{s} (f(t) + g(t))\, dt\right).$$

Noting that

$$w(x)/n(x) = p(x) \leq u(x),$$

we conclude

$$w(x) \leq n(x) \left\{1 + \int_{x^0}^{x} f(s) \exp\left(\int_{x^0}^{s} (f(t) + g(t))\, dt\right) ds\right\},$$

which completes the proof. □

Remark 5.4.8 The integral inequality obtained in Theorem 5.4.24 is a generalization of Gronwall and Bellman's inequality [75], and Pachpatte's inequality [454], to several variables.

The following result is due to Shih and Yeh [588], which generalizes Theorem 5.4.24.

Theorem 5.4.25 (Shih-Yeh [588]) *Let $w(x), f(x)$ and $g(x)$ be real-valued, non-negative and continuous functions on S, and $n(x)$ be a positive, non-decreasing continuous function on S. Suppose that the inequality holds for all x in S with $x \geq x^0$,*

$$w(x) \leq n(x) + \int_{x^0}^{x} f(s)w(s)ds + \int_{x^0}^{x} f(s) \left(\int_{x^0}^{s} g(t)w(t)dt\right) ds. \qquad (5.4.149)$$

Then for all x in S with $x \geq x^0$,

$$w(x) \leq n(x) \left\{ 1 + \int_{x^0}^{x} f(s) \exp\left(\int_{x^0}^{s} (f(t) + g(t))dt \right) ds \right\}.$$

Proof Since $n(x)$ is positive and non-decreasing, it follows from (5.4.149),

$$\frac{w(x)}{n(x)} \leq 1 + \int_{x^0}^{x} f(s) \frac{w(s)}{n(s)} ds + \int_{x^0}^{x} f(s) \left(\int_{x^0}^{s} g(t) \frac{w(t)}{n(t)} dt \right) ds. \quad (5.4.150)$$

Setting $w(x)/n(x) = p(x)$, we arrive at

$$p(x) \leq 1 + \int_{x^0}^{x} f(s)p(s)ds + \int_{x^0}^{x} f(s) \left(\int_{x^0}^{s} g(t)p(t)dt \right) ds. \quad (5.4.151)$$

Let

$$u(x) = 1 + \int_{x^0}^{x} f(s)p(s)ds + \int_{x^0}^{x} f(s) \left(\int_{x^0}^{x} g(t)p(t)dt \right) ds.$$

Then

$$p(x) \leq u(x), \ u(x) = 1 \ on \ x_i = x_i^0, \ i = 1, 2, \cdots, n, \quad (5.4.152)$$

and

$$D_1 \cdots D_n u(x) = f(x) \left(p(x) + \int_{x^0}^{x} g(t)p(t)dt \right). \quad (5.4.153)$$

It follows from (5.4.152) and (5.4.153) that

$$D_1 \cdots D_n u(x) \leq f(x) \left(u(x) + \int_{x^0}^{x} g(t)u(t)dt \right).$$

Let

$$v(x) = u(x) + \int_{x^0}^{x} g(t)u(t)dt.$$

Then

$$\begin{cases} u(x) \leq v(x), \\ v(x) = u(x) \ on \ x_i = x_i^0, \ i = 1, \cdots, n, \end{cases} \quad (5.4.154)$$

and

$$D_1 \cdots D_n v(x) = D_1 \cdots D_n u(x) + g(x)u(x).$$

Hence

$$D_1 \cdots D_n v(x) \le (f(x) + g(x))v(x),$$

which implies

$$\frac{v(x) D_1 \cdots D_n v(x)}{v^2(x)} \le f(x) + g(x).$$

Thus

$$\frac{v(x) D_1 \cdots D_n v(x)}{v^2(x)} \le f(x) + g(x) + \frac{(D_n v(x))(D_1 \cdots D_{n-1} v(x))}{v^2(x)},$$

i.e.,

$$D_n \left(\frac{D_1 \cdots D_{n-1} v(x)}{v(x)} \right) \le f(x) + g(x). \tag{5.4.155}$$

Integrating both sides of (5.4.155) with respect to the component x_n of x from x_n^0 to x_n, we get

$$\frac{D_1 \cdots D_{n-1} v(x)}{v(x)} \le \int_{x_n^0}^{x_n} \Big(f(x_1, \cdots, x_{n-1}, t_n) + g(x_1, \cdots, x_{n-1}, t_n) \Big) dt_n.$$

Therefore

$$\begin{aligned} \frac{v(x)(D_1 \cdots D_{n-1} v(x))}{v^2(x)} \le & \int_{x_n^0}^{x_n} \Big(f(x_1, \cdots, x_{n-1}, t_n) + g(x_1, \cdots, x_{n-1}, t_n) \Big) dt_n \\ & + \frac{(D_{n-1} v(x))(D_1 \cdots D_{n-2} v(x))}{v^2(x)}, \end{aligned}$$

i.e.,

$$D_{n-1} \left(\frac{D_1 \cdots D_{n-2} v(x)}{v(x)} \right) \le \int_{x_n^0}^{x_n} \Big(f(x_1, \cdots, x_{n-1}, t_n) + g(x_1, \cdots, x_{n-1}, t_n) \Big) dt_n. \tag{5.4.156}$$

Integrating both sides of (5.4.156) with respect to the component x_{n-1} of x from x_{n-1}^0 to x_{n-1}, we obtain

$$\frac{D_1 \cdots D_{n-2} v(x)}{v(x)} \leq \int_{x_{n-1}^0}^{x_{n-1}} \int_{x_n^0}^{x_n} \Big(f(x_1, \cdots, x_{n-2}, t_{n-1}, t_n) + g(x_1, \cdots, x_{n-2}, t_{n-2}, t_n) \Big) dt_n dt_{n-1}.$$

Continuing in this way, we deduce

$$\frac{D_1 D_2 v(x)}{v(x)} \leq \int_{x_3^0}^{x_3} \cdots \int_{x_n^0}^{x_n} \Big(f(x_1, x_2, t_3, \cdots, t_n) + g(x_1, x_2, t_3, \cdots, t_n) \Big) dt_n \cdots dt_3. \tag{5.4.157}$$

From (5.4.157) it follows that

$$D_2 \left(\frac{D_1 v(x)}{v(x)} \right) \leq \int_{x_3^0}^{x_3} \cdots \int_{x_n^0}^{x_n} \Big(f(x_1, x_2, t_3, \cdots, t_n) + g(x_1, x_2, t_3, \cdots, t_n) \Big) dt_n \cdots dt_3. \tag{5.4.158}$$

Integrating both sides of (5.4.158) with respect to the component x_2 of x from x_2^0 to x_2, we arrive at

$$\frac{D_1 v(t)}{v(x)} \leq \int_{x_2^0}^{x_2} \cdots \int_{x_n^0}^{x_n} \Big(f(x_1, t_2, \cdots, t_n) + g(x_1, t_2, \cdots, t_n) \Big) dt_n \cdots dt_2.$$

Integrating both sides of the above inequality with respect to the component x_1 of x from x_1^0 to x_1, we derive

$$\log \frac{v(x)}{v(x_1^0, x_2, \cdots, x_n)} \leq \int_{x^0}^{x} (f(t) + g(t)) \, dt,$$

which implies

$$v(x) \leq \exp \left(\int_{x^0}^{x} (f(t) + g(t)) dt \right).$$

Hence

$$D_1 \cdots D_n u(x) \leq f(x) v(x) \leq f(x) \exp \left(\int_{x^0}^{x} (f(t) + g(t)) dt \right).$$

Integrating first with respect to the component x_n of x from x_n^0 to x_n, then with respect to the component x_{n-1} of x from x_{n-1}^0 to x_{n-1}, and continuing in this way and finally

integrating with respect to the component x_1 of x from x_1^0 to x_1, we finally arrive at

$$u(x) \leq 1 + \int_{x^0}^{x} f(s) \exp\left(\int_{x^0}^{s} (f(t) + g(t))dt\right).$$

Since

$$w(x)/n(x) = p(x) \leq u(x),$$

we conclude

$$w(x) \leq n(x)\left\{1 + \int_{x^0}^{x} \exp\left(\int_{x^0}^{s} (f(t) + g(t))dt\right) ds\right\},$$

which completes the proof. □

Remark 5.4.9 The integral inequality obtained in Theorem 5.4.25 is a generalization of Gronwall-Bellman's inequality [75], and Pachpatte's inequality [181] to several variables.

Now let $I = [0, h)$ with $0 < h \leq +\infty$, and $C(N, M)$ be the class of all real-valued continuous functions defined on the set N and with range in the set M, and we set $S = C(I, \mathbb{R}_+)$, and for the convenience of the statement, we contract that all of the empty sums and empty produces are equal to zero and one, respectively, such as

$$\sum_{i=1}^{0} f_i = 0, \quad \prod_{i=0}^{0} f_i = 1.$$

The next result is due to Yang [657].

Theorem 5.4.26 (Yang [657]) *Let $x(t)$ and $p(t)$ belong to S with $p(t)$ positive and non-decreasing on I. Let $f_i(t, s) \in C(I \times I, \mathbb{R}_+)$ be non-decreasing in t when $s \in I$ fixed. Suppose that the inequality holds for all $t \in I$,*

$$x(t) \leq p(t) + \int_0^t f_1(t, t_1) \int_0^{t_1} f_2(t_1, t_2) \cdots \int_0^{t_{n-1}} f_n(t_{n-1}, t_n) x(t_n) dt_n \cdots dt_1. \tag{5.4.159}$$

Then for all $t \in I$,

$$x(t) \leq p(t)U(t), \tag{5.4.160}$$

where $U(t) = V_n(t,t)$ and $V_n(T,t)$ is defined successively by

$$\begin{cases} V_1(T,t) = \exp\left(\int_0^t \sum_{j=1}^{n} f_j(T,s)ds\right), \\ V_k(T,t) = F_{n-k+1}(T,t)\left\{1 + \int_0^t f_{n-k+1}\frac{V_{k-1}(T,s)}{F_{n-k+1}(T,s)}ds\right\}, \end{cases} \tag{5.4.161}$$

where T and $t \in I, k = 2,3,\cdots,n$ and

$$F_i(T,t) = \exp\left(\int_0^t \left[\sum_{j=1}^{i-1} f_j(T,s) - f_i(T,s)\right] ds\right), \; i = 1,2,\cdots,n-1. \tag{5.4.162}$$

Proof Obviously, we have $U(0) = V_n(0,0) = 1$ follows from (5.4.161) and (5.4.162), and hence the estimate in (5.4.160) trivially holds when $t = 0$. We define the following non-negative functionals on S by

$$J_k(c,t)(y) = f_k(c,t)\int_0^t f_{k+1}(c,t_{k+1})\int_0^{t_{k+1}} f_{k+2}(c,t_{k+2})\cdots\int_0^{t_{n-1}} f_n(c,t_n)y(t_n)$$
$$\times dt_n dt_{n-1}\cdots dt_{k+1}, \quad k = 1,2,\cdots,n-1$$

and

$$J_n(c,t)(y) = f_n(c,t)y(t), \tag{5.4.163}$$

here $y = y(t) \in S$, c is a constant and $t \in I$.

We note that here $J_i(c,t)(y), i = 1,2,\cdots,n$, are monotonic and non-decreasing in $y \in S$, that is, if $y_1, y_2 \in S$ and $y_1(t) \le y_2(t)$ on I, then for all $t \in I$,

$$J_i(c,t)(y_1) \le J_i(c,t)(y_2).$$

Now, fixing an arbitrary value T from $(0,h)$, then we derive from (5.4.159), for all $t \in [0,T]$,

$$x(t) \le p(T) + \int_0^t J_1(T,t_1)(x)dt_1.$$

If we set

$$\begin{cases} m_i(t) = P(T) + \int_0^t J_1(T,t_1)(x)dt_1, \\ m_k(t) = m_{k-1}(t) + \int_0^t J_k(T,t_k)(m_{k-1})dt, \quad k = 2,3,\cdots,n, \end{cases} \tag{5.4.164}$$

then we have the relations

$$\begin{cases} m_n(t) \geq m_{n-1}(t) \geq \cdots \geq m_1(t) \geq x(t), \quad t \in [0,T], & (5.4.165) \\ m_n(0) = m_{n-1}(0) = \cdots = m_1(0) = p(T) > 0. & (5.4.166) \end{cases}$$

We notice that the following differential inequalities for $m_i(t)$ hold:

$$m_i'(t) + f_i(T,t)m_i(t) \leq \sum_{j=1}^{i-1} f_j(T,t)m_i(t) + f_i(T,t)m_{i+1}(t), \ t \in [0,T], i = 1,2,\cdots,n-1, \tag{5.4.167}$$

which can be proved it by induction. First, using (5.4.165) and in view of the monotonicity of $J_k(T,t)(y)$, we obtain from the first equality in (5.4.164) that for all $t \in [0,T]$,

$$m_1'(t) \leq J_1(T,t)(m_1),$$

and adding $f_1(T,t)m_1(t)$ to both sides of the above inequality, by (5.4.164), we derive for all $t \in [0,T]$,

$$\begin{aligned} m_1'(t) + f_1(T,t)m_1(t) &\leq f_1(T,t)m_1(t) + J_1(T,t)(m_1) \\ &= f_1(T,t)\left[m_1(t) + \int_0^t J_2(T,t_2)(m_1)dt_2\right] \\ &= f_1(T,t)m_2(t). \end{aligned}$$

The above inequality indicates that (5.4.167) holds when $i = 1$. Now we suppose (5.4.167) holds for $i = k$, where $1 \leq k \leq n-2$. Then by differentiating, we obtain from (5.4.164) for all $t \in [0,T]$,

$$m_{k+1}'(t) = m_k' + J_{k+1}(T,t)(m_k).$$

Noting that (5.4.167) holds for $i = k, f_k(T,t)m_k(t) \geq 0$, and using (5.4.165), we obtain for all $t \in [0,T]$,

$$\begin{aligned} m_{k+1}'(t) &\leq \sum_{j=1}^{k-1} f_j(T,t)m_k(t) + f_k(T,t)m_{k+1}(t) + J_{k+1}(T,t)(m_k) \\ &\leq \sum_{j=1}^{k} f_j(T,t)m_{k+1}(t) + J_{k+1}(T,t)(m_{k+1}), \end{aligned}$$

since $J_{k+1}(T,t)(y)$ is non-decreasing in $y \in S$.

Adding $f_{k+1}(t,t)m_{k+1}(t)$ to both sides of the above inequality, we finally conclude for all $t \in [0,T]$,

$$\begin{aligned}
&m'_{k+1}(t) + f_{k+1}(t,t)m_{k+1}(t)\\
&\leq \sum_{j=1}^{k} f_j(T,t)m_{k+1}(t) + f_{k+1}(T,t)m_{k+1}(t) + J_{k+1}(T,t)(m_{k+1})\\
&= \sum_{j=1}^{k} f_j(T,t)m_{k+1}(t) + f_{k+1}(T,t)m_{k+2}(t),
\end{aligned}$$

which proves (5.4.167).

We now apply the relations (5.4.164) and (5.4.167) to derive the bounds on $m_i(t)$, here $i = 1, 2, \cdots, n$. We shall prove that the following estimates are true, for all $t \in [0,T]$,

$$m_{n-k}(t) \leq p(T)V_{k+1}(T,t), \quad k = 0, 1, \cdots, n-1 \tag{5.4.168}$$

where $V_{k+1}(T,t)$ are given by (5.4.161). First, we consider the last equality in (5.4.164),

$$m_n(t) = m_{n-1}(t) + \int_0^t f_n(T,t_n)m_{n-1}(t_n)dt_n, \;\; for \;\; all \;\; t \in [0,T],$$

Differentiating the above equality and using (5.4.164) and (5.4.167), in view of $f_{n-1}(T,t)$ and $m_{n-1}(t)$ being non-negative, we obtain for all $t \in [0,T]$,

$$\begin{aligned}
m'_n(t) &= m'_{n-1}(t) + f_n(T,t)m_{n-1}(t)\\
&\leq \sum_{j=1}^{n-2} f_j(T,t)m_{n-1}(T,t) + f_{n-1}(T,t)m_n(t) + f_n(T,t)m_{n-1}\\
&\leq \sum_{j=1}^{n} f_j(T,t)m_n(t).
\end{aligned}$$

Dividing both sides of the above inequality by $m_n(t) > 0$, and then integrating from 0 to t, using $m_n(0) = p(T)$, we obtain that for all $t \in [0,T]$,

$$m_n(t) \leq p(T)\exp\left(\int_0^t \sum_{j=1}^{n} f_j(T,s)ds\right) = p(T)V_1(T,t).$$

Here $V_1(T)$ is given by (5.4.161). Next, substituting this bound for $m_n(t)$ in (5.4.167) with $i = n-1$, we get for all $t \in [0, T]$,

$$m'_{n-1}(t) + \left[f_{n-1}(T,t) - \sum_{j=1}^{n-2} f_j(T,t) \right] m_{n-1}(t) \le f_{n-1}(T,t)p(T)V_1(T,t).$$

Multiplying first by $\exp\left(\int_0^t [f_{n-1}(T,s) - \sum_{j=1}^{n-2} f_j(T,s)]ds\right)$ both sides of the above inequality, and then integrating from 0 to t and using (5.4.166), we derive for all $t \in [0, T]$,

$$\begin{aligned} m_{n-1}(t) &\le p(T)F_{n-1}(T,t) \left\{ 1 + \int_0^t f_{n-1}(T,s) \frac{V_1(T,s)}{F_{n-1}(T,t)} ds \right\} \\ &= p(T)V_2(T,t), \end{aligned} \tag{5.4.169}$$

where $F_{n-1}(T,t)$ and $V_2(T,t)$ are defined by (5.4.161) and (5.4.162), respectively.

Suppose that the inequality (5.4.168) is proved for $1 \le k \le n-2$, then from (5.4.167), we derive

$$\begin{aligned} m'_{n-k-1}(t) + \left[f_{n-k-1}(T,t) - \sum_{j=1}^{n-k-2} f_j(T,t)(t) \right] m_{m-k-1}(t) \\ \le f_{n-k-1}(T,t)p(T)V_{k+1}(T,t). \end{aligned}$$

Multiplying by $\exp\left(\int_0^t [f_{n-k-1}(T,s) - \sum_{j=1}^{n-k-2} f_j(T,s)]ds\right)$ both sides of the above inequality and integrating from 0 to t, and using (5.4.166), we obtain for all $t \in [0, T]$,

$$\begin{aligned} m_{n-k-1}(t) &\le p(T)F_{n-k-1}(T,t) \left[1 + \int_0^t f_{n-k-1}(T,s) \frac{V_{k+1}(T,s)}{F_{n-k-1}(T,s)} ds \right] \\ &= p(T)V_{k+2}(T,t), \end{aligned}$$

where $F_{n-k-1}(T,t)$ and $V_{k+2}(T,t)$ are given by (5.4.161) and (5.4.162), respectively. Hence the inequality (5.4.168) follows immediately. Finally, we see from (5.4.165) that for all $t \in [0, T]$,

$$x(t) \le m_1(t) \le p(T)V_n(T,t).$$

Letting $t = T$ in the above inequality and in view of $U(t) = V_n(t,t)$, we finally obtain

$$x(T) \le p(T)U(T)$$

since $T \in (0, h)$ is arbitrary, thus the proof is complete. □

Corollary 5.4.10 (Yang [657]) *The case when $n = 1$ in Theorem 5.4.26 is a simple but useful generalization of an integral inequality due to Bellman and Cooke [75].*

Corollary 5.4.11 (Yang [657]) *Suppose the following integral inequality holds for all $t \in I$,*

$$x(t) \le p(t) + \int_0^t f_1(t, t_1) \int_0^{t_1} f_2(t_1, s)x(s)dsdt_1$$

where $x(t), p(t)$ and $f_i(t, s)$, $(i = 1, 2)$, are the same as defined in Theorem 5.4.27. Then for all $t \in I$,

$$x(t) \le p(t) \exp\left(-\int_0^t f_1(t, s)ds\right) \times \left[1 + \int_0^t f_1(t, s) \left\{\exp \int_0^s [2f_1(s, p) + f_2(s, q)]dq\right\} ds\right].$$

We shall give some further extensions of Theorem 5.4.26 which are also due to Yang [657]. To simplify the statement, in the sequel, we shall introduce several linear integral operators defined on class S.

We first define the operators $I_i(t)(y)$ by

$$I_i(t)(y) = \int_0^t f_{i1}(t, t_1) \int_0^{t_1} f_{i2}(t_1, t_2) \cdots \int_0^{t_{n-1}} f_{in}(t_{n-1}, t_n)x(t_n)dt_n \cdots dt_1,$$

where $y \in S, t \in I, i = 1, 2, \cdots, m$.

Theorem 5.4.27 (Yang [657]) *Let the functions $x(t)$ and $p(t)$ be the same as defined in Theorem 5.4.26, and let $f_{ij}(t, s) \in C(I \times I, \mathbb{R}_+), i = 1, 2, \cdots, m, j = 1, 2, \cdots, n$, be non-decreasing in t for $s \in I$ fixed. Suppose that the inequality holds for all $t \in I$,*

$$x(t) \le p(t) + \sum_{i=1}^{m} I_i(t)(x). \tag{5.4.170}$$

Then for all $t \in I$,

$$x(t) \le p(t) \prod_{i=0}^{m} W_i(t), \tag{5.4.171}$$

where $W_i(t) = V_{in}(t,t)$ *and* $V_{1n}(q,t), V_{2n}(q,t), \cdots, V_{mn}(q,t)$ *are defined successively by*

$$\begin{cases} V_{i1}(q,t) = \exp\left(\int_0^t \sum_{j=1}^{n} g_{ij}(q,s)ds\right), \\ V_{ik}(q,t) = G_{i,n-k+1}(q,t)\left[1 + \int_0^t g_{i,n-k+1}(q,s)\dfrac{V_{i,k-1}(q,s)}{G_{i,n-k+1}(q,s)}ds\right], \end{cases} \tag{5.4.172}$$

where

$$g_{i1}(q,t) = \left(\prod_{k=1}^{i-1} W_k(t)\right) f_{i1}(q,t), \quad g_{ij}(q,t) = f_{ij}(q,t), j = 2, \cdots, n, \ i = 1, 2, \cdots, m,$$

and

$$G_{ik}(q,t) = \exp\left(\int_0^t \left[\sum_{j=1}^{k-1} g_{ij}(q,s) - g_{ik}(q,s)\right] ds\right), \ k = 1, 2, \cdots, n-1, \ i = 1, 2, \cdots, m. \tag{5.4.173}$$

Proof Obviously, we may rewrite the inequality (5.4.170) as

$$x(t) \le p_1(t) + I_1(t)(x), \quad \text{for all } t \in I, \tag{5.4.174}$$

where

$$p_1(t) = p(t) + \sum_{i=2}^{m} I_i(t)(x).$$

It is easy to verify that the operators $I_i(t)(x)$ are non-negative, continuous and non-decreasing in t, and thus the function $p_1(t)$ satisfies the same conditions on $p(t)$. Now an application of Theorem 5.4.26 to (5.4.174) yields for all $t \in I$,

$$x(t) \le p_1(t) W_1(t), \tag{5.4.175}$$

where $W_1(t) = V_{1n}(q,t)$ and $V_{1n}(q,t)$ is given by (5.4.172) with $i = 1$. The above inequality can be rewritten as for all $t \in I$,

$$x(t) \le p_2(t) + I_2^*(t)(x), \tag{5.4.176}$$

where

$$I_2^*(t)(x) = W_1(t)I_2(t)(x), \quad p_2(t) = W_1(t)\left[p(t) + \sum_{i=3}^{m} I_i(t)(x)\right].$$

Clearly, here $I_2^*(t)$ is the same type of integral functional as $I_2(t)(x)$ except the function $f_{21}(t,s)$ is now replaced by $W_1(t)f_{21}(t,s)$. Applying Theorem 5.4.26 to the above inequality (5.4.176) again, we then conclude for all $t \in I$,

$$x(t) \le \left[p(t) + \sum_{i=3}^{m} I_i(t)(x)\right] W_1(t)W_2(t),$$

where $W_2(t) = V_{2n}(t,t)$ and $V_{2n}(q,t)$ is given by (5.4.172) with $i = 2$. Continuing in this way and combining an inductional argument, we can easily prove the estimate (5.4.171). Since this argument is obvious, we omit the details. □

We note that there is an interesting result, which can be seen as an extension to n-times integral case of an inequality of Willett [647], that can be derived from Theorem 5.4.27 by setting $f_{ij}(t) \equiv 1$ and $f_{in}(t,s) = f_i(t,s)$, here $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n-1$.

Next, we define the integral operators $I^{(j)}(t)(y)$ on S by

$$I^{(j)}(t)(y) = \int_0^t f_1^{(j)}(t,t_1) \int_0^{t_1} f_2^{(j)}(t_1,t_2) \cdots \int_0^{t_j-1} f_j^{(j)}(t_{j-1},t_j) y(t_j) dt_j \cdots dt_1,$$

where $y \in S$, $t \in I$ and $j = 1, 2, \cdots, n$.

Theorem 5.4.28 (Yang [657]) *Let $x(t)$ and $p(t)$ belong to class S with $p(t)$ positive and non-decreasing on I, and let $f_k^{(j)}(t,s) \in C(I \times I, \mathbb{R}_+)$, be non-decreasing in t for $s \in I$ fixed, here $k = 1, 2, \cdots, j$ and $j = 1, 2, \cdots, n$. Suppose that the inequality for all $t \in I$,*

$$x(t) \le p(t) + \sum_{j=1}^{m} I^{(j)}(t)(x), \textit{holds}. \tag{5.4.177}$$

Then for allt $\in I$,

$$x(t) \le p(t) \prod_{j=1}^{m} U^{(j)}(t), \tag{5.4.178}$$

where $U^{(j)}(t) = V_j^{(j)}(t,t)$ *and the functions* $V_1^{(1)}(q,t)$, $V_1^{(2)}(q,t), V_2^{(2)}(q,t), \cdots$, $V_1^{(n)}(q,t), \cdots, V_n^{(n)}(q,t)$ *are defined successively by*

$$\begin{cases} V_1^{(j)}(q,t) = \exp\left(\int_0^t \sum_{k=1}^{j} h_k^{(j)}(q,s)ds\right), \\ V_k^{(j)}(q,t) = H_{j-k+1}^{(j)}(q,t)\left[1 + \int_0^t h_{j-k+1}^{(j)}(q,s)\dfrac{V_{k-1}^{(j)}(q,s)}{H_{j-k+1}^{(j)}(q,s)}ds\right], \ k = 2,3,\cdots,j, \end{cases} \tag{5.4.179}$$

and

$$\begin{cases} H_r^{(j)}(q,t) = \exp(\int_0^t [\sum_{k=1}^{r-1} h_k^{(j)}(q,s) - h_r^{(j)}(q,s)]ds), r = 1,2,\cdots,j-1, & (5.4.180) \\ h_1^{(j)}(q,t) = (\prod_{k=1}^{j-1} U^{(k)}(q))f_1^{(j)}(q,t), h_k^{(j)}(q,t) = f_k^{(j)}(q,t), k = 2,3,\cdots,j. & (5.4.181) \end{cases}$$

Proof This result is a special case of Theorem 5.4.29 below. If we set $i = 1$ and $r_j = 1$, here $j = 1,2,\cdots,n$, then the desired bound for $x(t)$ in (5.4.178) follows immediately. □

The following corollaries are sometimes convenient for application.

Corollary 5.4.12 (Yang [657]) *Suppose that the inequality holds for all* $t \in I$,

$$x(t) \le p(t) + \int_0^t f_1^{(1)}(t,s)x(s)ds + \int_0^t f_1^{(2)}(t,s)\int_0^s f_2^{(2)}(s,u)x(u)duds$$

where $x(t), p(t)$ *and* $f_i^{(j)}(t,s)$ *are the same as in Theorem 5.4.28. Then for all* $t \in I$,

$$x(t) \le p(t)U^{(1)}(t)U^{(2)}(t),$$

where

$$\begin{cases} U^{(1)}(t) = \exp\left(\int_0^t f_1^{(1)}(t,s)x(s)ds\right), \\ U^{(2)}(t) = V^{(2)}(t,t) \\ \qquad = \exp\left(\int_0^t -f_1^{(2)}(t,s)\left[\exp(\int_0^t f_1^{(1)}(t,r)dr)\right]ds\right)\cdot\Big\{1 + \int_0^t f_1^{(2)}(t,s) \\ \qquad\quad \times \exp\int_0^t f_1^{(1)}(t,r)dr\left(\exp\left(\int_0^s f_1^{(2)}(t,u)\exp(\int_0^t f_1^{(1)}(t,r)dr)du\right)\right) \\ \qquad\quad \times\left(\exp\int_0^s\left[f_1^{(2)}(t,r)\exp(\int_0^t f_1^{(1)}(t,u)du) + f_2^{(2)}(t,r)\right]dr\right)ds\Big\}. \end{cases}$$

Corollary 5.4.13 (Yang [664]) *Let $u(t)$, $k(t)$, and $f_i(t)$ $(i = 1, 2, 3)$ be real-valued, non-negative and continuous functions defined on $I := [0, h)$, here $0 < h \leq +\infty$. Let further $k(t)$ be non-decreasing on I. If the linear integral inequality holds, for all $t \in I$,*

$$u(t) \leq k(t) + \int_0^t f_1(s)u(s)ds + \int_0^t f_2(s)\left(\int_0^s f_3(m)u(m)dm\right)ds, \quad (5.4.182)$$

then for all $t \in I$,

$$u(t) \leq k(t)r(t)q(t)\exp\left(-r(t)\int_0^t f_2(s)ds\right), \quad (5.4.183)$$

where

$$\begin{cases} r(t) := \exp\left(\int_0^t f_1(s)ds\right), \\ q(t) := 1 + r(t)\int_0^t f_2(s)\exp\left\{\int_0^s [2r(s)f_2(m) + f_3(m)]dm\right\} ds. \end{cases}$$

We define the linear integral operators $I_i^{(j)}(t)(y)$ on S by

$$I_i^{(j)}(t)(y) = \int_0^t f_{i1}^{(j)}(t, t_1)\int_0^{t_1} f_2^{(j)}(t_1, t_2)\cdots\int_0^{t_{j-1}} f_{ij}^{(j)}(t_{j-1}, t_j)y(t_j)dt_j\cdots dt_1,$$

where y belongs to $S, j = 1, 2, \cdots, n$ and $i = 1, 2, \cdots, r_j$, here $r_j \geq 0$ are some integers.

Theorem 5.4.29 (Yang [657]) *Let the functions $x(t)$ and $p(t)$ belong to the class S, and $p(x)$ is positive and non-decreasing on the interval I. Let the functions $f_{ik}^{(j)}(t, s) \in C(I \times I, \mathbb{R}_+)$ and be non-decreasing in t for $s \in I$ fixed, here $j = 1, 2, \cdots, n, k = 1, 2, \cdots, j$ and $i = 1, 2, \cdots, r_j$. Suppose that the inequality holds for all $t \in I$,*

$$x(t) \leq p(t) + \sum_{j=1}^{n}\sum_{i=1}^{r_j} I_i^{(j)}(t)(x). \quad (5.4.184)$$

Then for all $t \in I$,

$$x(t) \leq p(t)\prod_{j=1}^{n}\left(\prod_{i=1}^{r_j} W_i^{(j)}(t)\right), \quad (5.4.185)$$

where $W_i^{(j)}(t) = V_{ij}^{(j)}(t,t)$ *and the functions* $V_{i1}^{(j)}(q,t), V_{i2}^{(j)}(q,t), \cdots, V_{ij}^{(j)}(q,t)$ *are defined successively by*

$$\begin{cases} V_{i1}^{(j)}(q,t) = \exp\left(\int_0^t \sum_{k=1}^{j} g_{ik}^{(j)}(q,s)ds\right), \\ V_{ik}^{(j)}(q,t) = G_{i,j-k+1}^{(j)}(q,t)\left[1 + \int_0^t g_{i,j-k+1}^{(j)}(q,s)\frac{V_{i,k-1}^{(j)}(q,s)}{G_{i,j-k+1}^{(j)}(q,s)}ds\right],\ k = 2,3,\cdots,j, \end{cases} \tag{5.4.186}$$

in which

$$G_{ih}^{(j)}(q,t) = \exp\left(\int_0^t \left[\sum_{r=1}^{h-1} g_{ir}^{(j)}(q,s) - g_{ih}^{(j)}(q,s)\right]ds\right),\ h = 1,2,\cdots,j-1, \tag{5.4.187}$$

and

$$\begin{cases} g_{i1}^{(j)}(q,t) = \left[\prod_{k=1}^{j-1}\prod_{k=1}^{r_k} W_i^{(k)}(t)\right]\prod_{k=1}^{j-1} W_k^{(j)}(t) f_{i1}^{(j)}(q,t), \\ g_{ik}^{(j)}(q,t) = f_{ik}^{(j)}(q,t), k = 2,3,\cdots,j, i = 1,2,\cdots,r_j. \end{cases} \tag{5.4.188}$$

Proof According to the structure of the estimate (5.4.163) and in view of the contract we made before, without loss of the generality, we may assume here $r_j \geq 1, j = 1,2,\cdots,n$. We may rewrite the inequality (5.4.184) as for all $t \in I$,

$$x(t) \leq p_1(t) + \sum_{i=1}^{r_1} I_i^{(j)}(t)(x), \tag{5.4.189}$$

where

$$p_1(t) = p(t) + \sum_{j=2}^{n}\sum_{i=1}^{r_j} I_i^{(j)}(t)(x).$$

Clearly, under the conditions of this theorem, the function $p_1(t)$ belongs to S and it is positive and non-decreasing on I. Hence we can apply Theorem 5.4.27 to inequality (5.4.189) to obtain for all $t \in I$,

$$x(t) \leq p_1(t)\prod_{i=1}^{r_1} W_i^{(1)}(t),$$

i.e., for all $t \in I$,

$$x(t) \leq p_2(t) + \sum_{i=1}^{r_2} \left(\prod_{i=1}^{r_1} W_i^{(1)}(t) \right) I_i^{(2)}(t)(x), \tag{5.4.190}$$

where

$$p_2(t) = \left(\prod_{i=1}^{r_1} W_i^{(1)}(t) \right) \left[p(t) + \sum_{j=3}^{n} \sum_{i=1}^{r_j} I_i^{(j)}(t)(x) \right],$$

and $W_i^{(1)}(t) = V_{i1}^{(1)}(t,t), i = 1, 2, \cdots, r_1$, are given by (5.4.186) and (5.4.188) with $j = 1$.

Here $(\prod_{i=1}^{r_1} W_i^{(1)}(t)) I_{i1}^{(2)}(t)(x)$ is the same type of linear integral operator as the operator $I_i^{(2)}(t)(x)$ except the function $f_{21}^{(j)}(t,s)$ is now replaced by $(\prod_{i=1}^{r_1} W_i^{(1)}(t)) f_{21}^{(j)}(t,s)$. Thus, we may apply Theorem 5.4.27 once again to the inequality (5.4.190) to reach

$$\begin{aligned} x(t) &\leq p_2(t) \prod_{i=1}^{r_2} W_i^{(2)}(t) \qquad (5.4.191) \\ &= \left(\prod_{i=1}^{r_1} W_i^{(1)}(t)) \right) \left(\prod_{i=1}^{r_2} W_i^{(2)}(t)) \right) \left[p(t) + \sum_{j=3}^{n} \sum_{i=1}^{r_j} I_i^{(j)}(t)(x) \right], \quad \textit{for all } t \in I \end{aligned}$$

where $W_i^{(2)}(t) = V_{i2}^{(2)}(t,t)$, and $V_{i1}^{(2)}(q,t), V_{i2}^{(2)}(q,t)$ are defined by (5.4.187)–(5.4.188) with $j = 2, i = 1, 2, \cdots, r_2$.

Now, we rewrite the inequality (5.4.191) as for all $t \in I$,

$$x(t) \leq p_3(t) + \sum_{i=1}^{r_3} I_i^{(3)}(t)(x),$$

and proceeding in this way, then after n-times application of Theorem 5.4.26 we can obtain the desired estimate (5.4.185). Since the inductional argument is very easy, we leave it to the reader. □

Remark 5.4.10 If the function $p(t)$ in (5.4.184) belongs to the class S, but is not non-decreasing on I, then we can replace it by the function $P(t) = \max\{p(s) | 0 \leq s \leq t;\ t \in I\}$, and then use Theorem 5.4.29 to obtain the desired bound for $x(t)$.

Now if we write for $x \in \mathbb{R}^n$, $\underline{x}_j = (x_1, \cdots, x_j)$ and $\overline{x}_j = (x_j, x_{j+1}, \cdots, x_n)$ for $j \in \{2, \cdots, n-1\}$. For any functions $F : \mathbb{R}^n_+ \to \mathbb{R}_+$ and $G : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$, we define

$$\begin{cases} \hat{F}(x) := \sup\{F(y) : 0 \leq y \leq x; x, y \in \mathbb{R}^n_+\}, \\ \hat{G}(x,s) := \sup\{G(y,s) : 0 \leq y \leq x; x, y \in \mathbb{R}^n_+\}. \end{cases} \tag{5.4.192}$$

Clearly, the above functions $\hat{F}(x)$ and $\hat{G}(x,s)$ are non-decreasing with respect to x.

Let us consider the following integral inequality

$$u(x) \leq f(x) + \sum_{i=1}^{N} (T_i u)(x) + g(x) G\left[\int_0^x h(x,s) Q(u(s)) ds\right], \tag{5.4.193}$$

where $x \in \mathbb{R}^n_+$ and the integral operators T_i are defined by

$$(T_i u)(x) = \int_0^x k_{i1}(x, s_1) \int_0^{s_1} k_{i2}(s_1, s_2) \int_0^{s_2} k_{i3}(s_2, s_3) \cdots \int_0^{s_{i-1}} k_{ii}(s_{i-1}, s_i) u(s_i) ds_i ds_{i-1} \cdots ds_1, \tag{5.4.194}$$

where $u, f, g : \mathbb{R}^n_+ \to \mathbb{R}_+$ and $h, k_{ij} : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$; $(i, j = 1, \cdots, N, i \geq j)$ are continuous functions; and $G, Q : \mathbb{R}_+ \to \mathbb{R}_+$ are differentiable functions which verify some other assumptions.

To deal with (5.4.193), we shall use the following classes of functions.

Definition 5.4.1 A function $w : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to the function class $H(\varphi)$ if

(i) w is non-decreasing and continuous on $\mathbb{R}_+$ and positive on $(0, +\infty)$;
(ii) there exists a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $u, v \geq 0$,

$$w(uv) \leq \varphi(u) w(v). \tag{5.4.195}$$

It is known from [161] that any sub-multiplicative function w on $\mathbb{R}_+$ satisfying above condition (i) must belong to class $H(\varphi)$.

Definition 5.4.2 A function $w : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $F(\rho)$ if

(i) w is non-decreasing and continuous on $\mathbb{R}_+$;
(ii) there exists a function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $u \geq 0, v > 0$,

$$\frac{1}{v} w(u) \leq \rho(v) w\left(\frac{u}{v}\right). \tag{5.4.196}$$

We now first consider special linear cases of integral inequality (5.4.193).

Theorem 5.4.30 (Yang [662]) *Let $v, f : \mathbb{R}_+^n \to \mathbb{R}_+$ and $k : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+$ be continuous. Suppose the following integral inequality holds for all $x \in \mathbb{R}_+^n$,*

$$v(x) \leq f(x) + \int_0^x k(x,s)v(s)ds. \tag{5.4.197}$$

Then for all $x \in \mathbb{R}_+^n$,

$$v(x) \leq \hat{f}(x) \exp\left(\int_0^x \hat{k}(x,s)ds\right). \tag{5.4.198}$$

Proof Fixing any $X > 0$, $X \in \mathbb{R}_+^n$, we derive from (5.4.197), for all $x \in [0, X]$,

$$v(x) + \varepsilon \leq \hat{f}(x) + \varepsilon + \exp\left(\int_0^x \hat{k}(X,s)(v(s) + \varepsilon)ds\right),$$

where ε is an arbitrary positive number, since $\hat{k}(x,s)$ is non-decreasing in x. Thus it follows that for all $x \in [0, X]$,

$$\theta(x,\varepsilon) \leq J(x,\varepsilon) := 1 + \int_0^x \hat{k}(X,s)\theta(s,\varepsilon)ds, \tag{5.4.199}$$

where

$$\theta(x,\varepsilon) := \frac{v(x) + \varepsilon}{\hat{f}(x) + \varepsilon}. \tag{5.4.200}$$

We obtain by differentiation that for all $x \in [0, X]$,

$$D_n D_{n-1} \cdots D_1 J(x,\varepsilon) = \hat{k}(X,x)\theta(x,\varepsilon) \leq \hat{k}(X,x)J(x,\varepsilon), \tag{5.4.201}$$

since (5.4.199) holds, where $D_i = \partial/\partial x_i$ for $i \in \{1, \cdots, n\}$.

Obviously, $J(x,\varepsilon)$ possesses the following properties:

$$\begin{cases} J(x,\varepsilon) = 1,\ D_{j-1} \cdots D_1 J(x,\varepsilon) = 0, & \text{if } x_i = 0 \quad \text{for some } i \in \{1, \cdots, n\}; \quad (5.4.202) \\ D_{j-1} \cdots D_1 J(x,\varepsilon) \geq 0 \quad \text{on } [0,X] & \text{where } j \in \{2, \cdots, n\}. \quad (5.4.203) \end{cases}$$

Using (5.4.202)–(5.4.203), we easily derive from (5.4.201) that for all $x \in [0, X]$,

$$D_n \left[\frac{D_{n-1} \cdots D_1 J(x,\varepsilon)}{J(x,\varepsilon)}\right] \leq \hat{k}(X,x).$$

Keeping $\underline{x}_{n_1}$ fixed, letting $x_n = s_n$ and integrating both sides with respect to s_n over $[0, x_n]$, we obtain for all $x \in [0, X]$,

$$\frac{D_{n-1}\cdots D_1 J(x,\varepsilon)}{J(x,\varepsilon)} \leq \int_0^{x_n} \hat{k}(X, \underline{x}_{n-1}, s_n)ds_n.$$

Again using properties (5.4.202)–(5.4.203), we get from the last inequality, for all $x \in [0, X]$,

$$D_{n-1}\left[\frac{D_{n-2}\cdots D_1 J(x,\varepsilon)}{J(x,\varepsilon)}\right] \leq \int_0^{x_n} \hat{k}(X, \underline{x}_{n-1}, s_n)ds_n.$$

Keeping $\underline{x}_{n_2}$ and x_n fixed, letting $x_{n-1} = s_{n-1}$ and integrating both sides with respect to s_{n-1} over $[0, x_{n-1}]$, we have for all $x \in [0, X]$,

$$\frac{D_{n-2}\cdots D_1 J(x,\varepsilon)}{J(x,\varepsilon)} \leq \int_0^{\overline{x}_{n-1}} \hat{k}(X, \underline{x}_{n-2}, \overline{s}_{n-1})d\overline{s}_{n-1},$$

where $d\overline{s}_{n-1} = ds_n ds_{n-1}$. Continuing in the same way, we then arrive at for all $x \in [0, X]$,

$$\frac{D_1 J(x,\varepsilon)}{J(x,\varepsilon)} \leq \int_0^{\overline{x}_2} \hat{k}(X, x_1, \overline{s}_2)d\overline{s}_2,$$

where $\overline{s}_2 = ds_n \cdots ds_2$. Now keeping x_2 fixed, letting $x_1 = s_1$ and integrating both sides with respect to s_1 over $[0, x_1]$, we obtain from the last inequality, for all $x \in [0, X]$,

$$\log J(s_1, \bar{x}_2, \varepsilon)\Big|_0^{x_1} = \log J(x,\varepsilon) - \log 1 \leq \int_0^x \hat{k}(X, s)ds,$$

i.e., for all $x \in [0, X]$,

$$J(x,\varepsilon) \leq \exp\left(\int_0^x \hat{k}(X, s)ds\right).$$

Hence it follows from (5.4.199) and (5.4.200), that for all$x \in [0, X]$,

$$v(x) + \varepsilon \leq (\hat{f}(x) + \varepsilon)\exp\left(\int_0^x \hat{k}(X, s)ds\right).$$

Putting $x = X$ in the last inequality, and letting ε tend to zero, then we obtain the bound in (5.4.198), since $X \in \mathbb{R}_+^n$ is arbitrary. □

Remark 5.4.11 (i) When $n = 1$, Theorem 5.4.30 reduces to the well-known integral inequality of Bellman (Theorem 1.1.2). (ii) When the kernel $k(x, s)$ is directly

separate, the inequality (5.4.197) had been discussed by Abramovich [1], Conlan and Wang [142] and Beesack [55], by using the Neumann series method. (iii) In Yeh [668], an implicit upper bounds for the solutions of (5.4.197) was given in terms of Riemann functions. (iv) More precise bounds for the solutions of (5.4.197) when the kernel is a so-called good kernel can be found in Conlan and Wang [143].

The next result is a direct extension of Theorem 5.4.30.

Theorem 5.4.31 (Yang [662]) *Let $v, f : \mathbb{R}^n_+ \to \mathbb{R}_+$ and $k_{ij} : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}_+$ be continuous, $i, j = 1, 2, \cdots, N$ with $i \geq j$. Suppose the following integral inequality holds for all $x \in \mathbb{R}^n_+$,*

$$v(x) \leq f(x) + \sum_{m=1}^{N} (T_m v)(x), \tag{5.4.204}$$

where operators T_i are defined by (5.4.194). Then for all $x \in \mathbb{R}^n_+$,

$$v(x) \leq \hat{f}(x) \exp\left\{ \sum_{m=1}^{N} \int_0^x H_m(x) \hat{k}_{mm}(x, s) ds \right\}, \tag{5.4.205}$$

where

$$H_m(x) = \int_0^x \hat{k}_{m1}(x, s_1) ds_1 \int_{s_1}^x \hat{k}_{m2}(x, s_2) ds_2 \cdots \int_{s_{m-2}}^x \hat{k}_{mm-1}(x, s_{m-1}) ds_{m-1}.$$

Proof Since k_{ij} are continuous and non-negative, we obtain by changing the order of integration that for all $x \in \mathbb{R}^n_+$,

$$\begin{aligned}(T_m v)(x) \leq & \int_0^x \hat{k}_{m1}(x, s_1) \int_0^{s_1} \hat{k}_{m2}(s_1, s_2) ds_2 \cdots \int_0^{s_{m-1}} \hat{k}_{mm-1}(s_{m-2}, s_{m-1}) \\ & \times ds_{m-1} ds_{m-2} \cdots ds_1 \times \int_0^x \hat{k}_{mm}(x, s_m) v(s_m) ds_m \\ \leq & H_m(x) \int_0^x \hat{k}_{mm}(x, s) v(s) ds,\end{aligned}$$

which, along with (5.4.204), gives us for all $x \in \mathbb{R}^n_+$,

$$v(x) \leq f(x) + \int_0^x \left\{ \sum_{m=1}^{N} H_m(x) \hat{k}_{mm}(x, s) \right\} v(s) ds.$$

Thus applying Theorem 5.4.30 to the last inequality proves the desired inequality (5.4.205). ☐

Remark 5.4.12 The special case of (5.4.204) when $N = 2, k_{11}(x,s) = b(x)c(s)$, $k_{21}(x,s) = b(x)p(s)$ and $k_{22}(x,s) = q(s)$ had been considered by Yeh [668] and an implicit upper bound was given there in terms of Riemann functions. A similar result was also proved in Thandapani and Agarwal [621].

Remark 5.4.13 Inequality (5.4.204) was once studied by Conlon and Wang [143, Theorem 3.3]. But the method of proof in [143] is different from the method used above. In fact, the upper bound in (5.4.205) is much simpler in its former than that given in (3.13) of [143]. (iii) Theorem 1 of Mamekonyan [382] is exactly the special case of Theorem 5.4.31 when $N = 2, k_{11}(x,s) = b(x)K(s)$, $k_{21}(x,s) = q(x)l(s)$ and $k_{22}(x,s) = m(s)$.

The Riemann function $v(t,x)$ of a hyperbolic characteristic initial value problem was used to provide upper bounds for functions which satisfy the Gronwall-Bellman inequalities. The next result gives us a direct proof of the fact that v satisfies an inequality of the form $v(t,x) \le \exp(\int_t^x b(s)ds)$.

Since then, the method has been often used (see, for example, [4, 92, 312, 605, 636, 677]) to give us upper bounds for functions $u : \mathbb{R}^n \to \mathbb{R}$ which satisfy integral inequalities of the form

$$u(x) \le a(x) + \int_{x^0}^{x} b(t)u(t)dt,$$

where

$$\int_{x^0}^{x} \cdots dt = \int_{x_n^0}^{x_n} \cdots \int_{x_1^0}^{x_1} \cdots dt_1 \cdots dt_n.$$

In fact, under approximate hypotheses, Young's theorem [677] below (see, Theorem 5.4.32) gives us the best possible upper bound, where $v(t,x)$ is the Riemann function relative to the point x for the characteristic initial value problem.

Let Ω be an open bounded set in $\mathbb{R}^n$ and let a point $(x_1, \cdots, x_n)$ in Ω be denoted by x. Let x^0 and x $(x^0 < x)$ be any two points in Ω and denote by D the parallelepiped defined by $x^0 < \xi < x$ (that is, $x_i^0 < \xi_i < x_i,\ 1 \le i \le n$).

In 1973, Young [677] extended Snow's technique (see, Theorem 5.1.10) to the case of n independent variables, and later on Chandra and Davis [128] gave us a further extension to integral inequalities involving matrix functions. In [604], Snow noted that $v(t;x)$ is the generalization of an exponential function $\exp(\int_t^x b(r)dr)$ which appears in the corresponding bound for a one-dimensional Gronwall-Bellman inequality, see Theorems 1.1.1–1.1.2.

Theorem 5.4.32 (Young [677]) *Suppose $\phi(x), a(x)$, and $b(x) \ge 0$ are continuous functions in Ω. Let $v(\xi;x)$ be a solution of the characteristic initial value problem*

$$\begin{cases} (-1)^n v_\xi(\xi;x) - b(\xi)v(\xi;x) = 0 \quad \text{in } \Omega \\ \quad v(\xi;x) = 1 \quad \text{on } \xi_i = x_i, i = 1, \cdots, n, \end{cases} \tag{5.4.206}$$

where $v_\xi(\xi; x) = v_{\xi_1\cdots\xi_n}$ *and let* D^+ *be a connected sub-domain of* Ω *containing* x *such that* $v \geq 0$ *for all* $\xi \in D^+$. *If* $D \subset D^+$ *and*

$$\phi(x) \leq a(x) + \int_{x^0}^{x} b(\xi)\phi(\xi)d\xi, \tag{5.4.207}$$

then

$$\phi(x) \leq a(x) + \int_{x^0}^{x} a(\xi)b(\xi)v(\xi; x)d\xi. \tag{5.4.208}$$

Proof Set

$$u(x) = \int_{x^0}^{x} b(\xi)\phi(\xi)d\xi \tag{5.4.209}$$

so that

$$D_1 \cdots D_n u(x) = b(x)\phi(x),\ D_i = \partial/\partial x_i, \quad 1 \leq i \leq n. \tag{5.4.210}$$

Since $b(x) \geq 0$ in D, it follows from (5.4.207)–(5.4.209) that

$$\begin{cases} Lu \equiv D_1 \cdots D_n u(x) - b(x)\phi(x) \leq a(x)b(x), \\ u(x) = 0 \quad \text{on } x_i = x_i^0,\ 1 \leq i \leq n. \end{cases} \tag{5.4.211}$$

Furthermore, all pure mixed derivatives of u with respect to $x_1, \cdots x_{i-1}, x_{i+1}, \cdots, x_n$ up to order $n-1$ vanish on $x_i = x_i^0,\ 1 \leq i \leq n$. If w is a function which is n-times continuously differentiable in D, then

$$wLu - uMw = \sum_{k=1}^{n} (-1)^{k-1} D_k[(D_0 D_1 \cdots D_{k-1} w)(D_{k+1} \cdots D_n D_{n+1} u)] \tag{5.4.212}$$

where $Mw = (-1)^n D_1 \cdots D_n w(x) - b(x)w(x)$ with $D_0 \equiv D_{n+1} \equiv I$, the identity. Integrating (5.4.212) over D, using ξ as variables of integration, and notice that u vanishes together with all its mixed derivatives up to order $n-1$ on $\xi_k = x_k^0,\ 1 \leq k \leq n$, we then obtain

$$\int_D (wLu - uMw)d\xi = \sum_{k=1}^{n} (-1)^{k-1} \int_{\xi_k = x_k} (D_1 \cdots D_{k-1} w)(D_{k+1} \cdots D_n u)d\xi' \tag{5.4.213}$$

where $d\xi' = d\xi_1 \cdots d\xi_{k-1} d\xi_{k+1} \cdots d\xi_n$.

Now let w be chosen as the function v satisfying (5.4.206). Since $v = 1$ on $\xi_k = x_k,\ 1 \leq k \leq n$, it follows that $D_1 \cdots D_{k-1} v(\xi, x) = 0$ on $\xi_k = x_k$ for $2 \leq k \leq n$.

Thus (5.4.213) becomes

$$\int_D v(\xi;x)Lu(\xi)d\xi = \int_{\xi_1=x_1} v(\xi;x)D_2\cdots D_n u(\xi)d\xi' = u(x). \tag{5.4.214}$$

By the continuity of v and by the fact that $v = 1$ on $\xi = x$, there is a domain D^+ containing x on which $v \geq 0$. Hence multiplying (5.4.211) throughout by v and using (5.4.209) and (5.4.214), we can obtain the desired (5.4.208). □

We note that the problem (5.4.206) defines precisely the so-called Riemann function for the operator L. The existence and regularity property of v can be deduced from [140] (see, e.g., [607]). Indeed (5.4.206) is equivalent to the integral equation

$$v(\xi;x) = 1 + \int_\xi^x b(\eta)v(\eta;x)d\eta. \tag{5.4.215}$$

Now, the solution of (5.4.215) can be represented (see, e.g., [636]), by

$$v(\xi;x) = 1 + \int_\xi^x b(\eta)h^*(\eta;x)d\eta \tag{5.4.216}$$

where

$$h^*(\xi;x) = \sum_{i=1}^{+\infty} h_i(\xi;x) \tag{5.4.217}$$

with

$$h_1(\xi;x) = 1, \quad h_{i+1}(\xi,x) = \int_\xi^x b(\eta)h_i(\eta;x)d\eta. \tag{5.4.218}$$

From (5.4.215) and (5.4.216) it follows that $v(\xi;x) = h^*(\xi;x)$. Thus (5.4.208) can also be rewritten as

$$\phi(x) \leq a(x) + \int_{x^0}^x a(\xi)b(\xi)h^*(\xi;x)d\xi \tag{5.4.219}$$

with h^* defined by (5.4.217)–(5.4.218). This agrees with the result given in Walter [636, 637]. □

In the same manner, we can readily prove the following theorem.

Theorem 5.4.33 (Young [677]) *Let $x, \alpha \in D \subset \mathbb{R}^n$, and $\alpha \leq x$. Let $u(x), a(x), b(x) \geq 0, q(x) \geq 0$ be continuous functions in D. Let $v(s;x)$ be the*

solution of the characteristic initial value problem

$$\begin{cases} (-1)^n v_s(s;x) - q(s)b(s)v(s;x) = 0, & in \ \ D, \\ v(s;x) = 1 \ \ on \ \ s_i = x_i, & i = 1, \cdots, n, \end{cases} \tag{5.4.220}$$

and let D^+ be a connected sub-domain of D, containing x, on which $v \geq 0$ for all $s \in D^+$. If $[\alpha, x] \subset D^+$ and

$$u(x) \leq a(x) + q(x) \int_\alpha^x b(s)u(s)ds, \tag{5.4.221}$$

then

$$u(x) \leq a(x) + q(x) \int_\alpha^x a(s)b(s)v(s;x)ds. \tag{5.4.222}$$

Using Lemma 5.4.2, we also have the following theorem due to [479].

Theorem 5.4.34 (Pachpatte [479]) *Let $x, \alpha \in D \subset \mathbb{R}^n$, and $\alpha \leq x$. Let $u(x), a(x), b(x), c(x)$, and $\sigma(x)$ be non-negative continuous functions in D. Let $v(s;\ x)$ be the solution of the characteristic initial value problem*

$$\begin{cases} (-1)^n v_s(s;x) - [b(s) + c(s)]v(s;x) = 0, & in \ \ D, \\ v(s;x) = 1 \ \ on \ \ s_i = x_i, & i = 1, \cdots, n, \end{cases} \tag{5.4.223}$$

and let D^+ be a connected sub-domain of D, containing x, on which $v \geq 0$ for all $s \in D^+$. If $[\alpha, x] \subset D^+$ and

$$u(x) \leq a(x) + \int_\alpha^x b(s)ds + \int_\alpha^x b(s)\Big(\sigma(s) + \int_\alpha^s c(\tau)u(\tau)d\tau\Big)ds, \tag{5.4.224}$$

then

$$u(x) \leq a(x) + \int_\alpha^x b(s)\left[a(s) + \sigma(s) + \int_\alpha^s \Big(a(\tau)c(\tau) + b(\tau)[a(\tau) + \sigma(\tau)]\Big)v(s;x)d\tau\right]ds. \tag{5.4.225}$$

Proof Obviously, inequality (5.4.225) is equivalent to the following system

$$u(x) \leq a(x) + \phi(x), \tag{5.4.226}$$

where

$$\begin{cases} \phi(x) = \int_\alpha^x b(s)u(s)ds + \int_\alpha^x b(s)[\sigma(s) + \psi(s)]ds, & (5.4.227) \\ \psi(x) = \int_\alpha^x c(s)u(s)ds, & (5.4.228) \end{cases}$$

which implies

$$\begin{cases} \phi_x(x) \leq b(x)[a(x) + \sigma(x)] + b(x)[\phi(x) + \psi(x)], & (5.4.229) \\ \psi_x(x) \leq a(x)c(x) + c(x)\phi(x). & (5.4.230) \end{cases}$$

Thus adding (5.4.229) and (5.4.230) implies

$$[\phi(x) + \psi(x)]_x \leq a(x)c(x) + b(x)[a(x) + \sigma(x)] + [b(x) + c(x)][\phi(x) + \psi(x)], \tag{5.4.231}$$

whence, by Lemma 5.4.2,

$$\phi(x)+\psi(x) \leq \int_\alpha^x \Big(a(\tau)c(\tau)+b(\tau)[a(\tau)+\sigma(\tau)]\Big)v(\tau;x)d\tau \equiv R(x). \tag{5.4.232}$$

Thus it follows from (5.4.229)–(5.4.232) that

$$\phi_x(x) \leq b(x)[a(x) + \sigma(x)] + b(x)R(x)$$

which implies

$$\phi(x) \leq \int_\alpha^x b(s)[a(s) + \sigma(s) + R(s)]ds. \tag{5.4.233}$$

Hence (5.4.225) follows from (5.4.226)–(5.4.233). □

The next result, due to Beesack [54], gives us a direct proof of the fact that, when $b(t) \geq 0$, the Riemann function $v(t, x)$ is actually bounded above by an exponential function. This implies, for example, that if also $a \geq 0$ in (5.4.207), then a more explicit (if cruder) upper bound for u can be given, namely,

$$u(x) \leq a(x) + \int_{x_0}^x a(t)b(t)\exp\left(\int_t^x b(r)dr\right)dt. \tag{5.4.234}$$

Such as result, (5.4.234) had obtained by Beesack [55] for a more general inequality than

$$u(x) \leq a(x) + \int_{x_0}^x b(s)u(s)ds,$$

but in a context not involving the Riemann function. Moreover, inequality (5.4.235) below appears in the middle of a proof [128], of matrix version of an extension of (5.4.234). In view of the fact that in most cases where Riemann's method is employed us, only an upper bound on u is required (or obtained), it seems to be worthwhile to give a direct proof of the inequality

$$v(t;x) \leq \exp\left(\int_t^x b(r)dr\right). \tag{5.4.235}$$

We state the result in the form with $x^0 = 0$ and $x \geq 0$.

Theorem 5.4.35 (Beesack [54]) *Suppose $b : \mathbb{R}^n \to \mathbb{R}$ is continuous and non-negative for all $x \geq 0$, and $v(t,x)$, $0 \leq t \leq x$, is the Riemann function defined by (5.4.206). Then (5.4.235) holds for all* $0 \leq t \leq x$

Proof From (5.4.215) (see also, [605, 677]), $v(t;x)$ is the solution of the integral equation

$$v(t;x) = 1 + \int_t^x b(s)v(s,x)ds, \tag{5.4.236}$$

which also follows either by integration of (5.4.206) or by differentiation of (5.4.236). Moreover, $v(t;x)$ has the Riemann series representation

$$v(t,x) = \sum_{k=0}^{+\infty} u_k(t,x), \tag{5.4.237}$$

where $u_0(t,x) = 1$ and, for all $k \geq 1$,

$$u_k(t,x) = \int_t^x b(t)u_{k-1}(t_1,x)dt_1 = \cdots = \int_t^x \int_{t_1}^x \cdots \int_{t_{k-1}}^x \prod_{k=1}^{k} 1b(t_i)dt_k \cdots dt_1. \tag{5.4.238}$$

(see also, [677] and Walter [636]). To estimate the above integral, we use the technique of Fink [216], as used also by Beesack in [55]. Set $\tilde{t} = (t_1, \cdots, t_k) \in \mathbb{R}^{nk}$, let σ denote the set of all $k!$ permutations of $(1, 2, \cdots, k)$, $p \in \sigma$, and

$$\begin{cases} S = \{\tilde{t} : t \leq t_1 \leq t_2 \leq \cdots \leq t_{k-1} \leq t_k \leq x\}, \\ S_p = \{\tilde{t} : t \leq t_{p_1} \leq t_{p_2} \leq \cdots \leq t_{p_{k-1}} \leq t_{p_k} \leq x\}. \end{cases}$$

Because of the symmetry of the integrand $B(\tilde{t}) = \prod_{i=1}^{k} b(t_i)$, it follows that

$$u_k(t,x) = \sum_{p\in\sigma} \int_{S_p} B(\tilde{t})d\tilde{t}/k!.$$

If $p,\ q$ are different permutations in σ, then $A = S_p \cap S_q$ has nk-dimensional Lebesgue measure 0 since A is a subset of some hyperplane $t_i = t_j$ of dimension $n(k-1)$. In addition, we clearly have

$$C = \bigcup_{p\in\sigma} S_p \subset [t,x]^k = [t,x] \times \cdots \times [t,x].$$

Note the for $n > 1$, " $\subset$" cannot be replaced by " $=$" because " $\leq$" is only a partial order in $\mathbb{R}^n$. It now follows that for all $0 \leq t \leq x$,

$$u_k(t,x) = \int_C B(\tilde{t})d\tilde{t}/k! \leq \int_t^x \cdots \int_t^x \prod_{i=1}^{k} b(t_i)dt_k \cdots dt_1/k!,$$

or

$$u_k(t,x) \leq \left(\int_t^x b(s)ds\right)^k /k! \ \text{ for all } \ 0 \leq t \leq x,\ k = 0, 1, \cdots . \tag{5.4.239}$$

Thus the conclusion (5.4.235) now follows from (5.4.237) and (5.4.239). □

More generally, even if b changes sign, it is clear from the above analysis that for all $0 \leq t \leq x$,

$$|v(t,x)| \leq \exp\left(\int_t^x |b(s)|ds\right) \tag{5.4.240}$$

which extends the range of applicability of the Riemann method in dealing with integral inequalities (5.4.207).

Remark 5.4.14 As stated in (5.4.235), for $n = 1$, we have $v(t;x) = \exp\left(\int_t^x b(s)ds\right)$; but for $n \geq 2$, such an analytic representation is impossible, i.e., it holds that $v(t;x) < \exp\left(\int_t^x b(s)ds\right)$.

Next, we introduce some n-independent-variable generalizations in [587] of the integral inequalities of LaSalle [353], Gollwitzer [231], Langenhop [351], and Bondge and Pachpatte [91].

First, Gollwizer's inequality [231] and Bondge and Pachpatte's inequality [91] are unified and embodied in the following theorem.

Theorem 5.4.36 (Shin-Chih [587]) *Let $w(x), a(x)$ and $b(x)$ be real-valued, non-negative and continuous functions defined on $\mathbb{R}^n_+$; let $u(s)$ be a positive real-valued*

continuous function defined on $\mathbb{R}_+^n$. *Suppose that the following inequality holds for all* $0 \le x \le s$, *where* $s \in \mathbb{R}_+^n$,

$$u(s) \ge w(s) - a(s) \int_x^s b(t)w(t)dt. \tag{5.4.241}$$

Then for all $0 \le x \le s$,

$$u(x) \ge w(x) \exp\left(-a(s) \int_x^s b(t)dt\right).$$

Proof We first discuss the case when n is even. We rewrite (5.4.241) as

$$w(x) \le u(s) + a(s) \int_x^s b(t)w(t)dt. \tag{5.4.242}$$

For fixed s in $\mathbb{R}_+^n$, we define for all $0 \le x \le s$,

$$r(x) = u(s) + a(s) \int_x^s b(t)w(t)dt. \tag{5.4.243}$$

Then

$$r(x) = u(s) \qquad \text{on} \quad x_i = s_i, \quad i = 1, 2, \dots, n; \tag{5.4.244}$$

and

$$D_1 D_2 \dots D_n r(x) = a(s)b(x)w(x). \tag{5.4.245}$$

Then by (5.4.243)

$$D_1 D_2 \dots D_n r(x) \le a(s)b(x)r(x),$$

which implies

$$\frac{r(x) D_1 \cdots D_n r(x)}{r^2(x)} \le a(s)b(x) + \frac{D_n r(x)(D_1 \cdots D_{n-1} r(x))}{r^2(x)},$$

i.e.,

$$D_n \left(\frac{D_1 \dots D_{n-1} r(x)}{r(x)} \right) \le a(s)b(x). \tag{5.4.246}$$

Integrating both sides of (5.4.243) with respect to the component x_n of x from x_n to s_n, we conclude

$$\frac{D_1 \cdots D_{n-1} r(x_1, \cdots, x_{n-1}, s_n)}{r(x_1, \cdots, x_{n-1}, s_n)} - \frac{D_1 \cdots D_{n-1} r(x)}{r(x)} \leq a(s) \int_{x_n}^{s_n} b(x_1, \cdots, x_{n-1}, t_n) dt_n.$$

Since

$$D_1 \cdots D_{n-1} r(x_1, \cdots, x_{n-1}, s_n) = 0,$$

we have

$$-\frac{D_1 \cdots D_{n-1} r(x)}{r(x)} \leq a(s) \int_{x_n}^{s_n} b(x_1, \cdots, x_{n-1}, t_n) dt_n,$$

which implies

$$-D_{n-1}\left(\frac{D_1 \cdots D_{n-2} r(x)}{r(x)}\right) \leq a(s) \int_{x_n}^{s_n} b(x_1, \cdots, x_{n-1}, t_n) dt_n. \quad (5.4.247)$$

Integrating both sides of (5.4.247) with respect to the component x_{n-1} of x from x_{n-1} to s_{n-1}, we get

$$\frac{D_1 \cdots D_{n-2} r(x)}{r(x)} \leq a(s) \int_{x_{n-1}}^{s_{n-1}} \int_{x_n}^{s_n} f(x_1, \cdots, x_{n-2}, t_{n-1}, t_n) dt_n dt_{n-1}.$$

Computing in this way, we obtain

$$\frac{D_1 D_2 r(x)}{r(x)} \leq a(s) \int_{x_3}^{s_3} \cdots \int_{x^n}^{s_n} b(x_1, x_2, t_3, \cdots, t_n) dt_n \cdots dt_3. \quad (5.4.248)$$

It follows from (5.4.248) that

$$D_2\left(\frac{D_1 r(x)}{r(x)}\right) \leq a(s) \int_{x_3}^{s_3} \cdots \int_{x_0}^{s_n} b(x_1, x_2, t_3, \cdots, t_n) dt_n \cdots dt_3. \quad (5.4.249)$$

Integrating both sides of (5.4.249) with respect to the component x_2 of x from x_2 to s_2, we obtain

$$\frac{D_1 r(x_1, s_2, x_3, \cdots, x_n)}{r(x_1, s_2, x_3, \cdots, x_n)} - \frac{D_1 r(x)}{r(x)} \leq a(s) \int_{x_2}^{s_2} \cdots \int_{x_n}^{s_n} b(x_1, t_2, \cdots, t_n) dt_n \cdots dt_2,$$

whence

$$-\frac{D_1 r(x)}{r(x)} \leq a(s) \int_{x_2}^{s_2} \cdots \int_{x_n}^{s_n} b(x_1, t_2, \cdots, t_n) dt_n \cdots dt_2.$$

Integrating both sides of the above inequality with respect to the component x_1 of x from x_1 to s_1, we get

$$\log \frac{r(x)}{u(s)} \le a(s) \int_x^s b(t)dt,$$

which implies

$$w(x) \le r(x) \le u(s) \exp\left(a(s) \int_x^s b(t)dt\right)$$

and thus the theorem follows for n being even.

Next we discuss the case when n is odd. As in the proof of the first case, we obtain

$$D_1 D_2 \cdots D_n r(x) = -a(s)b(x)w(x)$$

whence

$$\frac{r(x)D_1 \cdots D_n r(x)}{r^2(x)} \ge -a(s)b(x) + \frac{D_n r(x)(D_1 \cdots D_{n-1} r(x)}{r^2(x)},$$

i.e.,

$$D_n(\frac{D_1 \cdots D_{n-1} r(x)}{r(x)}) \ge -a(s)b(x).$$

Integrating both sides of the above inequality with respect to the component x_n of x from x_n to s_n, we obtain

$$-\frac{D_1 \cdots D_{n-1} r(x)}{r(x)} \ge -a(s) \int_{x_n}^{s_n} b(x_1, \cdots, x_{n-1}, t_n)dt_n,$$

which implies

$$D_{n-1}\left(\frac{D_1 \cdots D_{n-2} r(x)}{r(x)}\right) \le a(s) \int_{x_n}^{s_n} b(x_1, \cdots, x_{n-1}, t_n)dt_n.$$

Similarly to the proof of the first case, we can obtain the desired result. □

As an application of Theorem 5.4.36, we now introduce the following n-independent-variable generalization for the lower bound on an unknown function.

Theorem 5.4.37 (Shin-Chih [587]) *Let $w(s), a(s)$ and $b(s)$ be as defined in Theorem 5.4.36; let $H(r)$ be a positive, continuous, strictly increasing, convex, and sub-multiplicative for all $r > 0$, $H(0) = 0$ and $\lim_{r\to+\infty} H(r) = +\infty$. Suppose*

that $g(s)$ and $h(s)$ are positive functions defined on $\mathbb{R}^n_+$ with $g(s) + h(s) = 1$ and

$$u(s) \geq w(x) - a(s)H^{-1}\left(\int_x^s b(t)H(w(t))dt\right) \tag{5.4.250}$$

holds for all $0 < x \leq s$, where $s \in \mathbb{R}^n_+$. Then for all $0 \leq x \leq s$,

$$u(s) \geq g(s)H^{-1}\left[g^{-1}(s)H(w(x))\exp\left(-h(s)H(a(s)h^{-1}(s))\int_x^s b(t)dt\right)\right]. \tag{5.4.251}$$

Proof In fact, we may rewrite (5.4.250) as

$$w(x) \leq g(s)u(s)g^{-1}(s) + h(s)a(s)h^{-1}(s)H^{-1}\left(\int_x^s b(t)H(w(t))dt\right).$$

Since H is convex, sub-multiplicative and increasing, we get

$$H(w(x)) \leq g(s)H(u(s)g^{-1}(s)) + h(s)H(a(s)h^{-1}(s))\int_x^s b(t)H(w(t))dt,$$

i.e.,

$$g(s)H(u(x)g^{-1}(s)) \leq H(w(x)) - h(s)H(a(s)h^{-1}(s))\int_x^s b(t)H(w(t))dt.$$

Applying Theorem 5.4.36 to the above inequality, we can prove (5.4.251). □

We next introduce the following n-independent-variable generalization in [587] of the integral inequality established by Langenhop [351] and Bondge and Pachpatte [91].

Theorem 5.4.38 (Shin-Chih [587]) *Let $u(s), a(s)$ and $b(s)$ be as defined in Theorem 5.4.36; let $W(r)$ be a positive, continuous, non-decreasing function for all $r > 0$, $W(0) = 0$ and $W'(r) \in C(\mathbb{R}_+, \mathbb{R}_+)$. Suppose that the inequality*

$$u(s) \geq u(x) - a(s)\int_x^s b(t)W(u(t))dt \tag{5.4.252}$$

holds for all $0 < x \leq s$, where $s \in \mathbb{R}^n_+$. Then for all $s^0 \in \mathbb{R}^n_+$, $0 \leq x \leq s \leq s^0$,

$$u(s) \geq Q^{-1}\left[Q(u(x)) - a(s)\int_x^s b(t)dt\right], \tag{5.4.253}$$

where

$$Q(r) = \int_{r_0}^{r} \frac{ds}{W(s)}, \qquad r \geq r_0 > 0, \tag{5.4.254}$$

and Q^{-1} *is the inverse function of* Q*, and for all* $0 \leq x \leq s$,

$$Q(u(x)) - a(s) \int_{x}^{s} b(t)dt \in Dom(Q^{-1}).$$

Proof We only prove the case when n is even. To this end, we may rewrite (5.4.252) as

$$u(x) \leq u(s) + a(s) \int_{x}^{s} b(t)W(u(t))dt. \tag{5.4.255}$$

For fixed s in $\mathbb{R}_{+}^{n}$, we define for all $0 \leq x \leq s$,

$$r(x) = u(s) + a(s) \int_{x}^{s} b(t)W(u(t))dt.$$

Then

$$\begin{cases} r(x) = u(s) \quad \text{on}\, x_i = s_i,\ i = 1, 2, \cdots, n; & (5.4.256) \\ D_1 D_2 \cdots D_n r(x) = a(s)b(x)W(u(x)), & (5.4.257) \end{cases}$$

and

$$u(x) \leq r(x).$$

Since W is non-decreasing, (5.4.257) implies

$$D_1 D_2 \cdots D_n r(x) \leq a(s)b(x)W(r(x)),$$

i.e.,

$$\frac{D_1 D_2 \cdots D_n r(x)}{W(r(x))} \leq a(s)b(x).$$

Thus

$$\frac{W(r(x))D_1 \cdots D_n r(x)}{W^2(r(x))} \leq a(s)b(x) + \frac{D_n W(r(x))D_1 \cdots D_{n-1} r(x)}{W^2(r(x))},$$

i.e.,

$$D_n\left(\frac{D_1\ldots D_{n-1}r(x)}{W(r(x))}\right)\le a(s)b(x). \tag{5.4.258}$$

Integrating both sides of (5.4.258) with respect to the component x_n of x from x_n to s_n, we get

$$\frac{D_1\cdots D_{n-1}r(x_1,\cdots,x_{n-1},s_n)}{W(r(x_1,\cdots,x_{n-1},s_n))}-\frac{D_1\cdots D_{n-1}r(x)}{W(r(x))}$$
$$\le a(s)\int_{x_n}^{s_n}b(x_1,\cdots,x_{n-1},t_n)dt_n$$

which implies

$$-D_{n-1}\left(\frac{D_1\cdots D_{n-2}r(x)}{W(r(x))}\right)\le a(s)\int_{x_n}^{s_n}b(x_1,\cdots,x_{n-1},t_n)dt_n.$$

Similarly to the proof of Theorem 5.4.36, we obtain

$$\frac{D_1r(x_1,s_2,x_3,\cdots,x_n)}{W(r(x_1,s_2,\cdots,x_n))}-\frac{D_1r(x)}{W(r(x))}\le a(s)\int_{x_2}^{s_2}\cdots\int_{x_n}^{s_n}b(x_1,t_2,\cdots,t_n)dt_n\cdots dt_2. \tag{5.4.259}$$

Thus it follows from (5.4.254) and (5.4.259) that

$$-D_1Q(r(x))\le a(s)\int_{x_2}^{s_2}\cdots\int_{x_n}^{s_n}b(x_1,t_2,\cdots,t_n)dt_n\cdots dt_2. \tag{5.4.260}$$

Integrating both sides of the above inequality with respect to the component x_1 of x from x_1 to s_1, we conclude

$$-Q(r(s_1,x_2,\cdots,x_n))+Q(r(x))\le a(s)\int_x^s b(t)dt.$$

It follows from (5.4.256) that

$$-Q(r(s))+Q(r(x))\le a(s)\int_x^s b(t)dt,$$

i.e.,

$$Q(r(s))\ge Q(r(x))-a(s)\int_x^s b(t)dt. \tag{5.4.261}$$

Therefore, from (5.4.261), (5.4.253) follows. □

We now introduce n-independent-variable generalizations in [587] of the integral inequalities established by Pachpatte [456, 457] and Bondge and Pachpatte [91].

Theorem 5.4.39 (Shin-Chih [587]) *Let $w(s)$, $a(s)$, $b(s)$, and $c(s)$ be real-valued non-negative continuous functions defined on $\mathbb{R}^n_+$, let $u(s)$ be a positive real-valued continuous function defined on $\mathbb{R}^n_+$. Suppose that the inequality*

$$u(s) \geq w(s) - a(s)\left[\int_x^s b(m)w(m)dm + \int_x^s b(m)(\int_m^s c(t)w(t)dt)dm\right], \tag{5.4.262}$$

holds for all $0 \leq x \leq s$, where $s \in \mathbb{R}^n_+$. Then for all $0 \leq x \leq s$,

$$u(x) \geq w(x)\left[1 + a(s)\int_x^s b(m)\exp\left(\int_m^s (a(s)b(t) + c(t))dt\right)dm\right]^{-1}. \tag{5.4.263}$$

Proof We only prove the case that n is even. We may rewrite (5.4.262) as

$$w(x) \leq u(s) + a(s)\left[\int_x^s b(m)w(m)dm + \int_x^s b(m)(\int_m^s c(t)w(t)dt)dm\right]. \tag{5.4.264}$$

For fixed s in $\mathbb{R}^n_+$, we define, for all $0 \leq x \leq s$,

$$r(x) = u(s) + a(s)\left[\int_x^s b(m)w(m)dm + \int_x^s b(m)(\int_m^s c(t)w(t)dt)dm\right]. \tag{5.4.265}$$

Then

$$r(x) = u(s) \qquad \text{on} \quad x_i = s_i, \quad i = 1, 2, \cdots, n; \tag{5.4.266}$$

and

$$w(x) \leq r(x).$$

Hence

$$\begin{aligned} D_1 D_2 \cdots D_n r(x) &= a(s)b(x)\left[w(x) + \int_x^s c(t)w(t)dt\right] \\ &\leq a(s)b(x)\left[r(x) + \int_x^s c(t)r(t)dt\right]. \end{aligned} \tag{5.4.267}$$

Define

$$v(x) = r(x) + \int_x^s c(t)r(t)dt.$$

Then

$$v(x) = r(x) = r(s), \qquad \text{on} \quad x_i = s_i,\ i = 1, 2, \cdots, n; \tag{5.4.268}$$

and

$$\begin{aligned} D_1 \cdots D_n v(x) &= D_1 \cdots D_n r(x) + c(x)r(x) \\ &\le [a(s)b(x) + c(x)]\, v(x). \end{aligned}$$

Similarly to that in the proof of Theorem 5.4.36, we obtain

$$v(x) \le u(s) \exp \int_x^s \Big(a(s)b(t) + c(t)\Big)dt$$

which, inserted into (5.4.267), yields

$$D_1 \cdots D_n r(x) \le a(s)b(x)u(x) \exp\left(\int_x^s (a(s)b(t) + c(t))dt\right).$$

Integrating both sides of the above inequality with respect to the component x_n of x from x_n to s_n, we get

$$\begin{aligned} & D_1 \cdots D_{n-1} r(x_1, \cdots, x_{n-1}, s_n) - D_1 \ldots D_{n-1} r(x) \\ \le\ & a(s)u(s) \int_{x_n}^{s_n} b(x_1, \cdots, x_{n-1}, m_n) \\ & \times \exp\left(\int_{x_1}^{s_1} \cdots \int_{x_{n-1}}^{s_{n-1}} \int_{x_n}^{s_n} (a(s)b(t) + c(t))dt\right) dm_n. \end{aligned}$$

Integrating both sides of the above inequality with respect to the component x_{n-1} of x from x_{n-1} to s_{n-1}, we obtain

$$\begin{aligned} & -D_1 \cdots D_{n-2} r(x_1, \cdots, x_{n-2}, s_{n-1}, x_n) + D_1 \cdots D_{n-2} r(x) \\ \le\ & a(s)u(s) \int_{x_{n-1}}^{s_{n-1}} \int_{x_n}^{s_n} b(x_1, \cdots, x_{n-2}, m_{n-1}, m_n) \\ & \times \exp\left(\int_{x_1}^{s_1} \cdots \int_{x_{n-2}}^{s_{n-2}} \int_{m_{n-1}}^{s_{n-1}} \int_{m_n}^{s_n} (a(s)b(t) + c(t))dt\right) dm_n dm_{n-1}. \end{aligned}$$

Computing in this way, we obtain

$$D_1 r(x_1, s_2, x_3, \cdots, x_n) - D_1 r(x) \leq a(s)u(s) \int_{x_2}^{s_2} \cdots \int_{x_n}^{s_n} b(x_1, m_2, \cdots, m_n) \times \exp\left(\int_{x_1}^{s_1} \int_{m_2}^{s_2} \cdots \int_{m_n}^{s_n} (a(s)b(t) + c(t))dt\right) dm_n \cdots dm_2.$$

Integrating both sides of the above inequality with respect to the component x_1 of x from x_1 to s_1, we conclude

$$r(x) \leq u(s)\left[1 + a(s) \int_x^s b(m) \exp\left(\int_m^s (a(s)b(t) + c(t))dt\right) dm\right], \tag{5.4.269}$$

which, together with (5.4.264) and (5.4.269), gives (5.4.263). □

Next, we apply Theorem 5.4.39 to establish the following n-independent-variable generalization in [587] of the integral inequality established by Pachpatte [456] and Bondge and Pachpatte [91].

Theorem 5.4.40 (Shin-Chih [587]) *Let* $w(x)$, $a(s)$, $b(s)$, $c(s)$, *and* $u(x)$ *be as defined in Theorem 5.4.39; let* $H(r)$, $g(s)$, *and* $h(s)$ *be as defined in Theorem 5.4.43. Suppose that the inequality*

$$u(x) \geq w(x) - a(s)H^{-1}\Big[\int_x^s b(m)H(w(m))dm + \int_x^s b(m)\left(\int_m^s c(t)H(w(t))dt\right) dm\Big] \tag{5.4.270}$$

holds for all $0 \leq x \leq s$, *where* $s \in \mathbb{R}_+^n$. *Then for all* $0 \leq x \leq s$,

$$u(s) \geq g(s)H^{-1}\Big(g^{-1}H(w(x))\Big[1 + h(s)H(a(s)h^{-1}(s)) \int_x^s b(m) \times \exp \int_m^s |h(s)H(a(s)h^{-1}(s))b(t) + c(t)|dtdm\Big]\Big)^{-1}. \tag{5.4.271}$$

Proof We may rewrite (5.4.270) as

$$w(x) \leq g(s)u(s)g^{-1}(s) + h(s)a(s)h^{-1}(s)H^{-1} \times \left[\int_x^s b(m)H(w(m))dm + \int_x^s b(m)(\int_m^s c(t)H(w(t))dt)dm\right].$$

Since H is convex, sub-multiplicative, and strictly increasing, we obtain

$$g(s)H(u(s)g^{-1}(s)) \geq H(w(x)) - h(s)H(a(s)h^{-1}(s)) \int_x^s b(m)H(w(m))dm$$

which, by Theorem 5.4.39, gives us (5.4.271). Thus the proof is complete. □

Next we introduce the following n-independent-variable generalization in [587] of the integral inequality established by Pachpatte [457] and Bondge and Pachpatte [91].

Theorem 5.4.41 (Shin-Chih [587]) *Let $u(s)$, $a(s)$, $b(s)$, and $c(s)$ be as defined in Theorem 5.4.39; let $G(r)$ be a positive, continuous, strictly increasing, sub-additive, and sub-multiplicative function for all $r > 0$, $r \in \mathbb{R}_+$ and $G(0) = 0$; let G^{-1} denote the inverse function of G. Suppose the inequality that for all $0 \leq x \leq s$,*

$$u(s) \geq u(x) - a(s)G^{-1}\Big[\int_x^s b(m)G(u(m))dm + \int_x^s b(m)(\int_m^s c(t)G(u(t))dt)dm\Big] \tag{5.4.272}$$

holds where $s \in \mathbb{R}_+^n$. Then for all $0 \leq x \leq s$,

$$u(s) \geq u(x)G^{-1}\left(\left[1 + G(a(s)) \int_x^s b(m) \times \exp\left(\int_m^s (b(t)G(a(s)) + c(t))dt\right) dm\right]^{-1}\right). \tag{5.4.273}$$

Proof Obviously, we may rewrite (5.4.272) as

$$u(x) \leq u(s) + a(s)G^{-1}\Big[\int_x^s b(m)G(u(m))dm + \int_x^s b(m)(\int_m^s c(t)G(u(t))dt)dm\Big]. \tag{5.4.274}$$

Since G is sub-additive, we derive from (5.4.274)

$$G(u(x)) \leq G(u(s)) + G(a(s))\Big[\int_x^s b(m)G(u(m))dm + \int_x^s b(m)(\int_m^s c(t)G(u(t))dt)dm\Big]. \tag{5.4.275}$$

Defining $r(x)$ by the right-hand side of (5.4.275) and following an argument similar to that in the proof of Theorem 5.4.39 with suitable modifications, we can obtain the desired bound in (5.4.273). □

The next result, due to Singare and Pachpatte [600], is the n independent variable generalizations of the integral inequalities of Gollwitzer [231], Langenhop [351] and Pachpatte [456, 457] which generalize the results of Bondge and Pachpatte [91].

First, we shall give some n independent variable generalizations of the integral inequalities of Gollwizer [231] and Langenhop [351]. To do this, we use the following notations.

Let

$$D(x,s) = \{\xi \in \mathbb{R}^n; x \le \xi \le s\} \subset \Omega.$$

A useful n independent variable generalization of Gollwitzer's inequality can be stated in the following theorem.

Theorem 5.4.42 (Singare-Pachpatte [600]) *Let $\phi(x), a(x), b(x)$ be real-valued non-negative continuous functions defined on Ω; let $u(x)$ be a positive real-valued continuous function defined on Ω, and suppose further that the inequality, for all $x \le s; x, s \in \Omega$,*

$$u(x) \ge \phi(x) - a(s) \int_x^s b(\xi)\phi(\xi)d\xi \tag{5.4.276}$$

holds. Then for all $x \le s; x, s \in \Omega$,

$$u(s) \ge \phi(x) \exp\Big(- a(s) \int_0^s b(\xi)d\xi\Big). \tag{5.4.277}$$

Proof We may rewrite (5.4.276) as

$$\phi(x) \ge u(s) + a(s) \int_x^s b(\xi)\phi(\xi)d\xi. \tag{5.4.278}$$

For fixed $s \in \Omega$, we define for all $x \le s$,

$$r(x) = u(s) + a(s) \int_x^s b(\xi)\phi(\xi)d\xi \tag{5.4.279}$$

whence

$$r(s_1, x_2, \cdots, x_n) = \cdots = r(x_1, \cdots, x_{n-1}, s_n) = u(s_1, \cdots, s_n).$$

Then from (5.4.278)–(5.4.279) it follows

$$D_1 r(x) = -a(s) \int_{x_2}^{s_2} \cdots \int_{x_n}^{s_n} b(x_1, \xi_2, \cdots, \xi_n)\phi(x_1, \xi_2, \cdots, \xi_n)d\xi_n \cdots d\xi_2 \tag{5.4.280}$$

and from (5.4.280) we derive

$$D_1D_2r(x) = a(s)\int_{x_3}^{s_3}\cdots\int_{x_n}^{s_n} b(x_1,x_2,\xi_3,\cdots,\xi_n)\phi(x_1,x_2,\xi_3,\cdots,\xi_n)d\xi_n\cdots d\xi_3 \tag{5.4.281}$$

and in general, we get

$$D_1\cdots D_kr(x) = (-1)^k a(s)\int_{x_{k+1}}^{s_{k+1}}\cdots\int_{x_n}^{s_n} b(x_1,\cdots x_k,\xi_{k+1},\cdots,\xi_n) \times\phi(x_1,\cdots,x_k,\xi_{k+1}.\cdots,\xi_n)d\xi_n\cdots d\xi_{k+1}. \tag{5.4.282}$$

Continuing in this way, we can obtain

$$D_1\cdots D_nr(x) = (-1)^n a(s)b(x)\phi(x). \tag{5.4.283}$$

Now, we consider the following two cases.

Case I If the order n of the derivatives in (5.4.283) is even, then from (5.4.283) it follows

$$D_1\cdots D_nr(x) = a(s)b(x)\phi(x) \tag{5.4.284}$$

which, in view of (5.4.278), implies

$$D_1\cdots D_nr(x) \le a(s)b(x)r(x) \tag{5.4.285}$$

i.e.,

$$\frac{D_1\cdots D_nr(x)}{r(x)} \le a(s)b(x). \tag{5.4.286}$$

From (5.4.286) we see that

$$\frac{r(x)[D_1\ldots D_nr(x)]}{r^2(x)} \le a(s)b(x) + \frac{D_nr(x)[D_1\ldots D_{n-1}r(x)]}{r^2(x)}. \tag{5.4.287}$$

We see from (5.4.279) that $D_nr(x)$ and $D_1\ldots D_{n-1}r(x)$ are both non-positive, which further implies that $D_nr(x)[D_1\cdots D_{n-1}r(x)]$ is non-negative and hence (5.4.287) is true. Now (5.4.287) is equivalent to

$$D_n\Big(\frac{D_1\cdots D_{n-1}r(x)}{r(x)}\Big) \le a(s)b(x).$$

Now keeping $x_1, \cdots, x_{n-1}$ fixed in the above inequality, setting $x_n = \xi_n$ and then integrating with respect to ξ_n from x_n to s_n, we obtain

$$\frac{D_1 \cdots D_{n-1} r(x)}{r(x)} \geq -a(s) \int_{x_n}^{s_n} b(x_1, \cdots, x_{n-1}, \xi_n) d\xi_n. \tag{5.4.288}$$

Again from (5.4.288), it follows

$$\frac{r(x)[D_1 \cdots D_{n-1} r(x)]}{r^2(x)} \geq -a(s) \int_{x_n}^{s_n} b(x_1, \cdots, x_{n-1}, \xi_n) d\xi_n + \frac{D_{n-1} r(x)[D_1 \cdots D_{n-2} r(x)]}{r^2(x)}. \tag{5.4.289}$$

As before, we can also see that $D_{n-1} r(x)$ is non-positive and $D_1 \cdots D_{n-2} r(x)$ is non-negative, which implies that $D_{n-1} r(x)[D_1 \cdots D_{n-2} r(x)]$ is non-positive and hence (5.4.289) is true. But (5.4.289) is equivalent to

$$D_{n-1}\Big(\frac{D_1 \cdots D_{n-2} r(x)}{r(x)}\Big) \geq -a(s) \int_{x_n}^{s_n} b(x_1, \cdots, x_{n-1}, \xi_n) d\xi_n.$$

Now keeping $x_1, \cdots, x_{n-2}, x_n$ fixed in the above inequality, setting $x_{n-1} = \xi_{n-1}$ and then integrating with respect to ξ_{n-1} from x_{n-1} to s_{n-1}, we obtain,

$$\frac{D_1 \cdots D_{n-2} r(x)}{r(x)} \leq a(s) \int_{x_{n-1}}^{s_{n-1}} \int_{x_n}^{s_n} b(x_1, \cdots, x_{n-2}, \xi_{n-1}, \xi_n) d\xi_n d\xi_{n-1}. \tag{5.4.290}$$

Proceeding in this way, we finally conclude

$$\frac{D_1 r(x)}{r(x)} \geq -a(s) \int_{x_2}^{s_2} \int_{x_n}^{s_n} b(x_1, \xi_2 \cdots, \xi_n) d\xi_n \cdots d\xi_2. \tag{5.4.291}$$

Now keeping $x_2, \cdots, x_n$ fixed in (5.4.291), setting $x_1 = \xi_1$ and then integrating with respect to ξ_1 from x_1 to s_1, we derive

$$r(x) \leq u(s) \exp\Big(a(s) \int_x^s b(\xi) d\xi\Big), \tag{5.4.292}$$

which, substituted into (5.4.278), gives us (5.4.277).

Case II If the order n of the derivatives in (5.4.283) is odd, then from (5.4.283) it follows

$$D_1 \cdots D_n r(x) = -a(s) b(x) \phi(x) \tag{5.4.293}$$

which, in view of (5.4.278), implies

$$D_1 \cdots D_n r(x) \geq a(s)b(x)r(x),$$

i.e.,

$$\frac{D_1 \cdots D_n r(x)}{r(x)} \geq a(s)b(x). \tag{5.4.294}$$

The rest of the proof for case II is the same as that for case I and the final inequality (5.4.292) remains unchanged since n is now odd. □

Remark 5.4.15 In Theorem 5.4.42, if we take $a(s) = M$, where $M > 0$ is a constant, then (5.4.277) reduces to

$$u(s) \geq \phi(x) \exp\Big(- M \int_x^s b(\xi)d\xi\Big).$$

In next Theorem 5.4.43, we introduce the following n independent variable generalization in [587] of the integral inequality established by Langenhop [351].

Theorem 5.4.43 (Singare-Pachpatte [600]) *Let $u(x), a(x)$ and $b(x)$ be as defined in Theorem 5.4.42; let $W(r)$ be a positive, continuous, monotonic non-decreasing function for all $r > 0, W(0) = 0$ and $W'(r)$ exist and is continuous, with $W'(r) \geq 0$ for all $r \geq 0$; and suppose further that the inequality*

$$u(s) \geq u(x) - a(s) \int_x^s b(\xi)W(u(\xi))d\xi \tag{5.4.295}$$

is satisfied for all $x \leq s; x, s \in \Omega$. Then, for $\Omega_1 \subset \Omega$,

$$u(s) \geq G^{-1}\Big[G(u(x)) - a(s)(\int_x^s b(\xi)d\xi)\Big] \tag{5.4.296}$$

where

$$G(r) = \int_{r_0}^r \frac{ds}{W(s)}, \quad r > 0 \tag{5.4.297}$$

where r_0 is any fixed positive number; G^{-1} is the inverse of function of G, and Ω_1 is such that

$$G(u(x)) - a(s)(\int_x^s b(\xi)d\xi) \in Dom \ \ (G^{-1})$$

for all $x \leq s, x, s \in \Omega_1 \subset \Omega$.

Proof We may rewrite (5.4.295) as

$$u(x) \leq u(s) + a(s)(\int_x^s b(\xi)W(u(\xi))d\xi). \tag{5.4.298}$$

For fixed $s \in \Omega$, we define for $x \leq s, x \in \Omega$,

$$r(x) = u(s) + a(s)(\int_x^s b(\xi)W(u(\xi))d\xi). \tag{5.4.299}$$

so

$$r(s_1, x_2, \ldots, x_n) = \ldots = r(x_1, \ldots, x_{n-1}, s_n) = u(s_1, \ldots, s_n).$$

Then by the same argument as in the proof of Theorem 5.4.42, we obtain in general from (5.4.299) that

$$\begin{aligned} D_1 \ldots D_k r(x) = (-1)^k a(s) \int_{x_{k+1}}^{s_{k+1}} \ldots \int_{x_n}^{s_n} b(x_1, \ldots x_k, \xi_{k+1}, \ldots, \xi_n) \\ \times W(u(x_1, \ldots, x_k \xi_{k+1}, \ldots, \xi_n)) d\xi_n \ldots d\xi_{k+1} \end{aligned} \tag{5.4.300}$$

and continuing in this way, we obtain

$$D_1 \ldots D_n r(x) = (-1)^n a(s) b(x) W(u(x)). \tag{5.4.301}$$

We now consider the following two cases.

Case I If the order n of the derivatives in (5.4.301) is even, then from (5.4.301) we have

$$D_1 \ldots D_n r(x) = a(s) b(x) W(u(x)) \tag{5.4.302}$$

which, in view of (5.4.298), implies

$$D_1 \ldots D_n r(x) \leq a(s) b(x) W(r(x)) \tag{5.4.303}$$

i.e.,

$$\frac{D_1 \ldots D_n r(x)}{W(r(x))} \leq a(s) b(x). \tag{5.4.304}$$

From (5.4.304) we derive that

$$\frac{W(r(x))[D_1 \ldots D_n r(x)]}{W^2(r(x))} \leq a(s) b(x) + \frac{W'(r(x)) \cdot D_n(r(x))[D_1 \ldots D_{n-1} r(x)]}{W^2(r(x))}. \tag{5.4.305}$$

For, by (5.4.307) we see that $D_n r(x)$ and $D_1 \ldots D_{n-1} r(x)$ are both non-positive which implies that $D_n r(x)[D_1 \ldots D_{n-1} r(x)]$ is non-negative and hence (5.4.305) is true. Now (5.4.305) is equivalent to

$$D_n\Big(\frac{D_1 \ldots D_{n-1} r(x)}{W(r(x))}\Big) \leq a(s)b(x).$$

Now keeping $x_1, \ldots, x_{n-1}$ fixed in the above inequality, setting $x_n = \xi_n$ and then integrating with respect to ξ_n from x_n to s_n, we have

$$\frac{D_1 \ldots D_{n-1} r(x)}{W(r(x))} \geq -a(s) \int_{x_n}^{s_n} b(x_1, \ldots, x_{n-1}, \xi_n) d\xi_n. \tag{5.4.306}$$

Again from (5.4.306), we observe that

$$\begin{aligned}\frac{W(r(x))[D_1 \ldots D_{n-1} r(x)]}{W^2(r(x))} \geq -a(s) \int_{x_n}^{s_n} & b(x_1, \ldots, x_{n-1}, \xi_n) d\xi_n \\ & + \frac{W'(r(x)) D_{n-1}(r(x))[D_1 \ldots D_{n-2} r(x)]}{W^2(r(x))}.\end{aligned} \tag{5.4.307}$$

For, (5.4.307) shows that $D_{n-1} r(x)$ is non-positive and $D_1 \ldots D_{n-2} r(x)$ is non-negative, which implies that $D_{n-1} r(x)[D_1 \ldots D_{n-2} r(x)]$ is non-positive and hence (5.4.307) is true. But (5.4.307) is equivalent to

$$D_{n-1}\Big(\frac{D_1 \ldots D_{n-2} r(x)}{W(r(x))}\Big) \geq -a(s) \int_{x_n}^{s_n} b(x_1, \ldots, x_{n-1}, \xi_n) d\xi_n.$$

Now keeping $x_1, \ldots, x_{n-2}, x_n$ fixed in the above inequality, setting $x_{n-1} = \xi_{n-1}$ and then integrating with respect to ξ_{n-1} from x_{n-1} to s_{n-1}, we have,

$$\frac{D_1 \ldots D_{n-2} r(x)}{W(r(x))} \leq a(s) \int_{x_{n-1}}^{s_{n-1}} \int_{x_n}^{s_n} b(x_1, \ldots, x_{n-2}, \xi_{n-1}, \xi_n) d\xi_n d\xi_{n-1}.$$

Proceeding in this way, we finally obtain

$$\frac{D_1 r(x)}{W(r(x))} \geq -a(s) \int_{x_2}^{s_2} \int_{x_n}^{s_n} b(x_1, \xi_2 \ldots, \xi_n) d\xi_n \ldots d\xi_2. \tag{5.4.308}$$

From (5.4.305) and (5.4.306), we observe that,

$$D_1 G(r(x)) \geq -a(s) \int_{x_2}^{s_2} \int_{x_n}^{s_n} b(x_1, \xi_2 \ldots, \xi_n) d\xi_n \ldots d\xi_2. \tag{5.4.309}$$

Now keeping $x_2, \ldots, x_n$ fixed in (5.4.309), setting $x_1 = \xi_1$ and then integrating with respect to ξ_1 from x_1 to s_1, we have

$$G(r(x)) \leq G(u(x)) + a(s) \int_x^s b(\xi) d\xi, \tag{5.4.310}$$

which implies

$$G(u(s)) \geq G(u(x)) - a(s) \int_x^s b(\xi) d\xi. \tag{5.4.311}$$

Then (5.4.296) follows from (5.4.311). The sub-domain Ω_1 of Ω is obvious.

Case II If the order n is odd; (5.4.301) becomes

$$D_1 \ldots D_n r(x) = -a(s) b(x) W(u(x))$$

and the proof proceeds exactly as in case I, again leading to (5.4.311). □

Remark 5.4.16 We note that in Theorem 5.4.43, if we take $W(u) = u$, then (5.4.296) reduces to

$$u(s) \geq u(x) \exp(-a(s) \int_x^s b(\xi) d\xi)$$

and if we set, $W(u) = u^\alpha, 0 < \alpha < 1$, then (5.4.296) reduces to

$$u(s) \geq \Big[u(x)^\beta - \beta a(s) \int_x^s b(\xi) d\xi \Big]$$

where $\alpha + \beta = 1$.

The next deals with the n independent variable generalization in [587] of the integral inequality.

Theorem 5.4.44 (Shih-Yeh [587]) *Let $\phi(x), a(x), b(x)$ and $c(x)$ be real-valued non-negative continuous functions defined on Ω; let $u(s)$ be a positive real-valued continuous function defined on Ω; and suppose further that the inequality*

$$u(s) \geq \phi(x) - a(s) \Big[\int_x^s b(\xi) \phi(\xi) d\xi + \int_x^s b(\xi) (\int_\xi^s c(\zeta) d\zeta) d\xi \Big] \tag{5.4.312}$$

is satisfied for all $x \leq s; x, s \in \Omega$. Then

$$u(s) \geq \phi(x) \Big[1 + a(s) (\int_x^s b(\xi) \exp(\int_\xi^s [a(s) b(\zeta) + c(\zeta)] d\zeta) d\xi \Big]^{-1} \tag{5.4.313}$$

for all $x \leq s; x, s \in \Omega$.

Proof We may rewrite (5.4.312) as

$$\phi(x) \leq u(s) + a(s)\Big[\int_x^s b(\xi)\phi(\xi)d\xi + \int_x^s b(\xi)(\int_\xi^s c(\zeta)d\zeta)d\xi\Big]. \quad (5.4.314)$$

For fixed $s \in \Omega$, we define for all $x \leq s; x \in \Omega$,

$$r(x) \leq u(s) + a(s)\Big[\int_x^s b(\xi)\phi(\xi)d\xi + \int_x^s b(\xi)(\int_\xi^s c(\zeta)d\zeta)d\xi\Big] \quad (5.4.315)$$

so

$$r(s_1, x_2, \dots, x_n) = \dots = r(x_1, \dots, x_{n-1}, s_n) = u(s_1, \dots, s_n).$$

Then, following the same argument as in the proof of Theorem 5.4.43, we obtain in general from (5.4.315) that

$$\begin{aligned} D_1 \dots D_k r(x) = (-1)^k \Big[& \int_{x_{k+1}}^{s_{k+1}} \dots \int_{x_n}^{s_n} b(x_1, \dots, x_k, \xi_{k+1}, \dots, \xi_n) \\ & \times \phi(x_1, \dots, x_k, \xi_{k+1}, \dots, \xi_n) d\xi_n \dots d\xi_{k+1} \\ & + \int_{x_{k+1}}^{s_{k+1}} \dots \int_{x_n}^{s_n} b(x_1, \dots, x_k, \xi_{k+1}, \dots, \xi_n) \\ & \times (\int_{\xi_{k+1}}^{s_{k+1}} \dots \int_{\xi_n}^{s_n} c(\xi_1, \dots, \xi_k, \zeta_{k+1}, \dots, \zeta_n) \\ & \times \phi(\xi_1, \dots, \xi_k, \zeta_{k+1}, \dots, \zeta_n) d\zeta_n \dots d\zeta_{k+1}) d\xi_n \dots d\xi_{k+1} \Big] \end{aligned} \quad (5.4.316)$$

and continuing in this way, we obtain

$$D_1 \dots D_n r(x) = (-1)^n a(s) b(x)\Big[\phi(x) + \int_x^s c(\xi)\phi(\xi)d\xi\Big]. \quad (5.4.317)$$

We now consider the following two cases.

Case I If the order n of the derivatives in (5.4.317) is even, then from (5.4.317) and (5.4.314), we have

$$D_1 \dots D_n r(x) \leq a(s)b(x)\Big[r(x) + \int_x^s c(\xi)r(\xi)d\xi\Big]. \quad (5.4.318)$$

In (5.4.318) if we put

$$v(x) = r(x) + \int_x^s c(\xi)r(\xi)d\xi \quad (5.4.319)$$

so

$$\nu(s_1, x_2, \ldots, x_n) = \ldots = \nu(x_1, \ldots, x_{n-1}, s_n) = r(s_1, \ldots, s_n) = u(s_1, \ldots, s_n).$$

Then we have

$$D_1 \ldots D_n \nu(x) = D_1 \ldots D_n r(x) + c(x) r(x) \tag{5.4.320}$$

since the order of the derivative is even. Using (5.4.318) and the fact that $r(x) \leq \nu(x)$ in (5.4.320), we have

$$D_1 \ldots D_n \nu(x) \leq [a(s)b(x) + c(x)]\nu(x).$$

Now repeating the argument used in the proof of Theorem 5.4.43, we obtain

$$\nu(x) \leq u(s) \exp\left(\int_x^s (a(s)b(\xi) + c(\xi)] d\xi\right).$$

Now substituting this bound for $\nu(x)$ into (5.4.318) and carrying out n successive integrations, using the fact that

$$D_1 \ldots D_k r(x_1, \ldots, \xi_{k+1}, \ldots, x_n) = 0$$

for $\xi_{k+1} = s_{k+1}$ by (5.4.318), we obtain

$$r(x) \leq u(s)\Big[1 + a(s) \int_x^s b(\xi) \exp(\int_\xi^s [a(s)b(\zeta) + c(\zeta)] d\zeta) d\xi\Big]. \tag{5.4.321}$$

Substituting this bound on $r(x)$ in (5.4.314), we obtain (5.4.313).

Case II If the order n is odd, (5.4.316) becomes

$$D_1 \ldots D_n r(x) = -a(s)b(x)\Big[r(x) + \int_x^s c(\xi) r(\xi) d\xi\Big] \tag{5.4.322}$$

and the proof proceeds exactly as in case I, again leading to (5.4.311). □

The next result deals with the n independent variable generalization in [600] of the integral inequality by Pachpatte [457].

Theorem 5.4.45 (Singare-Pachpatte [600]) *Let $\phi(x), a(x), b(x)$ and $c(x)$ be real-valued non-negative continuous functions defined on Ω; let $u(s)$ be a positive real-valued continuous function defined on Ω; and suppose further that the inequality for all $x \leq s; x, s \in \Omega$,*

$$u(s) \geq \phi(x) - a(s)\Big[\int_x^s b(\xi)\phi(\xi) d\xi + \int_x^s b(\xi)(\int_\xi^s c(\zeta) d\zeta) d\xi\Big] \tag{5.4.323}$$

holds. Then for all $x \le s; x, s \in \Omega$,

$$u(s) \ge \phi(x)\Big[1 + a(s)(\int_x^s b(\xi)\exp(\int_\xi^s [a(s)b(\zeta) + c(\zeta)]d\zeta)d\xi\Big]^{-1}. \quad (5.4.324)$$

Proof Obviously, we may rewrite (5.4.323) as

$$\phi(x) \le u(s) + a(s)\Big[\int_x^s b(\xi)\phi(\xi)d\xi + \int_x^s b(\xi)(\int_\xi^s c(\zeta)d\zeta)d\xi\Big]. \quad (5.4.325)$$

For fixed $s \in \Omega$, we define for all $x \le s; x \in \Omega$,

$$r(x) = u(s) + a(s)\Big[\int_x^s b(\xi)\phi(\xi)d\xi + \int_x^s b(\xi)(\int_\xi^s c(\zeta)d\zeta)d\xi\Big], \quad (5.4.326)$$

whence

$$r(s_1, x_2, \cdots, x_n) = \cdots = r(x_1, \cdots, x_{n-1}, s_n) = u(s_1, \cdots, s_n).$$

Then, by following the same argument as in the proof of Theorem 5.4.42, it follows from (5.4.326) that

$$\begin{aligned} D_1 \cdots D_k r(x) = (-1)^k \Big[& \int_{x_{k+1}}^{s_{k+1}} \cdots \int_{x_n}^{s_n} b(x_1, \cdots, x_k, \xi_{k+1}, \cdots, \xi_n) \\ & \times \phi(x_1, \cdots, x_k, \xi_{k+1}, \cdots, \xi_n) d\xi_n \cdots d\xi_{k+1} \\ & + \int_{x_{k+1}}^{s_{k+1}} \cdots \int_{x_n}^{s_n} b(x_1, \cdots, x_k, \xi_{k+1}, \cdots, \xi_n) \\ & \times \left(\int_{\xi_{k+1}}^{s_{k+1}} \cdots \int_{\xi_n}^{s_n} c(\xi_1, \cdots, \xi_k, \zeta_{k+1}, \cdots, \zeta_n \right) \\ & \times \phi(\xi_1, \cdots, \xi_k, \zeta_{k+1}, \cdots, \zeta_n) d\zeta_n \cdots d\zeta_{k+1}) d\xi_n \cdots d\xi_{k+1} \Big] \quad (5.4.327) \end{aligned}$$

and continuing in this way, we can obtain

$$D_1 \cdots D_n r(x) = (-1)^n a(s)b(x)\Big[\phi(x) + \int_x^s c(\xi)\phi(\xi)d\xi\Big]. \quad (5.4.328)$$

We now consider the following two cases.

Case I If the order n of the derivatives in (5.4.328) is even, then we infer from (5.4.328) and (5.4.325),

$$D_1 \cdots D_n r(x) \le a(s)b(x)\Big[r(x) + \int_x^s c(\xi)r(\xi)d\xi\Big]. \quad (5.4.329)$$

If we put in (5.4.329)

$$\nu(x) = r(x) + \int_x^s c(\xi)r(\xi)d\xi, \tag{5.4.330}$$

then

$$\nu(s_1, x_2, \cdots, x_n) = \cdots = \nu(x_1, \cdots, x_{n-1}, s_n) = r(s_1, \cdots, s_n) = u(s_1, \cdots, s_n).$$

Since the order of the derivative is even, we get

$$D_1 \cdots D_n \nu(x) = D_1 \cdots D_n r(x) + c(x)r(x). \tag{5.4.331}$$

Using (5.4.329) and the fact that $r(x) \le \nu(x)$ in (5.4.331), we obtain

$$D_1 \cdots D_n \nu(x) \le [a(s)b(x) + c(x)]\nu(x).$$

Now repeating the same argument as that used in the proof of Theorem 5.4.42, we conclude

$$\nu(x) \le u(s) \exp\left(\int_x^s (a(s)b(\xi) + c(\xi)]d\xi\right).$$

Now inserting the above bound for $\nu(x)$ into (5.4.329) and carrying out n successive integrations, using the fact that

$$D_1 \cdots D_k r(x_1, \cdots, \xi_{k+1}, \cdots, x_n) = 0$$

for $\xi_{k+1} = s_{k+1}$ by (5.4.327), we obtain

$$r(x) \le u(s)\Big[1 + a(s)(\int_x^s b(\xi) \exp\left(\int_\xi^s [a(s)b(\zeta) + c(\zeta)]d\zeta\right) d\xi\Big]. \tag{5.4.332}$$

Substituting this bound on $r(x)$ in (5.4.325), we can obtain (5.4.324).

Case II If the order n is odd, then (5.4.327) becomes

$$D_1 \cdots D_n r(x) = -a(s)b(x)\Big[r(x) + \int_x^s c(\xi)r(\xi)d\xi\Big] \tag{5.4.333}$$

and the proof proceeds exactly as in case I, also leading to (5.4.332). □

Now applying Theorem 5.4.45, we can establish the following n independent variable generalization in [600] of the integral inequality by Pachpatte [457].

Theorem 5.4.46 (Singare-Pachpatte [600]) *Let $\phi(x), a(x), b(x), c(x)$ and $u(x)$ be as defined in Theorem 5.4.45; let $H(r), \alpha(s)$ and $\beta(s)$ be as defined in Theorem 5.4.40, and suppose further that the inequality for all $x \leq s; x, s \in \Omega$,*

$$u(s) \geq \phi(x) - a(s)H^{-1}\Big[\int_x^s b(\xi)H(\phi(\xi))d\xi + \int_x^s b(\xi)(\int_\xi^s c(\zeta)H(\phi(\zeta))d\zeta)d\xi\Big] \tag{5.4.334}$$

holds. Then for all $x \leq s; x, s \in \Omega$,

$$u(s) \geq \alpha(s)H^{-1}\Bigg[\alpha^{-1}(s)H(\phi(x))\Big(1 + \beta(s)H(a(s)\beta^{-1}(s))\int_x^s b(\xi) \times \exp(\int_\xi^s \big[\beta(s)H(a(s)\beta^{-1}(s))b(\zeta) + c(\zeta)\big]d\zeta)d\xi\Big)^{-1}\Bigg]. \tag{5.4.335}$$

Proof In fact, we may rewrite (5.4.334) as

$$\phi(x) \leq \alpha(s)u(s)\alpha^{-1}(s) + \beta(s)a(s)\beta^{-1}(s)H^{-1}\Big[\int_x^s b(\xi)H(\phi(\xi))d\xi + \int_x^s b(\xi)(\int_\xi^s c(\zeta)H(\phi(\zeta))d\zeta)d\xi\Big].$$

Since H is convex, sub-multiplicative and monotonic, we get

$$\alpha(s)H(u(s)\alpha^{-1}(s)) \geq H(\phi(x)) - \beta(s)H(a(s)\beta^{-1}(s))H^{-1}\Big[\int_x^s b(\xi)H(\phi(\xi))d\xi + \int_x^s b(\xi)(\int_\xi^s c(\zeta)H(\phi(\zeta))d\zeta)d\xi\Big].$$

Now applying Theorem 5.4.45 yields the desired bound in (5.4.335). □

Remark 5.4.17 We note that, if $H(u) = u$, then Theorem 5.4.46 reduces to Theorem 5.4.45.

Next we introduce the following n independent variable generalization in [600] of the integral inequality established by Pachpatte [456].

Theorem 5.4.47 (Singare-Pachpatte [600]) *Let $a(x), b(x)$ and $c(x)$ be real-valued non-negative continuous functions defined on Ω; let $u(s)$ be a positive real-valued continuous function defined on Ω. Let $G(r)$ be positive, continuous, strictly increasing, sub-additive and sub-multiplicative function for all $r > 0$ with $G(0) = 0$, and let G^{-1} denote the inverse function of G. Suppose further that the inequality*

for all $x \le s; x, s \in \Omega$,

$$u(s) \ge \phi(x) - a(s)G^{-1}\Big[\int_x^s b(\xi)G(\phi(\xi))d\xi + \int_x^s b(\xi)(\int_\xi^s c(\zeta)G(\phi(\zeta))d\zeta)d\xi\Big] \tag{5.4.336}$$

holds, where $\phi(x)$ *is continuous and non-negative on* Ω. *Then for all* $x \le s; x, s \in \Omega$,

$$u(s) \ge G^{-1}\Big[G(\phi(x))\Big(1 + G(a(s))\int_x^s b(\xi) \times \exp(\int_\xi^s [G(a(s))b(\zeta) + c(\zeta)]d\zeta)d\xi\Big)^{-1}\Big]. \tag{5.4.337}$$

Proof The proof follows by the similar argument as that in the proof of Theorem 5.4.41 given in Pachpatte [456] and using Theorem 5.4.45 with suitable modifications. We omit here the details. □

The notation used in next theorem is following. Let Ω be an open bounded set in $\mathbb{R}^n$ and let a point $(x_1, \cdots, x_n)$ in Ω be denoted by x. Let x^0 and x $(x^0 < x)$ be any two points in Ω, let D denote the parallelepiped defined by $x^0 < \xi < x$, that is, $x_i^0 < \xi_i < x_i$, $1 \le i \le n$. Let $u_x(x) = D_1 \cdots D_n u(x)$, $D_i = \partial/\partial x_i$.

In next result, we shall denote the sum of all functions by $f_1(x) + \cdots + f_n(x) \cup g_1(x) \cup \cdots \cup g_r(x)$ except if any $g_i(x) = f_j(x)$; then $g_i(x)$ is taken to be zero.

Lemma 5.4.6 (Young [677]) *Suppose* $a(x)$, *and* $b(x) \ge 0$ *are continuous functions in* Ω. *Let* $v(\xi; x)$ *be the solution of the characteristic initial value problem (5.4.206) and let* D^+ *be a connected sub-domain of* Ω *containing* x *such that* $v \ge 0$ *for all* $\xi \in D^+$. *If* $D \subset D^+$ *and*

$$\phi_x(x) - b(x)u(x) \le a(x)b(x),$$

where u *vanishes together with all its mixed derivatives up to order* $n-1$ *on* $x_i = x_i^0$, $i = 1, \cdots, n$, *then*

$$\phi(x) \le \int_{x^0}^x a(\xi)b(\xi)v(\xi; x)d\xi.$$

Theorem 5.4.48 (Agarwal [4]) *Suppose that* $u(x)$, $a(x)$, $b(x)$, $c(x)$, $h(x)$, $p(x)$, *and* $q(x)$ *are continuous and non-negative functions on* Ω. *Let* $v(s, x)$, *and* $w(s, x)$ *be the solutions of the characteristic initial value problems*

$$\begin{cases} (-1)^n v_s(s, x) - [b(s) + h(s) + q(s) \cup c(s) \cup p(s)]v(s, x) = 0, \\ v(s, x) = 1 \qquad on \;\; s_i = x_i, \quad i = 1, \cdots, n, \end{cases} \tag{5.4.338}$$

and

$$\begin{cases} (-1)^n w_s(s,x) - [b(s) + h(s) - p(s) \cup c(s)]v(s,x) = 0, \\ w(s,x) = 1 \qquad on \;\; s_i = x_i, \quad i = 1, \cdots, n, \\ (-1)^n e_s(s,x) - [b(s) - c(s)]e(s,x) = 0, \\ e(s,x) = 1 \qquad on \;\; s_i = x_i, \quad i = 1, \cdots, n, \end{cases} \tag{5.4.339}$$

and let D^+ be a connected sub-domain of Ω containing x such that $v \geq 0$, $w \geq 0$ and $e \geq 0$ for all $s \in D^+$. If $D \subset D^+$ and

$$\begin{aligned} u(x) \leq a(x) &+ \int_{x_0}^{x} b(s)u(s)ds + \int_{x_0}^{x} c(s)\left(\int_{x_0}^{s} h(t)u(t)dt\right)ds \\ &+ \int_{x_0}^{x} c(s)\left(\int_{x_0}^{s} p(t)\left(\int_{x_0}^{t} q(\alpha)u(\alpha)d\alpha\right)dt\right)ds, \end{aligned} \tag{5.4.340}$$

then

$$\begin{aligned} u(x) \leq a(x) &+ \int_{x_0}^{x} e(s,x)\Big[a(s)b(s) + c(s)\int_{x_0}^{s} w(t,s) \\ &\times \Big\{a(t)b(t) + h(t) + p(t)\int_{x_0}^{t} v(\alpha,t)a(\alpha)[b(\alpha) + h(\alpha) + q(\alpha)]d\alpha\Big]dt\Big\}ds. \end{aligned} \tag{5.4.341}$$

Proof First we note that inequality (5.4.340) is equivalent to the system

$$\begin{cases} u_1(x) \leq a(x) + \displaystyle\int_{x_0}^{x} [b(s)u_1(s) + c(s)u_2(s)]ds, \\ u_2(x) = \displaystyle\int_{x_0}^{x} [h(s)u_1(s) + p(s)u_3(s)]ds, \\ u_3(x) = \displaystyle\int_{x_0}^{x} q(s)u_1(s)ds. \end{cases}$$

Define

$$\begin{cases} R_1(x) = \displaystyle\int_{x_0}^{x} [b(s)u_1(s) + c(s)u_2(s)]ds, \\ R_2(x) = \displaystyle\int_{x_0}^{x} [h(s)u_1(s) + p(s)u_3(s)]ds, \\ R_3(x) = \displaystyle\int_{x_0}^{x} q(s)u_1(s)ds, \end{cases}$$

then it follows that

$$\begin{cases} R_{1x}(x) \leq b(x)[a(x) + R_1(x)] + c(x)R_2(x), & (5.4.342) \\ R_{2x}(x) \leq h(x)[a(x) + R_1(x)] + p(x)R_3(x), & (5.4.343) \\ R_{3x}(x) \leq q(x)[a(x) + R_1(x)]. & (5.4.344) \end{cases}$$

Adding (5.4.342)–(5.4.344), we obtain

$$(R_1(x) + R_2(x) + R_3(x))_x \leq a(x)\Big(b(x) + h(x) + q(x)\Big) \\ + \Big(b(x) + h(x) + q(x) \cup c(x) \cup p(x)\Big)\Big(R_1(x) + R_2(x) + R_3(x)\Big).$$

Now from Lemma 5.4.6 it follows that

$$R_1(x) + R_2(x) + R_3(x) \leq \int_{x_0}^{x} v(s,x)a(s)\Big(b(s) + h(s) + q(s)\Big)ds, \tag{5.4.345}$$

whence

$$R_3(x) \leq \int_{x_0}^{x} v(s,x)a(s)\Big(b(s) + h(s) + q(s)\Big)ds - R_1(x) - R_2(x). \tag{5.4.346}$$

Adding (5.4.342) and (5.4.343) and using (5.4.343), we find

$$(R_1(x) + R_2(x))_x \leq a(x)\Big(b(x) + h(x)\Big) + p(x)\int_{x_0}^{x} v(s,x)a(s)\Big(b(s) + h(s) + q(s)\Big)ds \tag{5.4.347}$$

$$+\Big(b(x) + h(x) - p(x) \cup c(x)\Big)\Big(R_1(x) + R_2(x)\Big). \tag{5.4.348}$$

Using Lemma 5.4.6 once again, we obtain

$$R_1(x) + R_2(x) \\ \leq \int_{x_0}^{x}\left[a(s)(b(s) + h(s)) + p(s)\int_{x_0}^{s} v(t,s)a(t)(b(t) + h(t) + q(t))dt\right]w(s,x)ds. \tag{5.4.349}$$

Using the estimate for $R_2(x)$ from (5.4.349) in (5.4.342), we derive

$$R_{1x}(x) \leq a(x)b(x) + (b(x) - c(x))R_1(x) + c(x)\int_{x_0}^{x}\Big[a(s)(b(s) + h(s)) \\ + p(s)\int_{x_0}^{s} v(t,s)a(t)(b(t) + h(t) + q(t))dt\Big]w(s,x)ds,$$

whence, from Lemma 5.4.6 it follows that

$$R_1(x) \le \int_{x_0}^{x} e(s,x)\Big\{ a(s)b(s) + c(s)\int_{x_0}^{s} w(t,s)\Big[a(t)(b(t)+h(t))$$
$$+p(t)\int_{x_0}^{s} v(\alpha,t)a(\alpha)\Big(b(\alpha)+h(\alpha)+q(\alpha)\Big)d\alpha\Big]dt\Big\}\,ds$$

and now the result follows from $u(x) \le a(x) + R_1(x)$.

Therefore from (5.4.345) and (5.4.349), it follows that

$$u(x) \le a(x) + \int_{x_0}^{x} v(s,x)a(s)\Big(b(s)+h(s)+q(s)\Big)ds, \tag{5.4.350}$$

$$u(x) \;\le\; a(x) + \int_{x_0}^{x} w(s,x)\Big[a(s)(b(s)+h(s))$$
$$+p(s)\int_{x_0}^{s} v(t,s)a(t)(b(t)+h(t)+q(t))dt\Big]ds. \tag{5.4.351}$$

□

Theorem 5.4.49 (Yeh [669]) *Suppose that $u(x)$, $a(x)$, $b(x)$, $c(x)$ and $q(x)$ are real-valued non-negative continuous functions defined on Ω. Let $v(s;t)$ and $w(s;x)$ be the solutions of the characteristic initial value problems*

$$\begin{cases} (-1)^n \dfrac{\partial^n v(s;x)}{\partial s_1\cdots\partial s_n} - [p(s)+b(s)(c(s)+q(s))]v(s;x) = 0, \qquad in \quad \Omega, \\ v(s;x) = 1 \qquad on \;\; s_i = x_i,\; i = 1,\cdots,n, \end{cases} \tag{5.4.352}$$

and

$$\begin{cases} (-1)^n \dfrac{\partial^n w(s;x)}{\partial s_1\cdots\partial s_n} - [b(s)c(s)-p(s)]w(s;x) = 0, \qquad in \quad \Omega, \\ w(s;x) = 1 \qquad on \; s_i = x_i,\; i = 1,\cdots,n, \end{cases} \tag{5.4.353}$$

respectively, and let D^+ be a connected sub-domain of Ω which contains x such that $v \ge 0$ and $w \ge 0$ for all $s \in D^+$. If $D \subset D^+$ and

$$u(x) \le a(x) + b(x)\left[\int_{x^0}^{x} c(s)u(s)ds + \int_{x^0}^{x} p(s)\left(\int_{x^0}^{s} q(t)u(t)dt\right)ds\right], \tag{5.4.354}$$

then

$$u(x) \le a(x) + b(x)\Big[\int_{x^0}^{x} w(s;x)(a(s)c(s) + p(s)\int_{x^0}^{s} a(t)[c(t) + q(t)]v(t;s)dt)ds\Big]. \tag{5.4.355}$$

Proof Let

$$h(x) = \int_{x^0}^{x} c(s)u(s)ds + \int_{x^0}^{x} p(s)\left(\int_{x^0}^{s} q(t)u(t)dt\right)ds. \tag{5.4.356}$$

Then

$$\begin{cases} D_1 \cdots D_n h(x) = c(x)u(x) + p(x)\displaystyle\int_{x^0}^{x} q(t)u(t)dt, \\ h(x) = 0, \qquad on\ \ x_i = x_i^0,\ i = 1, \cdots n. \end{cases} \tag{5.4.357}$$

Thus it follows from (5.4.354) that

$$\begin{aligned} D_1 \cdots D_n h(x) \le{}& c(x)[a(x) + b(x)h(x)] \\ &+ p(x)\left(\int_{x^0}^{x} q(t)[a(t) + b(t)h(t)]dt\right). \end{aligned} \tag{5.4.358}$$

Adding $p(x)h(x)$ to both sides of the above inequality (5.4.358), we get

$$\begin{aligned} D_1 \ldots D_n h(x) + p(x)h(x) \le{}& c(x)[a(x) + b(x)h(x)] \\ &+ p(x)\left(h(x) + \int_{x^0}^{x} q(t)[a(t) + b(t)h(t)]dt\right). \end{aligned} \tag{5.4.359}$$

Set

$$k(x) = h(x) + \int_{x^0}^{x} q(t)[a(t) + b(t)h(t)]dt,$$

then

$$\begin{cases} k(x) = h(x) = 0, \qquad on\ x_i = x_i^0,\ \ i = 1, \cdots, n, \\ D_1 \cdots D_n k(x) = D_1 \cdots D_n h(x) + q(x)[a(x) + b(x)h(x)], \end{cases} \tag{5.4.360}$$

and

$$h(x) \le k(x). \tag{5.4.361}$$

It follows from (5.4.358)–(5.4.361) that

$$D_1 \cdots D_n k(x) \le a(x)[c(x) + q(x)] + (p(x) + b(x)[c(x) + q(x)])k(x),$$

which implies

$$\begin{aligned} L[k(x)] &\equiv D_1 \ldots D_n k(x) - (p(x) + b(x)[c(x) + q(x)])k(x) \\ &\le a(x)[c(x) + q(x)]. \end{aligned} \tag{5.4.362}$$

Furthermore, all pure mixed derivatives of $k(x)$ with respect to $x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$ up to order $n-1$ vanish on $x_i = x_i^0$ for $i = 1, \cdots, n$. If $e(x)$ is a function which is n times continuously differentiable in D, then

$$eL[k] - kL_1[e] = \sum_{j=1}^{n} (-1)^{j-1} D_j[(D_0 D_1 \cdots D_{j-1} e)(D_{j+1} \cdots D_n D_{n+1} k)], \tag{5.4.363}$$

where

$$L_1[e] = (-1)^n D_1 \cdots D_n e(x) - [p(x) + b(x)(c(x) + q(x))]e(x)$$

with $D_0 = D_{n+1} = I$, the identity. Integrating (5.4.362) over D and noting that $k(x)$ vanishes together with all its mixed derivatives up to order $n-1$ on $s_i = x_i^0$, $i = 1, \cdots, n$, we obtain

$$\int_D (eL[k] - kL_1[e])ds = \sum_{j=1}^{n} (-1)^{j-1} \int_{s_j = x_j} (D_1 \cdots D_{j-1} e)(D_{j+1} \cdots D_n k) ds', \tag{5.4.364}$$

where $ds' = ds_1 \cdots ds_{j-1} ds_{j+1} \cdots ds_n$.

We now choose $e(x)$ as the function v satisfying (5.4.352). Since $v(s;x) = 1$ on $s_j = x_j$, $j = 1, \cdots, n$, we have

$$D_0 \cdots D_{j-1} v(s;x) = 0, \qquad on \quad s_j = x_j,\ j = 2, \cdots, n.$$

Hence (5.4.364) can be rewritten as

$$\int_D v(s;x) L[k(s)]ds = \int_{s_1 = x_1} v(s;x) D_2 \ldots D_n k(s) ds' = k(x). \tag{5.4.365}$$

By the continuity of v and the fact that $v = 1$ on $s = x$, there exists a domain D^+ containing x on which $v \ge 0$. Now multiplying both sides of (5.4.362) by v and using (5.4.365), we get

$$k(x) \le \int_{x^0}^{x} a(s)[c(s) + q(s)] v(s;x) ds. \tag{5.4.366}$$

Thus it follows from (5.4.359) and (5.4.366) that

$$M[h(x)] \equiv D_1 \ldots D_n h(x) - [b(x)c(x) + p(x)]h(x) \\ \leq a(x)c(x) + p(x) \int_{x^0}^{x} a(s)\Big(c(s) + q(s)\Big)v(s;x)ds.$$

Following the same argument as above, we can obtain

$$h(x) \leq \int_{x^0}^{x} w(s;x) \left[a(s)c(s) + p(s) \int_{x^0}^{s} a(t)[c(t) + q(t)]v(t;s)dt \right] ds.$$

Inserting this bound on $h(x)$ into (5.4.354), we can derive (5.4.355). □

A slight different version of Theorem 5.4.49 is the following theorem in [669].

Theorem 5.4.50 (Yeh [669]) *Let $\alpha, x \in \mathbb{R}^n$ with $\alpha \leq x$. Suppose that $u(x)$, $a(x)$, $b(x)$, $q(x)$, $\gamma(x)$ and $\lambda(x)$ are real-valued non-negative continuous functions defined on $D \subset \mathbb{R}^n$. Let $V(s;x)$ and $W(s;x)$ be the solutions of the characteristic initial value problems*

$$\begin{cases} (-1)^n \dfrac{\partial^n V(s;x)}{\partial s_1 \cdots \partial s_n} - b(s)[\gamma(s) + \lambda(s) + q(s)]V(s;x) = 0 \qquad in \quad D, \\ V(s;x) = 1 \qquad on\ s_i = x_i,\ i = 1, \cdots, n, \end{cases} \tag{5.4.367}$$

and

$$\begin{cases} (-1)^n \dfrac{\partial^n W(s;x)}{\partial s_1 \cdots \partial s_n} - b(s)\gamma(s)W(s;x) = 0 \qquad in \quad D, \\ W(s;x) = 1 \qquad on\ s_i = x_i,\ i = 1, \cdots, n, \end{cases} \tag{5.4.368}$$

respectively, and let D^+ be a connected sub-domain of D, containing x, on which $V \geq 0$ and $W \geq 0$ for all $s \in D^+$. If $[\alpha, x] \subset D^+$ and

$$u(x) \leq a(x) + b(x) \left[\int_{\alpha}^{x} \gamma(s)u(s)ds + \int_{\alpha}^{x} \lambda(s)(u(s) + b(s) \int_{\alpha}^{s} q(\tau)u(\tau)d\tau)ds \right], \tag{5.4.369}$$

then

$$u(x) \leq a(x) + b(x)\Big[\int_{\alpha}^{x} W(s;x)\Big\{a(s)[\gamma(s) + \lambda(s)] \\ + b(s)\lambda(s) \int_{\alpha}^{s} a(\tau)[c(\tau) + \lambda(\tau) + q(\tau)]V(\tau;s)d\tau\Big\}ds\Big]. \tag{5.4.370}$$

Proof We may rewrite (5.4.369) as

$$u(x) \le a(x) + b(x) \left[\int_\alpha^x [\gamma(s) + \lambda(s)]u(s)ds + \int_\alpha^x \lambda(s)b(s) \left(\int_\alpha^s q(\tau)u(\tau)d\tau \right) ds \right]. \tag{5.4.371}$$

Thus (5.4.370) follows from (5.4.369) with $c(x) = \gamma(x) + \lambda(x), p(x) = \lambda(x)b(x)$, and w, E replaced by W, V, respectively, since $M(x) = b(x)[\gamma(x) + \lambda(x) + q(x)], b(x)c(x) - p(x) = b(x)\gamma(x)$, and $c(x) + q(x) = \gamma(x) + \lambda(x) + q(x)$. □

Another slightly different version of Theorem 5.4.49 is stated in the next theorem.

Theorem 5.4.51 (Yeh [669]) *Suppose that $u(x)$, $a(x)$, $b(x)$, $c(x)$, and $p(x)$ are real-valued non-negative continuous functions defined on Ω. Let $v(s;x)$ be the solution of the characteristic initial value problem*

$$\begin{cases} (-1)^n \dfrac{\partial^n v(s;x)}{\partial s_1 \cdots \partial s_n} - [b(s) + c(s)]v(s;x) = 0 \qquad in \quad \Omega, \\ v(s;x) = 1 \qquad on\ s_i = x_i,\ i = 1, \cdots, n, \end{cases} \tag{5.4.372}$$

and let D^+ be a connected sub-domain of Ω which contains x such that $v \ge 0$ for all $s \in D^+$. If $D \subset D^+$ and

$$u(x) \le a(x) \int_{x^0}^x b(s) \left(u(s) + p(s) + \int_{x^0}^s c(t)u(t)dt \right) ds, \tag{5.4.373}$$

then

$$u(x) \le a(x) + \int_{x^0}^x \left[b(s)\Big(a(s) + p(s)\Big) \right.$$
$$\left. + \int_{x^0}^s \Big(a(t)c(t) + b(t)(a(t) + p(t))\Big) v(t;s)dt \right] ds. \tag{5.4.374}$$

Proof Define

$$h(x) = \int_{x^0}^x b(s) \left[u(s) + p(s) + \int_{x^0}^s c(t)u(t)dt \right] ds. \tag{5.4.375}$$

Then,

$$h(x) = 0, \quad on \quad x_i = x_i^0, \quad i = 1, \cdots, n, \tag{5.4.376}$$

and

$$D_1 \cdots D_n h(x) = b(x)\left[u(x) + p(x) + \int_{x^0}^{x} c(t)u(t)dt\right].$$

It follows from (5.4.370) that

$$D_1 \cdots D_n h(x) \le b(x)\left[a(x) + h(x) + p(x) + \int_{x^0}^{x} c(t)(a(t) + u(t))dt\right]. \tag{5.4.377}$$

Let

$$k(x) = h(x) + \int_{x^0}^{x} c(t)(a(t) + h(t))dt. \tag{5.4.378}$$

Then

$$\begin{cases} h(x) = k(x) = 0 \qquad on \qquad x_i = x_i^0, \quad i = 1, \cdots, n, \\ h(x) \le k(x) \end{cases} \tag{5.4.379}$$

and

$$D_1 \cdots D_n k(x) = D_1 \cdots D_n h(x) + c(x)(a(x) + h(x)). \tag{5.4.380}$$

Thus we derive from (5.4.377)–(5.4.380) that

$$\begin{aligned} D_1 \ldots D_n k(x) &\le b(x)[a(x) + p(x) + k(x)] + c(x)[a(x) + h(x)] \\ &\le [b(x) + c(x)]k(x) + a(x)c(x) + b(x)[a(x) + p(x)], \end{aligned}$$

which implies

$$\begin{aligned} L[k(x)] &\equiv D_1 \ldots D_n k(x) - [b(x) + c(x)]k(x) \\ &\le a(x)c(x) + b(x)[a(x) + p(x)]. \end{aligned}$$

Following the same argument as in the proof of Theorem 5.4.49, we have

$$k(x) \le \int_{x^0}^{x} \Big(a(s)c(s) + b(s)[a(s) + p(s)])v(s; x\Big)ds$$

which, together with (5.4.349), imply

$$D_1 \cdots D_n h(x) \le b(x)\Big[a(x) + p(x) + \int_{x^0}^{x} (a(s)c(s) + b(s)[a(s) + p(s)])v(s; x)ds\Big].$$

Therefore, this and (5.4.376), imply

$$h(x) \leq \int_{x^0}^{x} b(s)\Big[a(s) + p(s) + \int_{x^0}^{x} \{a(t)c(t) + b(t)(a(t) + p(t))\}v(s;x)dt\Big]ds$$

which, substituted into (5.4.372), gives us (5.4.374). □

In next two theorems, we introduce some inequalities also due to Yeh [669] .

Theorem 5.4.52 (Yeh [669]) *Suppose that $u(x)$, $a(x)$, $b(x)$, $c(x)$, $p(x)$ and $q(x)$ are real-valued non-negative continuous functions defined on Ω. Let $v(s;x)$ and $w(s;x)$ be the solutions of the characteristic initial value problems*

$$\begin{cases} (-1)^n \dfrac{\partial^n v(s;x)}{\partial s_1 \cdots \partial s_n} - [b(s) + c(s) + p(s)]v(s;x) = 0 & in \quad \Omega, \\ v(s;x) = 1 & on \ s_i = x_i, \ i = 1, \cdots, n, \end{cases} \tag{5.4.381}$$

and

$$\begin{cases} (-1)^n \dfrac{\partial^n w(s;x)}{\partial s_1 \cdots \partial s_n} - [b(s) + c(s)]w(s;x) = 0 & in \quad \Omega, \\ w(s;x) = 1 & on \ s_i = x_i, \ i = 1, \cdots, n, \end{cases} \tag{5.4.382}$$

respectively, and let D^+ be a connected sub-domain of Ω which contains x such that $v \geq 0$, and $w \geq 0$ for all $s \in D^+$. If $D \subset D^+$ and

$$u(x) \leq a(x) + \int_{x^0}^{x} b(s)\Big[u(s) + \int_{x^0}^{x} c(t)u(t)dt + \int_{x^0}^{s} c(t)\left(\int_{x^0}^{t} p(m)u(m)dm\right)dt\Big]ds, \tag{5.4.383}$$

then

$$u(x) \leq a(x) + \int_{x^0}^{x} b(s)\Big\{a(s)\int_{x^0}^{s} w(t;s)\Big[a(t)b(t) + c(t)\Big\{a(t) \tag{5.4.384}$$

$$+ \int_{x^0}^{t} v(m;t)a(m)[b(m) + c(m) + p(m)]dm\Big\}\Big]dt\Big\}ds. \tag{5.4.385}$$

Proof The proof follows the proofs of Theorems 5.4.49 and 5.4.51 with suitable modifications. We omit the details here. □

Theorem 5.4.53 (Yeh [669]) *Suppose that $f(x)$, $a(x)$, $b(x)$, $c(x)$, $g(x)$, $p(x)$ and $q(x)$ are real-valued non-negative continuous functions defined on Ω. Let*

$u(s;x)$, $v(s;x)$ and $w(s;x)$ be the solutions of the characteristic initial value problems

$$\begin{cases} (-1)^n \dfrac{\partial^n u(s;x)}{\partial s_1 \ldots \partial s_n} - [b(s) + c(s) + g(s) + p(s) + q(s))]u(s;x) = 0 \ \ in \ \ \Omega, \\ u(s;x) = 1 \qquad on \ s_i = x_i, \ i = 1, \cdots, n, \end{cases} \tag{5.4.386}$$

$$\begin{cases} (-1)^n \dfrac{\partial^n v(s;x)}{\partial s_1 \cdots \partial s_n} - [b(s) + c(s) + g(s) - p(s)]v(s;x) = 0 \qquad in \quad \Omega, \\ v(s;x) = 1 \qquad on \ s_i = x_i, \ i = 1, \cdots, n, \end{cases} \tag{5.4.387}$$

and

$$\begin{cases} (-1)^n \dfrac{\partial^n w(s;x)}{\partial s_1 \cdots \partial s_n} - [b(s) - c(s)]w(s;x) = 0 \qquad in \quad \Omega, \\ w(s;x) = 1 \qquad on \ s_i = x_i, \ i = 1, \cdots, n, \end{cases} \tag{5.4.388}$$

respectively, and let D^+ be a connected sub-domain of Ω which contains x such that $u \geq 0$, $v \geq 0$, and $w \geq 0$ for all $s \in D^+$. If $D \subset D^+$ and

$$\begin{aligned} f(x) \leq a(x) &+ \int_{x^0}^{x} b(s)f(s)ds + \int_{x^0}^{x} c(s) \left(\int_{x^0}^{s} g(t)f(t)dt \right) ds \\ &+ \int_{x^0}^{s} c(s) \left(\int_{x^0}^{s} p(t)(\int_{x^0}^{t} q(m)f(m)dm)dt \right) ds, \end{aligned} \tag{5.4.389}$$

then

$$\begin{aligned} f(x) \leq a(x) + \int_{x^0}^{x} w(s;x) \Big\{ a(s)b(s) + c(s) \int_{x^0}^{s} v(t;s) \Big(a(t)[b(t) + g(t)] \\ + p(t) \int_{x^0}^{t} u(m;t)a(m)[b(m) + g(m) + q(m)]dm \Big) dt \Big\} ds. \end{aligned} \tag{5.4.390}$$

Proof The proof follows the proofs of Theorems 5.4.49 and 5.4.51 with suitable modifications. We omit the details here. □

The next result, due to Thandapani and Agarwal [621], extends the result of Young [677] to discuss the case when an inequality has repeated integrals. A unified result is also presented which covers several results of Pachpatte [477, 480].

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set and let a point $(x_1^i, \cdots, x_n^i) \in \Omega$ be denoted by x^i. Let $y = (y_1, \cdots, y_n), x = (x_1, \cdots, x_n) \in \Omega$ ($y < x$, *i.e.*, $y_i < x_i, i = 1, \cdots, n$) and denote by D parallelepiped defined by $y < s < x$. The $\int_y^x \cdot ds$ indicates the n-fold integral $\int_{y_1}^{x_1} \cdots \int_{y_n}^{x_n} ds_1 \cdots ds_n$, and $u_x(x)$ denotes $\partial^n u(x)/(\partial x_1 \cdots \partial x_n)$.

We shall assume that the functions which appear in the inequalities are real-valued, non-negative, continuous and defined in Ω.

Theorem 5.4.54 (Thandapani-Agarwal [621]) *Let $V(s,x)$ be the solution of characteristic initial value problem*

$$\begin{cases} (-1)^n V_s(s,x) - \sum_{i=1}^{m} E_s^r(s,b)V(s,x) = 0 \qquad in \quad \Omega, & (5.4.391) \\ V(s,x) = 1 \qquad on \;\; s_i = x_i, \;\; 1 \le i \le n, & (5.4.392) \end{cases}$$

and let D^+ be a connected sub-domain of Ω containing x such that $V \ge 0$ for all $s \in D^+$. If $D \subset D^+$ and

$$u(x) \le a(x) + b(x)\sum_{r=1}^{m} E^r(x,u), \tag{5.4.393}$$

where

$$E^r(x,u) = \int_y^x f_{r1}(x^1)\int_y^{x^1} f_{r2}(x^2)\cdots\int_y^{x^{r-1}} f_{rr}(x^r)\,dx^r\cdots dx^1, \tag{5.4.394}$$

$a, b, f_rj : D^+ \mapsto \mathbb{R}, j = 1, 2, \cdots, r$ are continuous non-negative functions. Then

$$u(x) \le a(x) + b(x)\int_y^x \sum_{r=1}^{m} E_s^r(s,a)V(s,x)\,ds. \tag{5.4.395}$$

Proof Define

$$\phi(x) = \sum_{r=1}^{m} E^r(x,u),$$

then we have

$$\phi_x(x) = \sum_{r=1}^{m} E_x^r(x,u)$$

and hence from (5.4.393) it follows

$$\phi_x(x) \le \sum_{r=1}^{m} E_x^r(x, a+b\phi) = \sum_{r=1}^{m} E_x^r(x,a) + \sum_{r=1}^{m} E_x^r(x,b\phi). \tag{5.4.396}$$

Using the non-decreasing nature of $\phi(x)$ in (5.4.396), we get

$$\phi_x(x) - \sum_{r=1}^{m} E_x^r(x,b)\phi(x) \le \sum_{r=1}^{m} E_x^r(x,a),$$

where ϕ vanishes together with all its mixed derivatives up to order $n-1$ on $x_i = y_i,\ 1 \le i \le n.$

Now applying Lemma 5.4.6 provides

$$\phi(x) \le \int_y^x \sum_{r=1}^{m} E_s^r(s,a)V(s,x)\,ds. \tag{5.4.397}$$

Therefore (5.4.395) now follows from (5.4.397) and $u(x) \le a(x) + b(x)\phi(x)$. □

Some particular cases of Theorem 5.4.54, $n = 2$ and m up to 3 have been considered by Pachpatte [477, 481], but his results cannot be compared with the following results. In the next theorem, we shall introduce a particular case of (5.4.393); the obtained result unifies all his six theorems for the general n.

We shall denote $\sum_{r=1}^{r_1} b(x)f_r(x) \bigcup_{l=1}^{r_2} g_l(x)$ as the sum of all functions except when $b(x)f_k(x) = g_l(x)$ for some $1 \le k \le r_1,\ 1 \le l \le r_2$; then $g_l(x)$ is taken to be zero, also $\bigcup_{l=1}^{0} g_l(x) = 0$.

Theorem 5.4.55 (Thandapani-Agarwal [621]) *Let $V(s,x),\ 1 \le i \le m$, be the solution of characteristic initial value problem*

$$\begin{cases} (-1)^n V_{1s}(s,x) - \left(\displaystyle\sum_{r=1}^{m} b(s)f_r(s) \bigcup_{i=1}^{m-1} g_i(s)\right) V_1(s,x) = 0 \quad in\ \Omega, \\ (-1)^n V_{js}(s,x) - \left(\displaystyle\sum_{r=1}^{m-j+1} b(s)f_r(s) \bigcup_{i=1}^{m-j} g_i(s) - g_{m-j+1}(s)\right) V_j(s,x) = 0 \quad in\ \Omega,\ 2 \le j \le m, \\ V_j(s,x) = 1 \qquad on\ \ s_i = x_i,\ 1 \le i \le n,\ 1 \le j \le m. \end{cases} \tag{5.4.398}$$

and let D^+ be a connected sub-domain of Ω containing x such that $V_j \ge 0,\ 1 \le j \le m$ for all $s \in D^+$. If $D \subset D^+$ and (5.4.393) holds where $f_{ii}(x) = f_i(x),\ 1 \le i \le m$; $f_{i+1,i}(x) = f_{i+2,i}(x) = \cdots = f_{m,i}(x) = g_i(x),\ 1 \le i \le m-1$, then

$$u(x) \le a(x) + b(x)P_j(x), \qquad 1 \le j \le m, \tag{5.4.399}$$

where

$$\begin{cases} P_1(x) = \displaystyle\int_y^x a(x^1) \sum_{r=1}^{m} f_r(x^1) V_1(x^1,x)\,dx^1, \\ P_j(x) = \displaystyle\int_y^x \left(a(x^1) \sum_{r=1}^{m-j+1} f_r(x^1) + g_{m-j+1}(x^1)P_{j-1}(x^1)\right) \times V_j(x^1,x)\,dx^1, \qquad 2 \le j \le m. \end{cases} \tag{5.4.400}$$

Proof The inequality (5.4.393) with functions $f_{ij}(x)$ is equivalent to the system

$$\begin{cases} u_1(x) \le a(x) + b(x) \displaystyle\int_y^x [f_1(s)u_1(s) + g_1(s)u_2(s)]\,ds, \\ u_{j-1}(x) = \displaystyle\int_y^x [f_{j-1}(s)u_1(s) + g_{j-1}(s)u_j(s)], \qquad 3 \le j \le m, \\ u_m(x) = \displaystyle\int_y^x f_m(s)u_1(s)\,ds. \end{cases} \tag{5.4.401}$$

Define

$$\begin{cases} \phi_1(x) = \displaystyle\int_y^x [f_1(s)u_1(s) + g_1(s)u_2(s)]\,ds, \\ \phi_{j_1}(x) = \displaystyle\int_y^x [f_{j-1}(s)u_1(s) + g_{j-1}(s)u_j(s)]ds, \qquad 3 \le j \le m, \\ \phi_m(x) = \displaystyle\int_y^x f_m(s)u_1(s)\,ds. \end{cases} \tag{5.4.402}$$

Then from (5.4.401) it follows that

$$\phi_{1x}(x) \le f_1(x)[a(x) + b(x)\phi_1(x)] + g_1(x)\phi_2(x), \tag{5.4.403}$$

$$\phi_{j-1x}(x) \le f_{j-1}(x)[a(x) + b(x)\phi_1(x)] + g_{j-1}(x)\phi_j(x), \quad 3 \le j \le m, \tag{5.4.404}$$

$$\phi_{mx} \le f_m(x)[a(x) + b(x)\phi_1(x)]. \tag{5.4.405}$$

We add (5.4.403)–(5.4.405) to obtain

$$\left(\sum_{r=1}^m \phi_r(x)\right)_x \le a(x)\sum_{r=1}^m f_r(x) + b(x)\sum_{r=1}^m f_r(x)\phi_1(x) + \sum_{r=1}^m g_r(x)\phi_{r+1}(x)$$

whence

$$\left(\sum_{r=1}^m \phi_r(x)\right)_x - \left(\sum_{r=1}^m b(x)f_r(x)\bigcup_{i=1}^{m-1} g_i(x)\right)\left(\sum_{r=1}^m \phi_r(x)\right) \le a(x)\sum_{r=1}^m f_r(x). \tag{5.4.406}$$

Using Lemma 5.4.6, we get

$$\sum_{r=1}^m \phi_r(x) \le P_1(x) \tag{5.4.407}$$

which yields

$$\phi_m(x) \le P_1(x) - \sum_{r=1}^{m-1} \phi_r(x). \tag{5.4.408}$$

Adding (5.4.405), (5.4.406), and using (5.4.380), we obtain

$$\begin{aligned}\left(\sum_{r=1}^{m} \phi_r(x)\right)_x &\le a(x)\sum_{r=1}^{m-1} f_r(x) + b(x)\sum_{r=1}^{m-1} f_r(x)\phi_1(x)\\ &\quad + \sum_{r=1}^{m-2} g_2(x)\phi_{r+1}(x) + g_{m-1}(x)[P_1(x) - \sum_{r=1}^{m-1} \phi_r(x)],\end{aligned}$$

which gives us

$$\begin{aligned}&\left(\sum_{r=1}^{m-1} \phi_r(x)\right)_x - \left(\sum_{r=1}^{m-1} b(x)f_r(x)\bigcup_{i=1}^{m-2} g_i(x) - g_{m-1}(x)\right)\left(\sum_{r=1}^{m-1} \phi_r(x)\right)\\ &\le a(x)\sum_{r=1}^{m-1} f_r(x) + g_{m-1}(x)P_1(x).\end{aligned}$$

Using again Lemma 5.4.6, we get

$$\sum_{r=1}^{m-1} \phi_r(x) \le P_2(x) \tag{5.4.409}$$

or

$$\phi_{m-1}(x) \le P_2(x) - \sum_{r=1}^{m-2} \phi_r(x). \tag{5.4.410}$$

Adding (5.4.403), (5.4.404), and using (5.4.410), we obtain

$$\begin{aligned}&\left(\sum_{r=1}^{m-2} \phi_r(x)\right)_x - \left(\sum_{r=1}^{m-2} b(x)f_r(x)\bigcup_{i=1}^{m-3} g_i(x) - g_{m-2}(x)\right)\left(\sum_{r=1}^{m-2} \phi_r(x)\right)\\ &\le a(x)\sum_{r=1}^{m-2} f_r(x) + g_{m-2}(x)P_2(x)\end{aligned}$$

which, together with Lemma 5.4.6, implies

$$\sum_{r=1}^{m-2} \phi_r(x) \le P_3(x). \tag{5.4.411}$$

Continuing in this way, we have

$$\sum_{r=1}^{m-j+1} \phi_r(x) \le P_j(x), \qquad 4 \le j \le m. \tag{5.4.412}$$

Since $u(x) = u_1(x) \le a(x) + b(x)\phi_1(x)$ and $\phi_1(x) \le \sum_{r=1}^{m-j+1} \phi_r(x)$, $1 \le j \le m$, the desired result (5.4.399) follows from (5.4.407), (5.4.409), (5.4.411), (5.4.412). □

We note that for the particular case when $m = 2,\ b = 1, f_{11} = f_{21} = f_1,\ f_{22} = f_2$ in (5.4.393), estimate (5.4.399) takes the form

$$u(x) \le a(x) + \int_y^x f_1(x^1) \times \left(a(x^1) + \int_y^x a(x^2)(f_1(x^2) + f_2(x^2))V_1(x^2, x^1)\, dx^2 \right) dx^1, \tag{5.4.413}$$

where $V_1(s, x)$ is the solution of characteristic initial value problem

$$\begin{cases} (-1)^n V_{1s}(s,x) - (f_1(s) + f_2(s))V_1(s,x) = 0 & \textit{in}\ \ \Omega, \\ V_1(s,x) = 1 \qquad \textit{on}\ s_i = x_i,\ 1 \le i \le n. \end{cases} \tag{5.4.414}$$

In the next result, we shall show that estimate (5.4.413) can be improved uniformly. In detail, the improved version of Theorem 1 in [477] is the following one, here we have taken $\sigma = 0$ since it dose not play any role, the term $\int_y^x b(s)\sigma(s)\, ds$ can always be merged in $a(x)$.

Theorem 5.4.56 (Thandapani-Agarwal [621]) *Let $V_1(s, x)$ be the solution of problem (5.4.414) and let D^+ be a connected sub-domain of Ω containing x such that $V_1 \ge 0$ for all $s \in D^+$. If $D \subset D^+$ and (5.4.393) holds where $m = 2,\ b = 1,\ f_{11} = f_{21} = f_1,\ f_{22} = f_2$, then*

$$u(x) \le a(x) + \int_y^x f_1(x^1) \times \left[a(x^1) + \int_y^{x^1} \left(a(x^2)\left(f_1(x^2) + f_2(x^2)\right) - c(x^2)\right) V_1(x^2, x_1)\, dx^2 \right] dx^1, \tag{5.4.415}$$

where

$$c(x) = f_2(x) \int_y^x a(x^1) f_2(x^1)\, dx^1.$$

Proof Define

$$\phi_1(x) = \int_y^x f_1(x^1) u(x^1)\, dx^1 + \int_y^x f_1(x^1) \int_y^{x^1} f_2(x^2) u(x^2)\, dx^2 dx^1, \tag{5.4.416}$$

then from (5.4.393) it follows that

$$\phi_{1x}(x) \le f_1(x) \left[a(x) + \phi_1(x) + \int_y^x f_2(x^1) \left(a(x^1) + \phi_1(x^1) \right) dx^1 \right]. \tag{5.4.417}$$

Let

$$\phi_2(x) = \phi_1(x) + \int_y^x f_2(x^1) \left(a(x^1) + \phi_1(x^1) \right) dx^1. \tag{5.4.418}$$

Then it follows that

$$\phi_{2x}(x) = \phi_{1x}(x) + f_2(x) \left(a(x) + \phi_1(x) \right),$$

which , from (5.4.417) and (5.4.418), gives us

$$\begin{aligned} \phi_{2x}(x) &\le f_1(x) \left(a(x) + \phi_1(x) \right) \\ &\quad + f_2(x) \left[a(x) + \phi_2(x) - \int_y^x a(x^1) f_2(x^1)\, dx^1 \right]. \end{aligned}$$

Therefore, using Lemma 5.4.6, we obtain

$$\phi_2(x) \le \int_y^x \left(a(x^1)(f_1(x^1) + f_2(x^1)) - c(x^1) \right) V_1(x^1, x)\, dx^1.$$

Inserting the above inequality in (5.4.419), we get

$$\begin{aligned} \phi_1(x) \le \int_y^x f_1(x^1) \Big[a(x^1) + \int_y^{x^1} & \left(a(x^2)(f_1(x^2) + f_2(x^2)) - c(x^2) \right) \\ & \times V_1(x^2, x^1)\, dx^2 \Big] dx^1 \end{aligned}$$

and hence the desired result (5.4.426) follows from $u(x) \le a(x) + \phi_1(x)$. □

In next result, we shall introduce a Wendroff type estimate for (5.4.393).

Theorem 5.4.57 (Thandapani-Agarwal [621]) *Let inequality (5.4.393) hold in Ω, where (i) $a(x)$ is positive and non-decreasing and (ii) $b(x) \geq 1$. Then*

$$u(x) \leq a(x)b(x)\exp\left(\sum_{r=1}^{m} E^{r}(x,b)\right). \tag{5.4.419}$$

Proof In fact, inequality (5.4.393) can be written as

$$\phi_1(x) \leq 1 + \sum_{r=1}^{m} E^{r}(x, b\phi_1), \tag{5.4.420}$$

where

$$\phi_1(x) = \frac{u(x)}{a(x)b(x)}.$$

Let $\phi_2(x)$ be the right-hand side of (5.4.410), then

$$\phi_{2x}(x) \leq \sum_{r=1}^{m} E_x^{r}(x, b\phi_1) \leq \sum_{r=1}^{m} E_x^{r}(x, b\phi_2) \tag{5.4.421}$$

and $\phi_2(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n) = 1$; all the partial derivatives up to order $n-1$ vanish when $x_i = y_i$, for any i, $1 \leq i \leq n$.

Since $\phi_2(x)$ is non-decreasing, it follows from (5.4.421) that

$$\phi_{2x}(x) \leq \sum_{r=1}^{m} E_x^{r}(x, b)\phi_2(x)$$

or

$$\frac{\phi_{2x}(x)}{\phi_2(x)} \leq \sum_{r=1}^{m} E_x^{r}(x, b) = \frac{(\phi_{2x_n}(x))(\phi_{2x_1\ldots x_{n-1}}(x))}{\phi_2^2(x)}.$$

Hence

$$\left(\frac{\phi_{2x_1\cdots x_{n-1}}(x)}{\phi_2^2(x)}\right)_{x_n} \leq \sum_{r=1}^{m} E_x^{r}(x, b).$$

Keeping $x_1, \cdots, x_n$ fixed in the above inequality and setting $x_n = s_n$ and integrating with respect to s_n from y_n to x_n, we get

$$\left(\frac{(\phi_{2x_1\cdots x_{n-1}}(x))}{\phi_2^2(x)}\phi_2(x)\right) \leq \int_{y_n}^{x_n} \sum_{r=1}^{m} E^r_{x_1\cdots x_{n-1}s_n}(x_1, \cdots, x_{n-1}, s_n, b)\, ds_n$$
$$= \sum_{r=1}^{m} E^r_{x_1\cdots x_{n-1}}(x, b).$$

Repeating the above argument for x_{n-1}, x_{n-2} to x_2, we derive

$$\frac{\phi_{2x_1}(x)}{\phi_2(x)} \leq \sum_{r=1}^{m} E^r_{x_1}(x, b).$$

Integrating the above inequality with respect to x_1 and using $\phi_2(y_1, x_2, \cdots, x_n) = 1$, we conclude

$$\phi_2(x) \leq \exp\left(\sum_{r=1}^{m} E^r(x, b)\right).$$

Thus, the required result (5.4.419) now follows readily from $\phi_1(x) \leq \phi_2(x)$ and the definition of $\phi_1(x)$. □

Note that, estimate (5.4.419) for $n = 2,\ m = 1$ is sharper than that given in [47], and the same as that obtained by Kasture and Deo [312]. Some results were given in [95] for $n = 2,\ b = 1,\ m$ up to 2 with different assumptions on $a(x)$.

The result do not require any condition on $a(x)$ and $b(x)$ as in Theorem 5.4.57, the estimate (5.4.419) can also be reobtained.

The domain of definition of a function f is denoted by $D(f)$. The classes $Z, Z_0, Z_c, \cdots$ refer to solutions or approximate solutions of the problems under consideration. The function classes $Z(f), Z_0(f), \cdots$ take this into account. If, for example, an ordinary differential equation $u' = f(t, u)$ is given for $0 < t \leq T$, then $u \in Z(f)$ means, first, that u is in the class Z defined above and, second, that u "can be substituted" in f, i.e., $(t, u(t)) \in D(f)$ for $0 < t \leq T$.

Let m be an integer and $G \subset E^m$ a bounded open set. The set of boundary points of G is denoted by ∂G, and the closure by $\overline{G} = G + \partial G$. We use $G(\overline{x})$ for the set of all points $x \in \overline{G}$ for which $x \leq \overline{x}$, R_v ("initial boundary") for the set of all $\overline{x}$ for which $G(\overline{x})$ consists only of the point $\overline{x}$, and G_v for the difference $\overline{G} - R_v$. We have $R_v \subset \partial G$.

The operator K is called a "monotone increasing operator" if it has the following property:

If $\varphi, \overline{\varphi} \in Z_c(K)$ and if for a point $x_0 \in G_v$ the inequality $\varphi(x) \leq \overline{\varphi}(x)$ holds in $G(x_0)$, then

$$(K\varphi)(x_0) \leq (K\overline{\varphi})(x_0).$$

Lemma 5.4.7 *Suppose that $k(t\tau, z)$ is a monotone increasing kernel, $v(t)$ and $w(t)$ are functions of the class $Z_c(k)$, $g(t)$ is a function defined in J, and in J*

$$v(t) \leq g(t) + \int_0^t k(t, \tau, v(\tau))d\tau, \ \ w(t) \geq g(t) + \int_0^t k(t, \tau, w(\tau))d\tau,$$

where for each t equality holds in at most one place. Then

$$v < w \ in \ J.$$

Proof The basic idea of the following simple proof will be seen below. For $t = 0$ it follows from the hypothesis that $v(0) \leq g(0), w(0) \geq g(0)$, where there cannot be equality in both places; hence $v(0) < w(0)$. If the assertion were false, there would be a first point $t_0 \in J_0$ such that $v(t_0) = w(t_0)$ and $v < w$ for $0 < t < t_0$. On the other band, because of the monotonicity of k,

$$v(t_0) \leq g(t_0) + \int_0^{t_0} k(t_0, \tau, v)d\tau \leq g(t_0) + \int_0^{t_0} k(t_0, \tau, w)d\tau \leq w(t_0),$$

where there is strict inequality in at least on position. The contradiction thus obtained proves the validity of the lemma. □

If both functions $\varphi, \overline{\varphi}$ are defined in an interval $a < t < a + \varepsilon$ $(\varepsilon > 0)$ and if there exists a $\delta > 0$ such that $\varphi < \overline{\varphi}$ for $a < t < a + \delta$, then we write simply

$$\varphi(a+) < \overline{\varphi}(a+).$$

Lemma 5.4.8 *For a monotone increasing operator K and two functions $v, w \in Z_c(K)$, suppose that we have*

(1) $v(0+) < w(0+)$,

(2)$v - Kv < w - Kw$ in J_0.

Then

$$v < w \ in \ J_0.$$

Proof The assumption (1) can be discarded if K has the property that $(K\varphi)(0) = 0$ for all $\varphi \in Z_c(K)$ and if (2) also holds for $t = 0$ (indeed it then follows from (2)).

The proof proceed in essence exactly as in Lemma 5.4.7. If the assertion is false, then there exists a $t_0 \in J_0$ with the properties noted in the proof of Lemma 5.4.7. Then because of the monotonicity of K

$$(Kv)(t_0) \leq (Kw)(t_0),$$

whence, with the aid of (2),

$$v = (v - Kv) + kv < (w - Kw) + Kv \leq w$$

at the point $t = t_0$, so that we have arrived at a contradiction to the assumption $v(t_0) = w(t_0)$. □

Lemma 5.4.9 *For a monotone increasing operator K and two functions $v, w \in Z_c(K)$, suppose that*

(1) $v < w$ on R_c^+,

(2)$v - Kv < w - Kw$ in G_v.

Then

$$v < w \text{ in } G_v.$$

Proof The hypothesis (1) can be omitted if $(K\varphi)(\overline{x}) = 0$ for $\overline{x} \in R_v$ and all $\varphi \in Z_c(K)$ and if (2) also holds on R_v (it then follows from (2)).

The proof goes through as in Lemma 5.4.8. Let us assume that the assertion is false and that A is the set of those points from G_v at which $v \leq w$. Let $s(x) = x_1 + \cdots + x_m$ and suppose that s_0 is the lower bound of this function relative to A. If this lower bound is achieved at a point of A, thus $v < w$. At the point x_0 by (2) and the monotonicity of K,

$$v = (v - Kv) + Kv < (w - Kw) + Kv \leq w$$

which contradicts the assumption $x_0 \in A$. Thus the function $s(x)$ does not assume its infimum (relative to A) on A. Then there exists a sequence $x_1, x_2, \cdots$ of points from A with $s(x_k) \to s_0$ as $k \to +\infty$. If $\overline{x} \in \overline{G}$ is an accumulation point of this sequence, then $v(\overline{x}) \leq w(\overline{x})$ because of the continuity of these functions, and furthermore $\overline{x} \notin A$ and thus $\overline{x} \in R_v$. But this leads to a contradiction of (1). Thus we have shown that the set A is empty and the assertion of the theorem is true.

Theorem 5.4.58 (Thandapani-Agarwal [621]) *Let inequality (5.4.413) hold in Ω. Then*

$$u(x) \leq a(x) + b(x) \int_y^x \sum_{r=1}^m E_s^r(s, a) \exp\left(\int_s^x \sum_{r=1}^m E_t^r(t, b)\, dt\right) ds. \tag{5.4.422}$$

Proof Define

$$w(s, x) = \exp\left(\int_s^x \sum_{r=1}^m E_t^r(t, b)\, dt\right).$$

Then it follows that

$$\begin{cases} (-1)^n w_s(s, x) - \displaystyle\sum_{r=1}^m E_s^r(s, b) w(s, x) \geq 0, \\ w(s, x) = 1 \qquad on \;\; s_i = x_i, \;\; 1 \leq i \leq n. \end{cases} \tag{5.4.423}$$

Thus, $w(s,x)$ satisfies a differential inequality (5.4.423) of which $V(s,x)$ is the exact solution (Theorem 5.4.54). It follows from Lemma 5.4.9 that $w(s,x) \geq V(s,x)$, and now (5.4.422) follows from (5.4.395). □

In case the conditions on $a(x)$ (which can be non-negative) and $b(x)$ of Theorem 5.4.57 are satisfied, then from (5.4.422), it follows

$$u(x) \leq a(x)b(x)\left[1 + \int_y^x \sum_{r=1}^m E_s^r(s,b) \exp\left(\int_s^x \sum_{r=1}^m E_t^r(t,b)\, dt\right) ds\right]. \tag{5.4.424}$$

Therefore, employing (5.4.423) in (5.4.424), we get

$$u(x) \leq a(x)b(x)\left[1 + (-1)^n \int_y^x w_s(s,x)\, ds\right]. \tag{5.4.425}$$

Now using the fact that the partial derivatives of $w(s,x)$ up to order $n-1$ vanishes on $s_i = x_i$, $1 \leq i \leq n$, it follows from (5.4.425) that

$$u(x) \leq a(x)b(x)\left[1 + (-1)^{2n-1} \int_{y_1}^{x_1} w_{s_1}(s_1, y_2, \ldots, y_n, x)\, ds_1\right]$$

whence

$$u(x) \leq a(x)b(x)\left[1 + (-1)^{2n-1}(w(x_1, y_2, \ldots, y_n, x) - w(y,x))\right]$$

or

$$u(x) \leq a(x)b(x)w(y,x),$$

which is the same as (5.4.419). Thus to obtain (5.4.419) in Theorem 5.4.58, we require $a(x)$ to be non-negative and non-decreasing.

In the sequel, we shall establish some new n-independent variables integral inequalities, due to Yang [659], which have unified and extended some known results due to Gollwitzer [231], Bondge and Pachpatte [91], Pachpatte [456, 457], and Shih and Yeh [587].

The following notations will be used.

Let $I = [0,h)$ where $0 < h \leq +\infty$. Let $C(I^n, \mathbb{R}_+)$ be the class of all continuous functions on I^n with range in $\mathbb{R}_+$. In what follows, we define the functions $E_k^{(j)}(s,x;v)$ on $C(I^n, \mathbb{R}_+)$ by

$$E_k^{(j)}(s,x;v) = \int_x^s a_k^{(j)}(s,t^k) \int_{t^k}^s a_{k+1}^{(j)}(s,t^{k+1}) \cdots \int_{t^{k-1}}^s a_j^{(j)}(s,t^j) v(t^j) dt^j dt^{j-1} \cdots dt^k,$$
$$j = 1,2,\cdots,m; \quad k = 1,2,\cdots,j; \quad 0 \leq x \leq s, s \in I^n,$$

where $v \in C(I^n, \mathbb{R}_+)$, $a_k^{(j)}(s,x) : I^n \times I^n \to \mathbb{R}_+$ are continuous functions.

Theorem 5.4.59 (Yang [659]) *Let the functions u and w be in the class $C(I^n, I^n)$, and let $a_k^{(j)}(s,x) : I^n \times I^n \to \mathbb{R}_+$ be continuous functions. Suppose that the inequality for all $0 \le x \le s$,*

$$u(s) \ge w(x) - \sum_{j=1}^{m} E_1^{(j)}(s,x;w) \tag{5.4.426}$$

holds where $s \in I^n$. Then the following two inequalities also hold for all $0 \le x \le s$,

$$u(s) \ge w(x) \exp\left(-\int_x^s \sum_{j=1}^{m} A_j(s,t)dt\right) \tag{5.4.427}$$

where

$$A_j(s,x) = \max\left[a_j^{(j)}(s,x), a_j^{(j+1)}(s,x), \cdots, a_j^{(m)}(s,x)\right], \tag{5.4.428}$$

and for each $s \in I^n$ fixed, $i = 1, 2, \cdots, m$;

$$u(s) \ge w(x)/q_m(s,x), \tag{5.4.429}$$

where the function $q_m(s,x)$ is defined by

$$\begin{cases} q_1(s,x) = \exp\left(\displaystyle\int_x^s \sum_{j=1}^{m} A_j(s,x)dt\right), \\ q_r(s,t) = 1 + \displaystyle\int_x^s \sum_{j=1}^{m-r-1} A_j(s,x)q_{r-1}(s,t)dt, \quad r = 2, 3, \cdots, m. \end{cases} \tag{5.4.430}$$

Proof We first prove (5.4.427). Fixing $s \in I^n$, the inequality (5.4.426) can be rewritten as for all $0 \le x \le s$,

$$w(t) \le r_1(x) \tag{5.4.431}$$

where

$$r_1(x) = u(s) + \sum_{j=1}^{m} E_1^{(j)}(s,x;w).$$

Therefore,

$$r_1(x) = u(s) \ \ on \ \ x_i = s, \ \ i = 1, 2, \cdots, n,$$

which, along with (5.4.431), yields

$$\begin{aligned}(-1)^n Dr_1(x) &= a_1^{(1)}(s,x)w(x) + \sum_{j=2}^{m} a_1^{(j)}(s,x)E_2^{(j)}(s,x;w) \\ &\leq A_1(s,x)\left\{r_1(x) + \sum_{j=2}^{m} E_2^{(m)}(s,x;r_1)\right\}, \end{aligned} \tag{5.4.432}$$

where $A_1(s,x)$ is given by (5.4.428). Define

$$r_2(x) = r_1(x) + \sum_{j=2}^{m} E_2^{(m)}(s,x;r_1),$$

then $r_1(x) \leq r_2(x)$ when $0 \leq x \leq s$, and $r_2(x) = u(s)$ on $x_i = s_i, i = 1, 2, \cdots, n$.
By applying (5.4.432), we derive

$$\begin{aligned}(-1)^n Dr_2(x) &= (-1)^n Dr_1(x) + \sum_{j=3}^{m} a_2^{(j)}(s,x)E_3^{(j)}(s,x;r_1) + a_2^{(2)}(s,x)r_1(x) \\ &\leq A_1(s,x)r_2(x) + A_2(s,x)r_3(x), \ \ 0 \leq x \leq s, \end{aligned} \tag{5.4.433}$$

where $A_2(s,x)$ is given by (5.4.428) and $r_3(x)$ is defined by

$$r_3(x) = r_2(x) + \sum_{j=3}^{m} E_3^{(j)}(s,x;r_2).$$

Continuing in this way, we then obtain

$$(-1)^n Dr_k(x) \leq r_{k+1}(x)\sum_{j=1}^{m} A_i(s,x), \ \ 0 \leq x \leq s, \ \ k = 1, 2, \cdots, m-1, \tag{5.4.434}$$

$$w(x) \leq r_1(x) \leq r_2(x) \leq \cdots r_m(x), \ \ 0 \leq x \leq s, \tag{5.4.435}$$

and

$$r_1(x) = r_2(x) = \cdots = r_m(x) = u(s) \ \ on \ \ x_i = s, i = 1, \cdots, n \tag{5.4.436}$$

where

$$\begin{cases} r_1(x) = u(s) + \sum_{j=1}^{m} E_1^{(j)}(s, x; w), \\ r_{k+1}(x) = r_k(x) + \sum_{j=k+1}^{m} E_{k+1}^{(j)}(s, x; r_k). \quad k = 1, 2, \cdots, m-1. \end{cases} \tag{5.4.437}$$

From (5.4.434)–(5.4.436) with $k = m - 1$, it follows, for all $0 \le x \le s$,

$$\begin{aligned} (-1)^n D r_m(x) &= (-1)^n D r_{m-1}(x) + a_m^{(m)}(s, x) r_{m-1}(x) \\ &\le r_m(x) \sum_{j=1}^{m-1} A_j(s, x) + a_m^{(m)}(s, x) r_{m-1}(x) \\ &\le r_m(x) \sum_{j=1}^{m} A_j(s, x). \end{aligned} \tag{5.4.438}$$

We derive from (5.4.437) that $r_i(x) \ge u(s) > 0$ is valid for $j = 1, 2, \cdots, m$ and $0 \le x \le s$. Hence, we obtain from (5.4.438) that for all $0 \le x \le s$, $s \in I^n$ *fixed*,

$$\frac{(-1)^n D r_m(x)}{r_m(x)} \le \sum_{j=1}^{m} A_j(s, x). \tag{5.4.439}$$

The above inequality (5.4.439) can be rewritten as

$$\frac{(-1)^n r_m(x) D_1 D_2 \cdots D_n r_m(x)}{r_m^2(x)} \le \sum_{j=1}^{m} A_j(s, x) + \frac{(-1)^n D_{n-1} r_m(x) D_1 D_2 \cdots D_{m-1} r_m(x)}{r_m^2(x)},$$

which implies

$$(-1)^n D_n \left[\frac{D_1 D_2 \cdots D_{n-1} r_m(x)}{r_m(x)} \right] \le \sum_{j=1}^{m} A_j(s, x),$$

by noting that $(-1)^n D_{n-1} r_m(x) D_1 D_2 \cdots D_{m-1} r_m(x) \ge 0$ holds.

Keeping $x_1, x_2, \cdots, x_{n-1}$ fixed in the above inequality, setting $x_n = t_n$ and integrating the both sides with respect to t_n from x_n to s_n, we conclude

$$\frac{(-1)^{n-1} D_1 D_2 \cdots D_{n-1} r_m(x)}{r_m(x)} \le \int_{x_n}^{s_n} \sum_{j=1}^{m} A_j(s, x_1, \cdots, x_{n-1}, t_n) dt_n,$$

which implies

$$(-1)^{n-1}D_{n-1}\left[\frac{D_1D_2\cdots D_{n-2}r_m(x)}{r_m(x)}\right] \leq \int_{x_n}^{s_n}\sum_{j=1}^{m}A_j(s,x_1,\cdots,x_{n-1},t_n)dt_n,$$

since $(-1)^{n-1}D_{n-1}r_m(x)D_1\cdots D_{n-2}r_m(x) \geq 0$.

Keeping $x_1,x_2,\cdots,x_{n-2},x_n$ fixed in the above inequality, setting $x_{n-1}=t_{n-1}$ and then integrating the both sides with respect to t_{n-1} from x_{n-1} to s_{n-1}, we derive

$$\frac{(-1)^{n-2}D_1\cdots D_{n-2}r_m(x)}{r_m(x)} \leq \int_{x_{n-1}}^{s_{n-1}}\int_{x_n}^{s_n}\sum_{j=1}^{m}A_j(s,x_1,\cdots,x_{n-2},t_{n-1},t_n)dt_ndt_{n-1},$$

since $D_1\cdots D_{n-2}r_m(x_1,\cdots,x_{n-2},s_{n-1},x_n)=D_1\cdots D_{n-2}u(s)=0$ for all $n\geq 3$.

Proceeding in this way, we easily obtain

$$D_2\left[\frac{D_1r_m(x)}{r_m(x)}\right] \leq \int_{x_3}^{s_3}\cdots\int_{x_n}^{s_n}\sum_{j=1}^{m}A_j(s,x_1,x_2,t_3,\cdots,t_n)dt_n\cdots dt_3.$$

Keeping $x_1,x_3,\cdots,x_n$ fixed in the above inequality, setting $x_2=t_2$ and integrating with respect to t_2 from x_2 to s_2, and in view of $D_1r_m(x_1,s_2,x_3,\cdots,x_n)=0$,

$$\frac{-D_1r_m(x)}{r_m(x)} \leq \int_{x_2}^{s_2}\cdots\int_{x_n}^{s_n}\sum_{j=1}^{m}A_j(s,x_1,t_2,\cdots,t_n)dt_ndt_{n-1}\cdots dt_2.$$

Now keeping $x_2,\cdots,x_n$ fixed in the above inequality, setting $x_1=t_1$ and integrating with respect to t_1 from x_1 to s_1, and using $r_m(s_1,x_2,\cdots,x_n)=u(s)$, we obtain

$$-\ln(u(s)/r_m(x)) \leq \int_x^s\sum_{j=1}^{m}A_j(s,t)dt,$$

or

$$r_m(x) \leq u(s)\exp\left(\int_x^s\sum_{j=1}^{m}A_j(s,t)dt\right) \equiv u(s)q_1(s,x). \tag{5.4.440}$$

We now prove the inequality (5.4.429). Substituting the bound for $r_m(x)$ in (5.4.440) in the inequality (5.4.434) with $k=m-1$, we can get, for all $0\leq x\leq s$,

$$(-1)^nDr_{m-1}(x) \leq u(s)q_1(s,x)\sum_{j=1}^{m-1}A_j(s,x).$$

Integrating the above inequality with respect to x_n from x_n to s_n, and using

$$D_1 \cdots D_{n-1} r_{m-1}(x_1, \cdots, x_{n-2}, x_{n-1}, s_n) = 0,$$

we get

$$(-1)^{n-1} D_1 \cdots D_{n-1} r_{m-1}(x) \le \int_{x_n}^{s_n} u(s) \sum_{j=1}^{m-1} A_j(s, x_1, \cdots, x_{n-1}, t_n) \times q_1(s, x_1, \cdots, x_{n-1}, t_n) dt_n.$$

Integrating the above inequality with respect to x_{n-1} from x_{n-1} to s_{n-1}, and using

$$D_1 \cdots D_{n-2} r_{m-1}(x_1, \cdots, x_{n-2}, s_{n-1}, x_n) = 0,$$

we conclude

$$(-1)^{n-2} D_1 \cdots D_{n-2} r_{m-1}(x) \le \int_{x_{n-1}}^{s_{n-1}} \int_{x_n}^{s_n} u(s) \sum_{j=1}^{m-1} A_j(s, x_1, \cdots, x_{n-2}, t_{n-1}, t_n) q_1(s, x_1, \cdots, x_{n-2}, t_{n-1}, t_n) dt_n dt_{n-1}.$$

Continuing in this way, we obtain

$$-D_1 r_{m-1}(x) \le \int_{x_2}^{s_2} \cdots \int_{x_n}^{s_n} u(s) \sum_{j=1}^{m-1} A_j(s, x_1, t_2 \cdots, t_n) \times q_1(s, x_1, t_2, \cdots, t_n) dt_2 d \cdots t_n.$$

Now integrating the above inequality with respect to x_1 from x_1 to s_1, and using $r_{m-1}(s_1, x_2, \cdots, x_n) = u(s)$, we obtain, when $0 \le x \le s$,

$$r_{m-1}(x) \le u(s) q_2(s, x).$$

Continuing in this way, we can easily derive inequality (5.4.429). The proof is now complete. □

Corollary 5.4.14 *In Theorem 5.4.59, if $m = 1$ and $a_1^{(1)}(s, x) = a(s)b(x)$, where $a(x)$ and $b(x)$ are non-negative continuous functions defined on I^n, then it reduces to Theorem 1 in Shih and Yeh [587] which, in turn, is an extension of Gollwitzer [231] and Bondge and Pachpatte [91].*

Corollary 5.4.15 *In Theorem 5.4.59, if $m = 2$, $a_1^{(i)}(s, x) = f(s, x)(i = 1, 2)$, and $a_2^{(2)}(s, x) = g(s, x)$, here f and $g : I^n \times I^n \to \mathbb{R}_+$ are continuous functions, then we*

may have the lower bound for $u(s)$ *such that*

$$u(s) \geq w(x)\left\{1+\int_x^s f(s,r)\left(\exp\int_r^s (f(s,t)+g(s,t))dt\right)dr\right\}^{-1},\ 0 \leq x \leq s.$$

We note that Theorem 4 in [587] is a special case of the above Corollary 5.4.14 where $f(s,x) = a(s)b(x)$ and $g(s,x) = c(x)$.

We shall next give some further extensions of Theorem 5.4.59 in [659], which unify and extend several known inequalities in [91, 231, 456, 457, 587].

Theorem 5.4.60 (Yang [659]) *Let all of the hypotheses in Theorem 5.4.59 hold, and let* $H(r)$ *be a positive, strictly increasing, convex, sub-multiplicative, and continuous functions defined for all* $r \geq 0, H(0) = 0$*, and* $\lim_{r\to+\infty} H(r) = +\infty$*. Let* $p(x), q(x)$ *be positive continuous functions on* I^n *with* $p(x) + q(x) = 1$*. Suppose that the inequality for all* $0 \leq x \leq s, s \in I^n$*,*

$$u(s) \geq w(x) - b(s)H^{-1}\left\{\sum_{j=1}^m E_1^{(j)}(s,x;H(w))\right\} \tag{5.4.441}$$

holds where H^{-1} *denotes the inverse of* H*, and* $b(x)$ *is a non-negative continuous function on* I^n*. Then for all* $0 \leq x \leq s$*,*

$$u(s) \geq p(s)H^{-1}\Big\{\frac{H(w(x))}{p(s)}\exp\big(-\int_x^s [q(s)H(b(s)/q(s))A_1(s,t)$$
$$+A_2(s,t)+\cdots+A_m(s,t)]dt\big)\Big\} \tag{5.4.442}$$

and

$$u(s) \geq p(s)H^{-1}\left[\frac{H(w(x))}{p(s)}\right] \tag{5.4.443}$$

where

$$\begin{cases} v_1(s,x) = \exp\displaystyle\int_x^s [q(s)H(b(s)/q(s))A_1(s,t)+A_2(s,t)+\cdots+A_m(s,t)]dt,\\ v_k(s,x) = 1+\displaystyle\int_x^s [q(s)H(b(s)/q(s))A_1(s,t)+A_2(s,t)+\cdots \\ \qquad\qquad + A_{m-k+1}(s,t)]v_{k-1}(s,t)dt,\quad k=2,3,\cdots,m-1,\\ v_m(s,x) = 1+\displaystyle\int_x^s q(s)H(b(s)/q(s))A_1(s,t)v_{m-1}(s,t)dt. \end{cases} \tag{5.4.444}$$

Proof We can rewrite the inequality (5.4.441) as

$$w(x) \leq p(s)(u(s)/p(s)) + q(s)(b(s)/q(s))H^{-1}\left\{\sum_{j=1}^{m} E_1^{(j)}(s,x;H(w))\right\}.$$

Since H is increasing, convex, and sub-multiplicative, from the above inequality it follows

$$H(w(x)) \leq p(s)H(u(s)/p(s)) + q(s)H(b(s)/q(s))\sum_{j=1}^{m} E_1^{(j)}(s,x;H(w)),$$

i.e.,

$$p(s)H(u(s)/p(s)) \geq H(w(x)) - q(s)H(b(s)/q(s))\sum_{j=1}^{m} E_1^{(j)}(s,x;H(w)).$$

Now applying Theorem 5.4.59 to the above inequality yields (5.4.442) and (5.4.443). □

The above Theorem 5.4.60 generalizes the results due to Bondge and Pachpatte [91], Gollwitzer [231], Pachpatte [456], and Shih and Yeh [587].

Theorem 5.4.61 (Yang [659]) *Let all of the hypotheses in Theorem 5.4.59 hold, and let the function $b(x)$ be the same as defined in Theorem 5.4.60. Let $G(r)$ be a positive, continuous, strictly increasing, sub-additive, and sub-multiplicative functions for all $r > 0, G(0) = 0$, and $G(r) \to +\infty$ as $r \to +\infty$. Suppose that the inequality for all $0 \leq x \leq s, s \in I^n$,*

$$u(s) \geq w(x) - b(s)G^{-1}\left\{\sum_{j=1}^{m} E_1^{(j)}(s,x;G(w))\right\} \tag{5.4.445}$$

holds where G^{-1} denotes the inverse function of G. Then for all $0 \leq x \leq s$, we have

$$\begin{aligned} u(s) \geq G^{-1}\Big\{G(w(x))\exp\Big(-\int_x^s [G(b(s))A_1(s,t) + A_2(s,t) \\ + \cdots + A_m(s,t)]dt\Big)\Big\}, \end{aligned} \tag{5.4.446}$$

and

$$u(s) \geq G^{-1}[G(w(x))/z_m(s,x)] \tag{5.4.447}$$

where the function $z_m(s,x)$ *is defined by*

$$\begin{cases} z_1(s,x) = \exp\left(\int_x^s [G(b(s))A_1(s,t) + A_2(s,t) + \cdots + A_m(s,t)]dt\right), \\ z_k(s,x) = 1 + \int_x^s [G(b(s))A_1(s,t) + A_2(s,t) + \cdots \\ \qquad + A_{m-k+1}(s,t)]z_{k-1}(s,t)dt, \; k = 2,3,\cdots,m-1, \\ z_m(s,x) = 1 + \int_x^s G(b(s))A_1(s,t)z_{m-1}(s,t)dt. \end{cases} \tag{5.4.448}$$

Proof We may rewrite inequality (5.4.445) as

$$w(x) \le u(s) + b(s)G^{-1}\left\{\sum_{j=1}^m E_1^{(j)}(s,x;G(w))\right\}. \tag{5.4.449}$$

Since G is increasing, sub-additive, and sub-multiplicative, we derive from the above inequality

$$G(w(x)) \le G(u(s)) + G(b(s))\sum_{j=1}^m E_1^{(j)}(s,x;G(w)), \tag{5.4.450}$$

i.e.,

$$G(u(s)) \ge G(w(x)) - G(b(s))\sum_{j=1}^m E_1^{(j)}(s,x;G(w)). \tag{5.4.451}$$

Applying Theorem 5.4.59 to (5.4.451) yields (5.4.446) and (5.4.447). □

Note that Theorem 5.4.60 has generalized the results obtained in Bondge and Pachpatte [91, Theorem 6], Pachpatte [457], and Shih and Yeh [587]. Moreover, we should point out the additional condition $G(r) \to +\infty$ as $r \to +\infty$ should be added to Theorem 6 in [587] to ensure the desired lower bound for $u(s)$.

Remark 5.4.18 Applying the inequalities in Theorems 5.4.59–5.4.61, according to the choice of the unknown function from u and w. If u is unknown and we set $w(x) = F(u(x))$, here $F : \mathbb{R}_+ \to \mathbb{R}_+$ is any known continuous function, then as discussed in [91, 231, 587], we may use these inequalities to obtain the lower bounds for $u(s)$ from the corresponding integral inequalities for u.

In the sequel, we shall introduce a result on a singular integral inequality in n variable. To achieve this result, we apply the method of desingularization of weakly singular inequalities and the result by Thandapani and Agarwal [621].

We use the notations: $e^x := e^{|x|}, x^\gamma := x_1^{\gamma^1} \cdots x_n^{\gamma^n}$ for $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{R}_+^n = \{(k_1, \cdots, k_n) | k_i \in \mathbb{R}, k_i \geq 0, i = 1, \cdots, n\}$, where $|x| = x_1 + x_2 + \cdots + x_n$. We also denote by $[\beta]$ the vector $(\beta, \beta, \cdots, \beta) \in \mathbb{R}^n$, by $\mathbf{1}$, $\mathbf{2}, \cdots$ the vectors $(1, 1, \cdots, 1) \in \mathbb{R}^n$, $(2, 2, \cdots, 2) \in \mathbb{R}^n, \cdots$ and by $\mathbf{p}/\mathbf{q}$ we mean the vector $(p/q, \cdots, p/q)$.

Theorem 5.4.62 (Medveď [398]) *Let* $\Omega, D, D^+, V(s,x), a(x), b(x), f_{r1}(x), \cdots, f_{rr}(x)$ *be as in Theorem 5.4.54 and let* $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}_+^n, 0 < \alpha < 1$ *(i.e.,* $0 < \alpha_i < 1, i = 1, 2, \cdots, n$*). Let* $u : D^+ \mapsto \mathbb{R}$ *be a continuous, non-negative function satisfying the inequality for all* $x \geq 0$,

$$u(x) \leq a(x) + b(x) \sum_{r=1}^{m} F^r(x, u), \tag{5.4.452}$$

where

$$F^r(x, u) = \int_0^x f_{r1}(x^1) \int_0^{x^1} f_{r2}(x^2) \cdots \int_0^{x^{r-1}} (x^{r-1} - x^r)^{\alpha-1} f_{rr}(x^r) u(x^r) dx^r \cdots dx^1. \tag{5.4.453}$$

Then the following assertions hold:

(i) Suppose $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) > \frac{\mathbf{1}}{\mathbf{2}}$. *Then*

$$\begin{aligned} u(x) \leq e^x \Bigg[& 2a^2(x) + 4b^2(x) S^2 \sum_{r=1}^{m} \left(\prod_{j=1}^{r-1} \int_0^x f_{rj}^2(\sigma) d\sigma \right) \\ & \times \int_0^x \left(\int_0^{x^1} \cdots \int_0^{x^{r-2}} f_{rr}^2(x^{r-1}) a^2(x^{r-1}) dx^r dx^{r-1} \cdots dx^2 \right) W(x^1, x) dx^1 \Bigg]^{1/2} \end{aligned} \tag{5.4.454}$$

where

$$S = \frac{1}{2^{2|\alpha|-2}} \prod_{i=1}^{n} \Gamma(2\alpha_i - 1)$$

and $W(s,x)$ *is the solution of characteristic initial value problem*

$$\begin{cases} (-1)^n W_s(s,x) - \sum_{r=1}^{m} K_s^r(s, B) W(s,x) = 0 \ \ in \ \ \Omega \\ W(s,x) = 1 \ \ on \ \ s_i = x_i, \ \ i = 1, 2, \cdots, n, \end{cases} \tag{5.4.455}$$

with

$$B(x) = 2b^2(x)$$

$$K^r(s, B) = S^2 \left(\prod_{j=1}^{r-1} \int_0^x f_{rj}^2(\sigma) d\sigma \right) \times \int_0^x \int_0^{x^1} \cdots \int_0^{x^{r-1}} f_{rr}^2(x^r) B^2(x^r) dx^r \cdots dx^1.$$

(ii) Suppose $\alpha = [1/(z+1)]$ *for a real number* $z > 1$ *and let* $q = z+2, p = (z+2)/(z+1)$, *i.e.,* $1/p + 1/q = 1$. *Then*

$$u(x) \le 2^{1-1/q} e^x \Bigg[a^q(x) + b^q(x) T_p^q \sum_{r=1}^m \left(\prod_{j=1}^{r-1} \int_0^x f_{rj}^p d(\sigma) \sigma \right)^{q/p} \times \int_0^x \left(\int_0^{x^1} \cdots \int_0^{x^{r-1}} f_{rr}^q(x^r) 2^{q-1} a^q(x) dx^r \cdots dx^2 \right) Z(x^1, x) dx^1 \Bigg]^{1/q}, \tag{5.4.456}$$

where

$$T_p := \left(\frac{\Gamma(1 - p\delta)}{p^{n(1-p\delta)}} \right)^{1/p}, \quad \delta = 1 - \alpha, \tag{5.4.457}$$

and $Z(s, x)$ *is the solution of characteristic initial value problem*

$$\begin{cases} (-1)^n Z_s(s, x) - \sum_{r=1}^r R_s^r(s, C) Z(s, x) = 0 \ \ in \ \ \Omega, \\ Z(s, x) = 0 \ \ on \ \ s_i = x_i, \ \ i = 1, 2, \cdots n, \end{cases} \tag{5.4.458}$$

where $C(x) = 2^{q-1} a(x)^q$, *and*

$$R^r(s, x) = T_p^q \left(\prod_{j=1}^{r-1} \int_0^x f_r j^p(\sigma) d\sigma \right)^{q/p} \int_0^x \left(\int_0^{x^1} \cdots \int_0^{x^{r-1}} f_{rr}^q(x^r) 2^{q-1} b^q(x^r) dx^r \cdots dx^2 \right) dx^1.$$

Proof We shall prove (i). Let us estimate the function $F^r(x, u)$ by using the Cauchy-Schwarz inequality and the inequality

$$\int_0^x (x^1 - \sigma)^{2\alpha - 2} e^{2\sigma} d\sigma \le e^{2x} S,$$

where S is as in theorem. On the other hand, it is easy to check

$$
\begin{aligned}
&F(x,u)\\
&\le \int_0^x f_{r1}(x^1)\int_0^{x^1} f_{r2}(x^2)\cdots\int_0^{x^{r-2}} f_{rr-1}(x^{r-1})\left[\int_0^{x^{r-1}}(x^{r-1}-x^r)^{2\alpha-2}e^{2x^r}dx^r\right]^{1/2}\\
&\quad\times\left[\int_0^{x^{r-1}} f_{rr}^2(x^r)e^{-2x^r}u^2(x^r)dx^r\right]^{1/2}dx^{r-1}\cdots dx^1\\
&\le S^{1/2}e^x\int_0^x f_{r1}(x^1)\int_0^{x^1} f_{r2}(x^2)\cdots\int_0^{x^{r-2}} f_{rr-1}(x^{r-1})\\
&\quad\times\left[\int_0^{x^{r-1}} f_{rr}^2(x^r)e^{-2x^r}u^2(x^r)dx^r\right]^{1/2}dx^{r-1}\cdots dx^1\\
&\le S^{1/2}e^x\int_0^x f_{r1}(x^1)\int_0^{x^1} f_{r2}(x^2)\cdots\int_0^{x^{r-3}} f_{rr-2}(x^{r-2})\left[\int_0^{x^{r-2}} f_{rr-1}(x^{r-1})^2dx^{r-1}\right]^{1/2}\\
&\quad\times\left[\int_0^{x^{r-2}}\int_0^{x^{r-1}} f_{rr}^2(x^r)e^{-2x^r}u^2(x^r)dx^{r-1}\right]^{1/2}dx^{r-2}\cdots dx^1\\
&\le S^{1/2}e^x(\int_0^x f_{rr-1}^2(\sigma)d\sigma)^{1/2}\cdots\int_0^x f_{r1}(x^1)\int_0^{x^1} f_{r2}(x^2)\cdots\int_0^{x^{r-3}} f_{rr-2}(x^{r-2})\\
&\quad\times\left(\int_0^{x^{r-2}}\int_0^{x^{r-1}} f_{rr}^2(x^r)e^{-2x^2}u^2(x^r)dx^{r-1}\right)^{1/2}dx^{r-2}\cdots dx^1.
\end{aligned}
\tag{5.4.459}
$$

Proceeding in this way, by using the Cauchy-Schwarz inequality, we derive

$$
\begin{aligned}
F(x,u)\le S^{1/2}e^x&\left(\prod_{j=1}^{r-1}\int_0^x f_{ij}^2(\sigma)d\sigma\right)^{1/2}\\
&\times\left[\int_0^x\int_0^{x^1}\cdots\int_0^{x^{r-1}} f_{rr}^2(x^r)e^{-2x^r}u^2(x^r)dx^r dx^{r-1}\cdots dx^1\right]^{1/2}
\end{aligned}
\tag{5.4.460}
$$

which, along with (5.4.452), implies

$$
\begin{aligned}
v(x)\le a(x)+S^{1/2}e^x b(x)\sum_{r=1}^m&\left(\prod^{r-1} j=1\int_0^x f_{rj}^2(\sigma)d\sigma\right)^{1/2}\\
&\times\left[\int_0^x\int_0^{x^1}\cdots\int_0^{x^{r-1}} f_{rr}^2(x^r)v^2(x^r)dx^r\cdots dx^1\right]^{1/2},
\end{aligned}
$$

where $\nu(x) = e^{-x}u(x)$. Then using the Jensen inequality, we obtain

$$\nu^2(x) \leq 2a^2(x) + 2Sb^2(x) \sum_{j=1}^{m} \left(\prod_{j=1}^{r-1} \int_0^x f_{rj}(\sigma) d\sigma \right) \times \left[\int_0^x \int_0^{x^1} \cdots \int_0^{x^{r-1}} f_{rr}^2(x^{r-1}) \nu^2(x^r) dx^r \cdots dx^1 \right].$$

Thus from Theorem 5.4.54 it follows that

$$\nu^2(x) \leq 2a^2(x) + 2Sb^2(x) \sum_{j=1}^{m} \left(\prod_{j=1}^{r-1} \int_0^x f_{rj}(\sigma) d\sigma \right) \times \int_0^x \left(\int_0^{x^1} \cdots \int_0^{x^{r-1}} f_{rr}^2(x^r) \left(2a(x^{r-1}) \right)^2 dx^{r-1} \cdots dx^2 \right) W(x^1, x) dx^1 \tag{5.4.461}$$

where $W(s, x)$ is as in theorem and from definition of $\nu(x)$, we obtain the inequality (5.4.454). Now let us prove the assertion (ii). We shall estimate the function $F^r(x, u)$ by using the Hölder inequality,

$$\begin{aligned} F^r(x, u) &\leq \int_0^x f_{r1}(x^1) \int_0^{x^1} f_{r2}(x^2) \cdots \int_0^{x^{r-1}} f_{rr-1}(x^{r-1}) \left[\int_0^{x^{r-1}} (x^{r-1} - x^r)^{[p\alpha - p]} e^{px^r} dx^r \right]^{1/p} \\ &\quad \times \left[\int_0^{x^{r-1}} f^q(x^r) e^{-qx^r} u^q(x^r) dx^r \right]^{1/q} dx^{r-1} \cdots dx^1 \\ &\leq T_p^{1/p} e^x \int_0^x f_{r1}(x^1) \int_0^{x^1} f_{r2}(x^2) \cdots \int_0^{x^{r-2}} f_{rr-1}(x^{r-1}) \\ &\quad \times \left[\int_0^{x^{r-1}} f^q(x^r) e^{-qx^r} u^q(x^r) dx^r \right]^{1/q} dx^{r-1} \cdots dx^1 \end{aligned}$$

where T_p is as in theorem. Similarly as in the case (i) by using the Hölder inequality, we can prove that

$$F^r(x, u) \leq T_p e^x \left(\prod_{j=1}^{r-1} \int_0^x f_{rj}^p(s) ds \right)^{1/p} \left[\int_0^x \int_0^{x^1} \cdots \int_0^{x^{r-1}} f_{rr}^q(x^r) \nu^q(x^r) dx^r \cdots dx^1 \right]^{1/q}.$$

From this inequality, (5.4.452) and the Jensen inequality, it follows that

$$\nu^q(x) \le 2^{q-1}\Big[a^q(x) + b^q(x)T_p^q \sum_{r=1}^{m} \left(\prod_{j=1}^{r-1} \int_0^x f_{rj}(\sigma)d\sigma\right)^{q/p}$$
$$\times \int_0^x \int_0^{x^1} \cdots \int_0^{x^{r-1}} f_{rr}^q(x^r)\nu^q(x^r)dx^r \cdots dx^1\Big]$$

which, together with Theorem 5.4.54, yields

$$\nu^q(x) \le 2^{q-1}\Big[a^q(x) + b^q(x)T_p^q \sum_{r=1}^{m} \left(\prod_{j=1}^{r-1} \int_0^x f_{rj}(\sigma)d\sigma\right)^{q/p}$$
$$\times \int_0^x \int_0^{x^1} \cdots \int_0^{x^{r-1}} f_{rr}^q(x^r)(2^{q-1}a^q(x^r))dZ^r \cdots dx^2 Z(x^1,x)dx^1\Big]$$

where $Z(s,x)$ is as in the theorem and from the definition of $\nu(x)$, we can obtain (5.4.456). □

Remark 5.4.19 The case $\boldsymbol{\alpha} < \frac{\mathbf{1}}{\mathbf{2}}, \alpha$ not equal to some $[\beta]$ is much more complicated than the case (ii) from the above theorem and we do not discuss it here.

5.4.2 Multi-Dimensional Gronwall-Bellman-Bihari Integral Inequalities with Delays

In this section, we shall introduce some multi-dimensional Gronwall-Bellman-Bihari integral inequalities with delays.

Definition 5.4.3 Denote by $\mathcal{G}$ the class of continuous functions $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following conditions:

(i) $\sigma(x) = (\sigma_1(x), \sigma_2(x), \cdots, \sigma_n(x))$ where $\sigma_j(x) : \mathbb{R}^n \to \mathbb{R}$ are continuous functions on $\mathbb{R}^n$ into $\mathbb{R}$,
(ii) $\sigma(x) \le x$, and
(iii) $\lim_{|x|\to+\infty} \sigma_j(x) = +\infty$ for each $j = 1, 2, \cdots, n$.

The next result, due to Akinyele [24], unifies the results of Yeh and Shih [673] and Pachpatte [458].

Theorem 5.4.63 (Akinyele [24]) *Assume that $x^0 \in \mathbb{R}^n$ is fixed and the region $B = \{y \in \mathbb{R}^n : x^0 \le y \le x^1\}$. Let $\phi(x)$, $f(x)$ and $g(x)$ be real-valued non-negative continuous functions on B and $n(x)$ be a positive, non-decreasing continuous function on B. Suppose σ and $\rho \in \mathcal{G}$ and $q(x) \ge 1$ is a real-valued*

continuous function defined on B. If the functional inequality holds for all $x \in B$, with $x \geq x^0$,

$$\phi(x) \leq n(x) + q(x)\Big[\int_{x^0}^{x} f(s)\phi(\sigma(s))ds + \int_{x^0}^{x} f(s)\left(\int_{x^0}^{s} g(s)\phi(\rho(t))dt\right)ds\Big], \tag{5.4.462}$$

then for all $x \in B$ with $x \geq x^0$,

$$\phi(x) \leq n(x)q(x)\exp\left(\int_{x^0}^{x}\frac{f(t)q(\sigma(t))n(\sigma(t)) + g(t)q(\rho(t))n(\rho(t))}{n(\sigma(t))}dt\right) \tag{5.4.463}$$

and

$$\phi(x) \leq n(x)q(x)\Big[1 + \int_{x^0}^{x} f(s)q(\sigma(s)) \times \exp\left(\int_{x^0}^{x}\frac{f(t)q(\sigma(t))n(\sigma(t)) + g(t)q(\rho(t))n(\rho(t))}{n(\sigma(t))}dt\right)ds\Big]. \tag{5.4.464}$$

Proof Since $n(x)$ is positive and non-decreasing, from (5.4.462) we derive

$$\begin{aligned}\frac{\phi(x)}{n(x)} &\leq 1 + q(x)\left[\int_{x^0}^{x} f(s)\frac{\phi(\sigma(s))}{n(s)}ds + \int_{x^0}^{x} f(s)\left(\int_{x^0}^{s} g(t)\frac{\phi(\rho(t))}{n(t)}dt\right)ds\right]\\ &\leq q(x)\left[1 + \int_{x^0}^{x} f(s)\frac{\phi(\sigma(s))}{n(s)}ds + \int_{x^0}^{x} f(s)(int_{x^0}^{s} g(t)\frac{\phi(\rho(t))}{n(t)}dt)ds\right].\end{aligned} \tag{5.4.465}$$

Define

$$u(x) = 1 + \int_{x^0}^{x} f(s)\frac{\phi(\sigma(s))}{n(s)}ds + \int_{x^0}^{x} f(s)\left(\int_{x^0}^{s} g(t)\frac{\phi(\rho(t))}{n(t)}dt\right)ds. \tag{5.4.466}$$

Then

$$u(x) = 1 \quad if \quad x_i = x_i^0, \;\; 1 \leq i \leq n$$

and inequality (5.4.465) becomes, for all $x \in B$,

$$\frac{\phi(x)}{n(x)} \leq q(x)u(x). \tag{5.4.467}$$

Now

$$u(\sigma(x)) = 1 + \int_{x^0}^{\sigma(x)} f(s)\frac{\phi(\sigma(s))}{n(s)}ds + \int_{x^0}^{\sigma(x)} f(s)\left(\int_{x^0}^{s} g(t)\frac{\phi(\rho(t))}{n(t)}dt\right) ds \leq u(x).$$

Similarly, $u(\rho(x)) \leq u(x)$, hence, from (5.4.467) it follows

$$\begin{cases} u(\sigma(x)) \leq q(\sigma(x))n(\sigma(x))u(x), \\ u(\rho(x)) \leq q(\rho(x))n(\rho(x))u(x). \end{cases} \tag{5.4.468}$$

Using (5.4.467), the non-decreasing property of $n(x)$, and the property of q, we can arrive at

$$\begin{aligned} D_1D_2\cdots D_n u(x) &= f(x)\frac{\phi(\sigma(x))}{n(x)} + f(s)\left(\int_{x^0}^{x} g(t)\frac{\phi(\rho(t))}{n(t)}dt\right) \\ &\leq f(x)q(\sigma(x))\left[u(x) + \int_{x^0}^{x} g(t)q(\rho(t))\frac{n(\rho(t))}{n(\sigma(t))}u(t)dt\right]. \end{aligned} \tag{5.4.469}$$

Define

$$V(x) = u(x) + \int_{x^0}^{x} g(t)q(\rho(t))\frac{n(\rho(t))}{n(\sigma(t))}u(t)dt,$$

then

$$V(x) = u(x) \ \ if \ \ x_i = x_i^0, \quad 1 \leq i \leq n,$$

and for all $x \geq x^0$,

$$V(x) \geq u(x). \tag{5.4.470}$$

Hence, in view of (5.4.469),

$$\begin{aligned} D_1D_2\cdots D_n V(x) &= D_1D_2\cdots D_n u(x) + g(x)q(\rho(x))\frac{n(\rho(x))}{n(\sigma(x))}u(x) \\ &\leq f(x)q(\sigma(x))V(x) + g(x)q(\rho(x))\frac{n(\rho(x))}{n(\sigma(x))}V(x). \end{aligned}$$

Thus

$$D_1 D_2 \cdots D_n V(x) \le \left\{ \frac{f(x)q(\sigma(x))n(\sigma(x)) + g(x)q(\rho(x))n(\rho(x))}{n(\sigma(x))} \right\} V(x). \tag{5.4.471}$$

Set

$$W(x) = \frac{f(x)q(\sigma(x))n(\sigma(x)) + g(x)q(\rho(x))n(\rho(x))}{n(\sigma(x))},$$

then (5.4.471) gives us

$$D_1 D_2 \cdots D_n V(x) \le W(x)V(x)$$

and

$$\frac{V(x)D_1 D_2 \cdots D_n V(x)}{|V(x)|^2} \le W(x) + \frac{D_n V(x) D_1 D_2 \cdots D_{n-1} V(x)}{|V(x)|^2},$$

i.e.,

$$D_n \left(\frac{D_1 D_2 \cdots D_{n-1} V(x)}{V(x)} \right) \le W(x). \tag{5.4.472}$$

Integrating both sides of (5.4.472) with respect to the component x_n of x from x_n^0 to x_n, we have

$$\frac{D_1 D_2 \cdots D_{n-1} V(x)}{V(x)} \le \int_{x_n^0}^{x_n} W(x_1, \cdots, x_{n-1}, t_n) dt_n.$$

Therefore

$$\frac{V(x)D_1 D_2 \cdots D_{n-1} V(x)}{|V(x)|^2}$$
$$\le \int_{x_n^0}^{x_n} W(x_1, \cdots, x_{n-1}, t_n) dt_n + \frac{D_{n-1} V(x) D_1 D_2 \cdots D_{n-2} V(x)}{|V(x)|^2},$$

i.e.,

$$D_{n-1} \left(\frac{D_1 D_2 \cdots D_{n-2} V(x)}{V(x)} \right) \le \int_{x_n^0}^{x_n} W(x_1, \cdots, x_{n-1}, t_n) dt_n. \tag{5.4.473}$$

Integrating both sides of (5.4.473) with respect to the component x_{n-1} of x from x_{n-1}^0 to x_{n-1}, we obtain

$$\frac{D_1 D_2 \cdots D_{n-2} V(x)}{V(x)} \leq \int_{x_{n-1}^0}^{x_{n-1}} \int_{x_n^0}^{x_n} W(x_1, \cdots, x_{n-2}, t_{n-1}, t_n) dt_n dt_{n-1}.$$

Proceeding in this manner, we arrive at

$$\frac{D_1 V(x)}{V(x)} \leq \int_{x_2^0}^{x_2} \cdots \int_{x_n^0}^{x_n} W(x_1, t_2, \cdots, t_n) dt_n dt_{n-1} \cdots dt_2. \tag{5.4.474}$$

Finally integrating both sides of (5.4.474) with respect to the component x_1 of x from x_1^0 to x_1, we conclude

$$\ln V(x) \leq \int_{x^0}^{x} W(t) dt,$$

i.e.,

$$V(x) \leq \exp(\int_{x^0}^{x} W(t) dt). \tag{5.4.475}$$

Using (5.4.467) and (5.4.470), we have for all $x \geq x^0$,

$$\phi(x) \leq n(x) q(x) \exp\left(\int_{x^0}^{x} \frac{f(t) q(\sigma(t)) n(\sigma(t)) + g(t) q(\rho(t)) n(\rho(t))}{n(\sigma(t))} dt\right)$$

which implies (5.4.463).

To establish (5.4.465), substituting (5.4.475) into (5.4.469), we obtain

$$D_1 D_2 \cdots D_n u(x) \leq f(x) q(\sigma(x)) \exp\left(\int_{x^0}^{x} W(t) dt\right). \tag{5.4.476}$$

Integrating (5.4.476) first with respect to the component x_n of x from x_n^0 to x_n, then with respect to the component x_{n-1} of x from x_{n-1}^0 to x_{n-1} and continuing in this way up to the component x_1 of x from x_1^0 to x_1, we conclude

$$u(x) \leq 1 + \int_{x^0}^{x} f(s) q(\sigma(s)) \exp\left(\int_{x^0}^{s} W(t) dt\right) ds. \tag{5.4.477}$$

Thus inserting (5.4.477) into (5.4.467), we obtain

$$\phi(x) \leq q(x) n(x) \Big[1 + \int_{x^0}^{x} f(s) q(\sigma(s)) \exp\left(\int_{x^0}^{s} \frac{f(t) q(\sigma(t)) n(\sigma(t)) + g(t) q(\rho(t)) n(\rho(t))}{n(\sigma(t))} dt\right) ds\Big]$$

which yields (5.4.464). □

Remark 5.4.20 When $\sigma(x) = \rho(x) = x$ and $q(x) = 1$, Theorem 5.4.63 with the estimate (5.4.464) reduces to a result of Yeh and Shih [673].

Remark 5.4.21 When $\sigma(x) = \rho(x) = x$, $g(t) = 0$, and $q(x) = 1$, the estimate (5.4.464) is due to Zahariev and Bainov [683] if $n(x)$ is taken as $\phi(x^0)$. For $\sigma(x) = \rho(x) = x$, $g(t) = 0$ and $x^0 = 0$, the estimate (5.4.464) reduces to a result due to Yeh [668].

Remark 5.4.22 When $n = 1$, $q(x) = 1$ and $g(t) = 0$, an estimate (5.4.463) of Theorem 5.4.63 gives us a functional integral inequality obtained by Dhongade-Deo [182].

Remark 5.4.23 When $n = 1$ and $x^0 = 0$, Theorem 5.4.63 with estimate (5.4.462) is another generalization of the inequality due to Pachpatte [458] and if, in addition, $n(x)$ is a constant, we can obtain a generalization of [445].

Corollary 5.4.16 (Akinyele [24]) *Let the inequality (5.4.460) hold for all $x \in B$, with $\sigma(x) = \rho(x)$. Then*

$$\phi(x) \le q(x)n(x)\exp\left(\int_{x^0}^{x} q(\sigma(t))\{f(t) + g(t)\}dt\right) \tag{5.4.478}$$

and for all $x \in B$ with $x \ge x^0$,

$$\phi(x) \le q(x)n(x)\left[1 + \int_{x^0}^{x} f(s)q(\sigma(s))\exp\left(\int_{x^0}^{s} q(\sigma(t))(f(t) + g(t))dt\right)ds\right]. \tag{5.4.479}$$

Remark 5.4.24 If $q(x) \equiv 1$, the two estimates in Corollary 5.4.16 are independent of the delay $\sigma(t)$. The second estimate in that case coincides with that of Yeh and Shih [673]. If $n = 1$ and $\sigma(t) = t$, then Corollary 5.4.16 reduces to a unified version of the results of Pachpatte [445].

Theorem 5.4.64 (Akinyele [24]) *Let $\phi(x)$, $n(x)$, $g(x)$, $q(x)$, $\sigma(x)$ and $\rho(x)$ be as defined in Theorem 5.4.63. Let Ω be a continuous function defined on $\mathbb{R}$ into $\mathbb{R}$ such that $\Omega(u)$ is positive, non-decreasing, and sub-multiplicative for all $u > 0$ and $\Omega(0) = 0$. If for all $x \ge x^0$, and $D_k\Omega(u) \ge 0$ for $k = 2, 3, \cdots, n$,*

$$\phi(x) \le n(x) + q(x)\left[\int_{x_0}^{x} g(s)\left(\phi(\sigma(s)) + \int_{x_0}^{s} g(t)\Omega(\phi(\rho(t)))dt\right)ds\right], \tag{5.4.480}$$

then for all $x \in B$ with $x \ge x^0$,

$$\phi(x) \le n(x)q(x)G^{-1}\left[G(1) + \int_{x_0}^{x} \frac{g(y)p(y)}{n(\sigma(y))}dy\right] \tag{5.4.481}$$

and

$$\phi(x) \le n(x)q(x)\left[1 + \int_{x_0}^{x} g(s)q(\sigma(s))G^{-1}[G(1) + \int_{x_0}^{s} \frac{g(t)p(t)}{n(\sigma(t))}dt]ds\right] \tag{5.4.482}$$

where $p(t) = \max\{q(\sigma(t))n(\sigma(t)), \Omega(q(\rho(t))n(\rho(t)))\}$,

$$G(r) = \int_{r^0}^{r} \frac{ds}{s + \Omega(s)}, \qquad r \ge r^0 > 0, \tag{5.4.483}$$

and G^{-1} *is the inverse of* G *and* $x \ge x^0$ *such that*

$$G(1) + \int_{x_0}^{s} \frac{g(t)p(t)}{n(\sigma(t))}dt \in Dom\ (G^{-1}).$$

Proof Since $n(x)$ is positive and non-decreasing, (5.4.480) becomes

$$\frac{\phi(x)}{n(x)} \le q(x)\left[1 + \int_{x_0}^{x} g(s)\left(\frac{\phi(\sigma(s))}{n(x)} + \int_{x_0}^{s} \frac{g(t)}{n(t)}\Omega(\phi(\rho(t)))dt\right)ds\right]. \tag{5.4.484}$$

Define

$$u(x) = 1 + \int_{x_0}^{x} g(s)\left(\frac{\phi(\sigma(s))}{n(x)} + \int_{x_0}^{s} \frac{g(t)}{n(t)}\Omega(\phi(\rho(t)))dt\right)ds, \tag{5.4.485}$$

then

$$\begin{cases} u(x) = 1 \ \ for \ \ x_i = x_i^0, 1 \le i \le n, \\ \phi(\sigma(x)) \le n(\sigma(x))q(\sigma(x))u(x) \le n(x)q(\sigma(x))u(x), \end{cases}$$

and

$$\phi(\rho(x)) \le n(\rho(x))q(\rho(x))u(x) \le n(x)q(\rho(x))u(x).$$

Now

$$\begin{aligned} D_1D_2\cdots D_nu(x) &= g(x)\left(\frac{\phi(\sigma(s))}{n(x)} + \int_{x_0}^{s} \frac{g(t)}{n(t)}\Omega(\phi(\rho(t)))dt\right) \\ &\le g(x)\left(q(\sigma(x))u(x) + \int_{x_0}^{x} \frac{g(t)}{n(\sigma(t))}\Omega(q(\rho(t))n(\rho(t)))\Omega(u(t))dt\right) \\ &\le g(x)q(\sigma(x))\left[u(x) + \int_{x^0}^{x} \frac{g(t)}{n(\sigma(t))}\Omega(q(\rho(t))n(\rho(t)))\Omega(u(t))dt\right]. \end{aligned}$$

Define

$$w(x) = u(x) + \int_{x^0}^{x} \frac{g(t)}{n(\sigma(t))} \Omega(q(\rho(t))n(\rho(t)))\Omega(u(t))dt,$$

then

$$w(x) = u(x), \quad if \;\; x_i = x_i^0 \;\; for \;\; 1 \le i \le n,$$

and

$$u(x) \le w(x) \quad for\ all \quad x \in B. \tag{5.4.486}$$

Moreover,

$$D_1 D_2 \cdots D_n u(x) \le g(x)q(\sigma(x))w(x) \tag{5.4.487}$$

whence

$$\begin{aligned} D_1 D_2 \cdots D_n w(x) &= D_1 D_2 \cdots D_n u(x) + \frac{g(x)}{n(\sigma(x))} \Omega[q(\rho(x))n(\rho(x))]\Omega(u(x)) \\ &\le g(x)q(\sigma(x))w(x) + \frac{g(x)}{n(\sigma(x))} \Omega[q(\rho(x))n(\rho(x))]\Omega(w(x)) \\ &\le \frac{g(x)}{n(\sigma(x))}[q(\sigma(x))n(\sigma(x))w(x) + \Omega(q(\rho(x))n(\rho(x)))\Omega(w(x))]. \end{aligned}$$

Let

$$p(x) = \max\{q(\sigma(x))n(\sigma(x)), \Omega(q(\rho(x))n(\rho(x)))\}, \tag{5.4.488}$$

then we get

$$D_1 D_2 \cdots D_n w(x) \le \frac{g(x)}{n(\sigma(x))} p(x)[w(x) + \Omega(w(x))]$$

and

$$\begin{aligned} &\frac{[w(x) + \Omega(w(x))]D_1 D_2 \cdots D_n w(x)}{[w(x) + \Omega(w(x))]^2} \\ &\quad \le \frac{g(x)p(x)}{n(\sigma(x))} + \frac{D_n(w(x) + \Omega(w(x)))D_1 D_2 \cdots D_{n-1} w(x)}{[w(x) + \Omega(w(x))]^2}. \end{aligned}$$

That is,

$$D_n\left[\frac{D_1D_2\cdots D_{n-1}w(x)}{w(x)+\Omega(w(x))}\right]\le\frac{g(x)p(x)}{n(\sigma(x))}.$$

Using the same arguments as in Theorem 5.4.63, we arrive at

$$\frac{D_1w(x)}{w(x)+\Omega(w(x))}\le\int_{x_2^0}^{x_2}\cdots\int_{x_n^0}^{x_n}\frac{g(x_1,y_2\cdots y_n)p(x_1,y_2\cdots y_n)}{n(\sigma(x_1,y_2\cdots y_n))}dy_n\cdots dy_2.$$

By (5.4.483), we obtain

$$D_1[G(w(x))]\le\int_{x_2^0}^{x_2}\cdots\int_{x_n^0}^{x_n}\frac{g(x_1,y_2\cdots y_n)p(x_1,y_2\cdots y_n)}{n(\sigma(x_1,y_2\cdots y_n))}dy_n\cdots dy_2.$$

Integrating with respect to the component x_1 of x from x_1^0 to x_1, we have

$$G(w(x))\le G(1)+\int_{x^0}^{x}\frac{g(y)p(y)}{n(\sigma(y))}dy,$$

i.e.,

$$w(x)\le G^{-1}[G(1)+\int_{x^0}^{x}\frac{g(y)p(y)}{n(\sigma(y))}dy]. \tag{5.4.489}$$

Thus (5.4.484)–(5.4.486) and (5.4.489) readily imply

$$\phi(x)\le n(x)q(x)G^{-1}[G(1)+\int_{x^0}^{x}\frac{g(y)p(y)}{n(\sigma(y))}dy]$$

which gives us (5.4.481), where p is as defined in (5.4.488). Now, putting (5.4.489) into inequality (5.4.487), we obtain

$$D_1D_2\cdots D_nu(x)\le g(x)q(\sigma(x))G^{-1}[G(1)+\int_{x^0}^{x}\frac{g(y)p(y)}{n(\sigma(y))}dy]. \tag{5.4.490}$$

Integrating both sides of (5.4.490) by using same arguments as in Theorem 5.4.63, we conclude

$$u(x)\le 1+\int_{x^0}^{x}g(s)q(\sigma(s))G^{-1}[G(1)+\int_{x^0}^{s}\frac{g(y)p(y)}{n(\sigma(y))}dy]. \tag{5.4.491}$$

Therefore, using (5.4.484), (5.4.485) and (5.4.491), we can obtain (5.4.482), and the proof is now complete. □

Corollary 5.4.17 (Akinyele [24]) *Let all the hypotheses of Theorem 5.4.64 hold. If inequality (5.4.478) holds, then*

$$\phi(x) \leq n(x)q(x)G^{-1}[G(1) + \int_{x^0}^{x} g(y)q(\sigma(y))dy], \tag{5.4.492}$$

or

$$\phi(x) \leq n(x)q(x)G^{-1}\left[G(1) + \int_{x^0}^{x} \frac{g(y)}{n(\sigma(y))}\Omega(q(\rho(y))n(\rho(y)))dy\right] \tag{5.4.493}$$

or

$$\phi(x) \leq n(x)q(x)\left[1 + \int_{x^0}^{x} g(s)q(\sigma(s))G^{-1}\left[G(1) + \int_{x^0}^{s} g(t)q(\sigma(t))dt\right]ds\right] \tag{5.4.494}$$

or

$$\phi(x) \leq n(x)q(x)\Big\{1 + \int_{x^0}^{x} g(s)q(\sigma(s))G^{-1}\Big[G(1) + \int_{x^0}^{s} \frac{g(t)}{n(\sigma(t))}\Omega(q(\rho(t))n(\rho(t)))dt\Big]\Big\}. \tag{5.4.495}$$

Remark 5.4.25 The integral inequalities of Corollary 5.4.17 have extended Pachpatte's result [458] to n-independent variables with delays. For $\rho(x) = \sigma(x) = x$, we also have a new generalization of Yeh and Shih's result [673].

We now establish other useful n-independent variable generalization due to [24], of the Bellman-Bihari type inequality with delay.

Theorem 5.4.65 (Akinyele [24]) *Let $\phi(x)$, $n(x)$, $g(x)$, $q(x)$ and $\sigma(x)$ be as defined in Theorem 5.4.63. Suppose $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function such that $\Omega(u)$ is positive, non-decreasing, and sub-multiplicative for all $u > 0$ and $\Omega(0) = 0$. If for all $x \in B$, and $D_k\Omega(u) \geq 0$ for $k = 2, 3, \cdots, n$,*

$$\phi(x) \leq n(x) + q(x)\int_{x_0}^{x} g(s)\Omega(\phi(\sigma(s)))ds, \tag{5.4.496}$$

then for all $x^0 \leq x < x^*$,

$$\phi(x) \leq n(x)q(x)G^{-1}\left[G(1) + \int_{x_0}^{x} \frac{g(y)}{n(y)}dy\Omega(q\sigma(y))n(\sigma(y))\right], \tag{5.4.497}$$

where

$$G(r) = \int_{r^0}^{r} \frac{ds}{\Omega(s)}, \qquad r \geq r^0 > 0,$$

and G^{-1} is the inverse of G and x^* is chosen so that for all x satisfying $x^0 \le x < x^*$,

$$G(1) + \int_{x_0}^{s} \frac{g(y)}{n(y)} \Omega(q(\sigma(y))n(\sigma(y)))dy \in Dom\,(G^{-1}).$$

Proof As before, we have

$$\frac{\phi(x)}{n(x)} \le q(x) \left[1 + \int_{x_0}^{x} \frac{g(y)}{n(y)} \Omega(\phi(\sigma(y)))dy \right]. \tag{5.4.498}$$

Define

$$u(x) = 1 + \int_{x_0}^{x} \frac{g(y)}{n(y)} \Omega(\phi(\sigma(y)))dy,$$

then

$$u(x) = 1 \quad \textit{if} \quad x_i = x_i^0, \quad 1 \le i \le n$$

and

$$\phi(\sigma(x)) \le q(\sigma(x))n(\sigma(x))u(x),$$

whence

$$D_1 D_2 \cdots D_n u(x) \le \frac{g(x)}{n(x)} \Omega(q(\sigma(x))n(\sigma(x)))\Omega(u(x))$$

and

$$\frac{\Omega(u(x))D_1 D_2 \cdots D_n u(x)}{|\Omega(u(x))|^2} \le \frac{g(x)}{n(x)} \Omega(q(\sigma(x))n(\sigma(x))) + \frac{D_n \Omega(u(x)) D_1 D_2 \ldots D_{n-1} u(x)}{\Omega(u(x))^2}.$$

Thus

$$D_n \left(\frac{D_1 D_2 \cdots D_{n-1} u(x)}{\Omega(u(x))} \right) \le \frac{g(x)}{n(x)} \Omega(q(\sigma(x))n(\sigma(x))).$$

Integrating as before from x^0 to x, we obtain

$$G(u(x)) \le G(1) + \int_{x_0}^{x} \frac{g(y)}{n(y)} \Omega(q(\sigma(y))n(\sigma(y)))dy,$$

i.e.,

$$u(x) \leq G^{-1}[G(1) + \int_{x_0}^{x} \frac{g(y)}{n(y)} \Omega(q(\sigma(y))n(\sigma(y)))dy]. \tag{5.4.499}$$

Inserting (5.4.499) into (5.4.498) gives us (5.4.495). □

5.4.3 Multi-Dimensional Gronwall-Bellman-Bihari Inequalities with General Kernels

In this section, we shall introduce some multi-dimensional Gronwall-Bellman-Bihari inequalities with general kernels.

The next result, due to Oguntuase [436], is to establish some new integral inequalities in n independent variables with general kernels.

We shall assume that S is any bounded open set in $\mathbb{R}^n$ and that our integrals are on $\mathbb{R}^n$ ($n \geq 1$), all functions considered are functions of n-variables which are non-negative and continuous on $[x_0, x]$, $x \geq x^0 \geq 0$ and $x \in S$ unless otherwise specified.

We shall obtain bounds to the linear Gronwall-Bellman-Bihari type integral inequalities for a more general kernel $k(x,t)$ and a product kernel $k(x,t) = h(x)f(t)$.

Definition 5.4.4 A function $k(x,t)$ of the $2n$ variables $x_1, \cdots x_n, t_1, \cdots, t_n$ is called a good kernel if

(1) $k(\cdot, \cdot) \geq 0$.
(2) $k(\cdot, \cdot)$ is a continuous function of its $2n$ variables.
(3) $k(\cdot, \cdot)$ is monotone non-decreasing in its first n variables, i.e., $k(x,t) \geq k(y,t)$ whenever $x \geq y$.

Theorem 5.4.66 (Oguntuase [436]) *Let $k(x,t)$ be a good kernel, $u(x)$ be a real-valued non-negative continuous function on S, and $g(x)$ be a positive, non-decreasing continuous function on S. Suppose that the following inequality holds for all $x \in S$ with $x \geq x^0$,*

$$u(x) \leq g(x) + \int_{x^0}^{x} k(x,t)u(t)dt, \tag{5.4.500}$$

then

$$u(x) \leq g(x) \left[1 + \int_{x^0}^{x} k(s,s) \exp\left(\int_{x^0}^{s} k(t,t)dt \right) ds \right]. \tag{5.4.501}$$

Proof Since $g(x)$ is positive and non-decreasing, we can rewrite (5.4.500) as

$$\frac{u(x)}{g(x)} \leq 1 + \int_{x_0}^{x} k(x,t)\frac{u(t)}{g(t)}dt.$$

Setting $\frac{u(x)}{g(x)} = r(x)$, then

$$r(x) \leq 1 + \int_{x^0}^{x} k(x,t)r(t)dt.$$

Let

$$v(x) = 1 + \int_{x^0}^{x} k(x,t)r(t)dt.$$

Then

$$r(x) \leq v(x)$$

and $v(x^0) = 1$ or $x_i = x_i^0, \quad i = 1, 2, \cdots, n$. Hence

$$D_1 \cdots D_n v(x) = k(x,x)r(x) \leq k(x,x)v(x). \tag{5.4.502}$$

From (5.4.502) it follows

$$\frac{v(x)D_1 \cdots D_n v(x)}{v^2(x)} \leq k(x,x),$$

that is

$$\frac{v(x)D_1 \cdots D_n v(x)}{v^2(x)} \leq k(x,x) + \frac{D_n v(x) D_1 \cdots D_{n-1} v(x)}{v^2(x)},$$

whence

$$D_n \left(\frac{D_1 \cdots D_{n-1} v(x)}{v(x)} \right) \leq k(x,x).$$

Integrating with respect to x_n from x_n^0 to x_n, we get

$$\frac{D_1 \cdots D_{n-1} v(x)}{v(x)} \leq \int_{x_n^0}^{x_n} k(x_1, x_2, \cdots, x_{n-1}, t_n, x_1, x_2, \cdots, x_{n-1}, t_n)dt_n. \tag{5.4.503}$$

Thus

$$\frac{v(x)D_1\cdots D_{n-1}v(x)}{v^2(x)}$$

$$\leq \int_{x_n^0}^{x_n} k(x_1,x_2,\cdots,x_{n-1},t_n,x_1,x_2,\cdots,x_{n-1},t_n)dt_n + \frac{(D_{n-1}v(x))(D_1\cdots D_{n-2}v(x))}{v^2(x)},$$

i.e.,

$$D_{n-1}\left(\frac{D_1\cdots D_{n-2}v(x)}{v(x)}\right) \leq \int_{x_n^0}^{x_n} k(x_1,x_2,\cdots,x_{n-1},t_n,x_1,x_2,\cdots,x_{n-1},t_n)dt_n.$$

Integrating with respect to x_{n-1} from x_{n-1}^0 to x_{n-1}, we have

$$\frac{D_1\cdots D_{n-2}v(x)}{v(x)} \leq \int_{x_{n-1}^0}^{x_{n-1}}\int_{x_n^0}^{x_n} k(x_1,x_2,\cdots,x_{n-2},t_{n-1},t_n,x_1,x_2,\cdots,x_{n-2},t_{n-1},t_n)dt_n dt_{n-1}.$$

Continuing this process, we obtain

$$\frac{D_1D_2v(x)}{v(x)} \leq \int_{x_3^0}^{x_3}\cdots\int_{x_n^0}^{x_n} k(x_1,x_2,t_3,\cdots,t_n,x_1,x_2,t_3,\cdots,t_n)dt_n\cdots dt_3$$

which gives us

$$D_2\left(\frac{D_1v(x)}{v(x)}\right) \leq \int_{x_3^0}^{x_3}\cdots\int_{x_n^0}^{x_n} k(x_1,x_2,t_3,\cdots,t_n,x_1,x_2,t_3,\cdots,t_n)dt_n\cdots d_{t_3}.$$

Integrating with respect to the x_2 component from x_2^0 to x_2, we get

$$\frac{D_1v(x)}{v(x)} \leq \int_{x_2^0}^{x_2}\cdots\int_{x_n^0}^{x_n} k(x_1,t_2,t_3,\cdots,t_n,x_1,t_2,t_3,\cdots,t_n)dt_n\cdots dt_2. \tag{5.4.504}$$

Integrating the above inequality with respect to the x_1 component from x_1^0 to x_1, we obtain

$$\log\frac{v(x)}{v(x_1^0,x_2,\cdots,x_n)} \leq \int_{x^0}^{x} k(t,t)dt$$

which implies

$$v(x) \leq \exp\left(\int_{x^0}^{x} k(t,t)dt\right). \tag{5.4.505}$$

Substituting (5.4.505) into (5.4.502), we get

$$D_1 \cdots D_n r(x) \le k(x,x)v(x) \le k(x,x)\exp\left(\int_{x^0}^{x} k(t,t)dt\right). \tag{5.4.506}$$

Integrating the above inequality with respect to x_n component from x_n^0 to x_n, then with respect to x_{n-1} component from x_{n-1}^0 to x_{n-1}, and continuing until finally x_1^0 to x_1, and noting that $r(x) = 1$ at $x_i = x_i^0$, we conclude

$$r(x) \le 1 + \int_{x^0}^{x} k(s,s)\exp\left(\int_{x^0}^{x} k(t,t)dt\right) ds.$$

Since $\frac{u(x)}{g(x)} = r(x)$, we get

$$u(x) \le g(x)\left\{1 + \int_{x^0}^{x} k(s,s)\exp\left(\int_{x^0}^{x} k(t,t)dt\right) ds\right\},$$

which thus completes the proof. □

Next, we shall consider the case in which $k(x,t) = h(x)f(t)$. Then we have the following result.

Theorem 5.4.67 (Oguntuase [436]) *Let $h(x)$, $f(t)$, $u(x)$ be real-valued non-negative continuous functions on S and $g(x)$ be a positive, non-decreasing continuous function on S. If $h'(x) = 0$, where the prime denotes $\frac{\partial^n}{\partial_{x_1}\cdots\partial_{x_n}}$ and the following inequality holds for all $x \in S$ with $x \ge x^0$,*

$$u(x) \le g(x) + h(x)\int_{x^0}^{x} f(t)u(t)dt, \tag{5.4.507}$$

then

$$u(x) \le g(x)\left[1 + \int_{x^0}^{x} h(s)f(s)\exp\left(\int_{x^0}^{s} h(t)f(t)dt\right) ds\right]. \tag{5.4.508}$$

Proof Similar to the proof of Theorem 5.4.66, we can prove the theorem. □

Remark 5.4.26 If $k(x,t) = f(t)$ in Theorem 5.4.67, then estimate (5.4.501) reduces to

$$u(x) \le g(x)\left[1 + \int_{x^0}^{x} f(s)\exp\left(\int_{x^0}^{x} f(t)dt\right) ds\right]. \tag{5.4.509}$$

5.4.4 Linear Multi-Dimensional Continuous Integral Inequalities of Volterra Type

The next result, due to Defranco [171], establishes a generalization of the Gronwall inequality (Theorem 1.1.1) for a class of systems of multiple Volterra integral equations.

Unless specified otherwise, we shall adopt the following notation.

Let $|\cdot|$ be an arbitrary vector norm on $\mathbb{R}^n$ and let $\| M \| = \sup_{|x|=1}|Mx|$ be the norm of an $m \times n$ matrix M. Let α_k denote any combination of the integers $\{1, 2, \cdots, n\}$ taken k at a time.

We shall assume that the elements in any combination $\alpha_k = \{i_1, i_2, \cdots, i_k\}$ have been ordered (i.e., $i_1 < i_2 < \cdots < i_k$) . Given the combination α_k, we let $\alpha'_k = \{1, 2, \cdots, n\} - \alpha_k$.

Let $\alpha_k = \{i_1, i_2, \cdots, i_k\}$. For all $x \in \mathbb{R}^n$, we define $x_{\alpha_k} = (x_{i_1}, x_{i_2}, \cdots, x_{i_k})$. We denote the multiple integral symbol $\int_{a_{i_1}}^{x_{i_1}} \int_{a_{i_2}}^{x_{i_2}} \cdots \int_{a_{i_k}}^{x_{i_k}}$ by $\int_{a_{\alpha_k}}^{x_{\alpha_k}}$ and the sequence of differentials $dr_{i_k} dr_{i_{k-l}} \cdots dr_{i_1}$ by $dr_{\alpha k}$. If $g : \mathbb{R}^n \to \mathbb{R}^m$, we define the partial derivative $g_{x_{i_1} x_{i_2} \cdots x_{i_k}}$. For $x, y \in \mathbb{R}^n$, we introduce the following: let

$$w_i(x, y; \alpha_k) = \begin{cases} x_i, & if \quad i \bar{\in} \alpha_k, \\ y_i, & if \quad i \in \alpha_k \end{cases}$$

and $w(x, y; \alpha_k) = (w_1(x, y; \alpha_k), w_2(x, y; \alpha_k), \cdots, w_n(x, y; \alpha_k))$.

Let $u, \phi : [a, b] \to \mathbb{R}^m$ and for each α_k , $1 \leq k \leq n$, let $K_{\alpha_k}(x, r_{\alpha_k})$ be an $m \times m$ matrix function for $x \in [a, b], a_{\alpha_k} \leq r_{\alpha_k} \leq x_{\alpha_k} \leq b_{\alpha_k}$.

We now consider the linear system of m Volterra integral equations in n independent variables of the form

$$u(x) = \phi(x) + \sum_{\alpha_k, 1 \leq k \leq n} \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k}) u(w(x, r; \alpha_k)) dr_{\alpha_k}. \tag{5.4.510}$$

It may be shown that if $\phi(x)$ and the matrix functions $K_{\alpha k}(x, r_{\alpha k})$ are continuous, then equation (5.4.510) has a unique continuous solution on $[a, b]$. Also, we observe that there are $2^n - 1$ integrals appearing in Eq. (5.4.510). We simplify the notation further by using $\sum$ to mean $\sum_{\alpha_k, 1 \leq k \leq n}$.

We now define the fundamental solution for Eq. (5.4.510).

Definition 5.4.5 Suppose the matrix function $A(x; \xi)$ satisfies the equation

$$A(x; \xi) = I + \sum_{\alpha_k, 1 \leq k \leq n} \int_{\xi_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha k}(x, r_{\alpha_k}) A(w(x, r; \alpha_k); \xi) dr_{\alpha_k} \tag{5.4.511}$$

for $a \leq \xi \leq x \leq b$. Then $A(x; \xi)$ is called a fundamental solution for equation (5.4.510).

If the functions $K_{\alpha_k}(x, r_{\alpha_k})$ are continuous, it is easy to check that $A(x;\xi)$ exists, is unique, and is continuous for $a \le \xi \le x \le b$. The following theorem, which may be verified directly, gives us the solution of Eq. (5.4.510) in terms of the fundamental solution. The two forms for the solution of Eq. (5.4.510) given in this theorem allow us to establish analogues for inequalities (1.2.5) and (1.2.2).

Theorem 5.4.68 (Defranco [171]) *Suppose that $\phi(x)$ is continuous on $[a, b]$ and each matrix function $K_{\alpha_k}(x, r_{\alpha_k})$ is continuous for $x \in [a, b]$, $a_{\alpha_k} \le r_{\alpha_k} \le x_{\alpha_k} \le b_{\alpha_k}$.*

(i) If each $A_{\xi_{\alpha k}}(x;\xi)$ is continuous for $a \le \xi \le x \le b$, then the unique continuous solution of Eq. (5.4.510) on $[a, b]$ is

$$u(x) = \phi(x) + \sum_{\alpha_k, 1\le k\le n} (-1)^k \int_{a_{\alpha_k}}^{x_{\alpha_k}} A_{r_{\alpha_k}}(x; w(x, r; \alpha_k))\phi(w(x, r; \alpha_k))dr_{\alpha_k}. \tag{5.4.512}$$

(ii) If each $\phi_{x_{\alpha k}}$ is continuous on $[a, b]$, then the unique continuous solution of Eq. (5.4.510) can be represented as

$$u(x) = A(x; a)\phi(a) + \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} A(x; w(a, r; \alpha_k))\phi_{r_{\alpha_k}}(w(a, r; \alpha_k))dr_{\alpha_k}. \tag{5.4.513}$$

The above result is based on Theorem 5.4.68 and the following theorem may be found in [281].

Theorem 5.4.69 (Hille [281]) *Suppose F is a complete metric space and is partially ordered in such a way, that if an increasing sequence $(g_n) \subset F$ has the limit g_0, then $g_n < g_0$ for all n. Suppose T is an order preserving ($f_1 < f_2 \Rightarrow Tf_1 < Tf_2$) contraction on F and f_0 is the unique fixed point of T. Then $f \in F$ and $f < Tf$ implies $f < f_0$.*

For any real λ, consider the space $C_\lambda[a, b]$ consisting of the set of continuous functions $g : [a, b] \to \mathbb{R}^m$ normed by $\| g \|_\lambda = \sup_{x\in[a,b]}\left\{|g(x)| \exp[-\lambda(\sum_{i=1}^n x_i)]\right\}$. It may be shown that $C_\lambda[a, b]$ is a Banach space. Let $K \subset C_\lambda[a, b]$ be the positive cone of functions such that $\phi(x) = (\phi_1(x), \phi_2(x), \cdots, \phi_m(x)) \in K$ iff $\phi_i(x) \ge 0$, $1 \le i \le m$. We consider the partial order on $C_\lambda[a, b]$ defined such that $g_1 < g_2$ iff $g_2 - g_1 \in K$. We notice that this partial order has the property that if (g_n) is an increasing sequence in $C_\lambda[a, b]$ converging to g_0, then $g_n < g_0$ for all n.

Theorem 5.4.70 (Defranco [171]) *Suppose $\phi(x)$ is continuous on $[a, b]$ and for each α_k with $1 \le k \le n$, the $m\times m$ matrix function $K_{\alpha_k}(x, r_{\alpha_k})$ is continuous and has non-negative elements. If $u : [a, b] \to \mathbb{R}^m$ is continuous and for all $x \in [a, b]$,*

$$u(x) \le \phi(x) + \sum_{\alpha_k, 1\le k\le n} \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k})u(w(x, r; \alpha_k))dr_{\alpha_k}, \tag{5.4.514}$$

then $u(x) < v(x)$ *where* $v(x)$ *is the unique continuous solution of Eq. (5.4.510) on* $[a,b]$. *If, in addition, each* $A_{\xi_{\alpha_k}}(x;\xi)$ *is continuous on* $a \le \xi \le x \le b$, *then*

$$u(x) < \phi(x) + \sum_{\alpha_k, 1\le k\le n} (-1)^k \int_{a_{\alpha k}}^{x_{\alpha_k}} A_{r_{\alpha_k}}(x; w(x,r;\alpha_k))\phi(w(x,r;\alpha_k))dr_{\alpha_k} \tag{5.4.515}$$

or if, instead, we make the additional assumption that each $\phi_{x_{\alpha_k}}(x)$ *is continuous on* $[a,b]$, *then*

$$u(x) < A(x;a)\phi(a) + \sum_{\alpha_k, 1\le k\le n} \int_{a_{\alpha_k}}^{x_{\alpha_k}} A(x; w(a,r;\alpha_k))\phi_{r_{\alpha_k}}(w(a,r;\alpha_k))dr_{\alpha_k}. \tag{5.4.516}$$

Proof The continuity of each $K_{\alpha_k}(x, r_{\alpha_k})$ on the compact domain implies there is a constant $M > 0$ such that $\| K_{\alpha_k}(x, r_{\alpha_k}) \| \le M$ for each α_k. Choose λ_0 so that $\lambda_0 > 1$ and $\frac{M(2^n-1)}{\lambda_0} < 1$. Define T on $C_{\lambda_0}[a,b]$ such that for $g \in C_{\lambda_0}[a,b]$, then

$$(Tg)(x) = \phi(x) + \sum_{\alpha_k, 1\le k\le n} \int_{a_{\alpha_k}}^{x_{\alpha_k}} K_{\alpha_k}(x, r_{\alpha_k})g(w(x,r;\alpha_k))dr_{\alpha_k}.$$

It follows from the continuity of the functions $\phi(x)$, $g(x)$, and $K_{\alpha_k}(x, r_{\alpha_k})$ that $(Tg)(x)$ is continuous on $[a,b]$.

Take $g_1, g_2 \in C_{\lambda_0}[a,b]$. Then we have

$$\begin{aligned}
&| (Tg_1)(x) - (Tg_2)(x) | \\
&\le \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} M \mid g_1(w(x,r;\alpha_k)) - g_2(w(x,r;\alpha_k)) \mid \\
&\quad \times \exp\Big[-\lambda_0\Big(\sum_{i=1}^{n} w_i(x,r;\alpha_k)\Big)\Big] \exp\Big[\lambda_0\Big(\sum_{i=1}^{n} w_i(x,r;\alpha_k)\Big)\Big] dr_{\alpha_k} \\
&\le M \| g_1 - g_2 \|_{\lambda_0} \sum \int_{a_{\alpha_k}}^{x_{\alpha_k}} \exp\Big[\lambda_0\Big(\sum_{i=1}^{n} w_i(x,r;\alpha_k)\Big)\Big] dr_{\alpha_k} \\
&\le M \| g_1 - g_2 \|_{\lambda_0} \sum \frac{1}{\lambda_0^k} \exp\Big[\lambda_0\Big(\sum_{i=1}^{n} x_i\Big)\Big] \\
&\le \frac{M(2^n-1)}{\lambda_0} \| g_1 - g_2 \|_{\lambda_0} \exp\Big[\lambda_0\Big(\sum_{i=1}^{n} x_i\Big)\Big].
\end{aligned}$$

Therefore $\| Tg_1 - Tg_2 \|_{\lambda_0} \leq \frac{M(2^n-1)}{\lambda_0} \| g_1 - g_2 \|_{\lambda_0}$ and T is a contraction on $C_{\lambda_0}[a, b]$.

Now suppose $g_1, g_2 \in C_{\lambda_0}[a, b]$ such that $g_1 < g_2$. Then since the elements in each $K_{\alpha_k}(x, r_{\alpha_k})$ are non-negative, we see that for each α_k,

$$K_{\alpha_k}(x, r_{\alpha_k}) g_1(w(x, r; \alpha_k)) < K_{\alpha_k}(x, r_{\alpha_k}) g_2(w(x, r; \alpha_k)).$$

Thus $Tg_1 < Tg_2$ and T is order preserving. Thus the result now follows directly from Theorems 5.4.68 and 5.4.69. □

We note that the above method of proof depends on the proper choice of norm on the space of continuous functions. For further discussion of this idea and related topics, the reader is referred to Chu and Diaz [134].

In the case when $m = n = 1$ and $K_1(x, r) = k(r)$ (x, r real), we see the $A(x; r) = \exp\left(\int_r^x k(s)ds\right)$ and estimates (5.4.515) and (5.4.516) reduce to estimates (1.2.2) and (1.2.5) respectively. The form of the estimates (5.4.515) and (5.4.516) suggest that the fundamental solution defined above is the natural generalization of the exponential function appearing in Theorems 1.2.1–1.2.2; in fact the fundamental solution defined here is a generalization of the fundamental matrix in the theory of ordinary differential equations.

We now turn to a discuss several generalizations of Theorems 1.2.1–1.2.2 and show how some of these may be obtained from Theorem 5.4.70.

Chu and Metcalf [135] had given an extension of the Gronwall's inequality for scalar integral equations in one variable, see Theorem 1.2.38.

It has been shown [625] that the resolvent kernel $H(x, r)$ (see, Theorem 1.2.38) satisfies the integral equation

$$H(x, \xi) = K(x, \xi) + \int_\xi^x K(x, r) H(r, \xi) dr.$$

Under the assumption that $A_\xi(x; \xi)$ is continuous for $0 \leq \xi \leq x \leq 1$, in this case, from Eq. (5.4.511) it follows that

$$A_\xi(x, \xi) = -K(x, \xi) + \int_\xi^x K(x, r) H(r, \xi) dr.$$

Hence, by uniqueness, we see that $A_\xi(x, \xi) = -H(x, \xi)$. Using this fact and Theorem 5.4.68 in the form of inequality (5.4.515) (with $m = n = 1$ and $a = 0$), we can obtain the result given by Chu and Metcalf [135], i.e., Theorem 1.2.38.

Conlan and Diaz [140] have used the following generalization of the Gronwall's inequality to study existence and uniqueness for an n-th order hyperbolic partial differential equation.

Theorem 5.4.71 (Conlan-Diaz [140]) *If γ, M, and L are non-negative constants, if in the region* $0 \leq x \leq b$ $(b \in \mathbb{R}^n, 0 < b < +\infty)$, *the real-valued function $u(x)$ is*

continuous and non-negative, and if for all $x \in [0, b]$,

$$u(x) \leq \gamma + L \sum_{\alpha_k, 1 \leq k \leq n-1} \int_{0_{\alpha k}}^{x_{\alpha_k}} u(w(x, r; \alpha_k)) dr_{\alpha_k} + M \int_0^x u(r) dr, \quad (5.4.517)$$

then for all $x \in [0, b]$,

$$u(x) \leq \gamma K \quad (5.4.518)$$

where K is a constant depending on L, M, and b.

Proof Under the assumptions, Theorem 5.4.70 may be applied directly and inequality (5.4.516) implies $u(x) \leq A(x; 0)\gamma$. The assertion then follows from the continuity of $A(x; 0)$ on $[0, b]$. □

Other generalizations have been given when $u(x)$ satisfies the following special inequality ($K_{\alpha_k} \equiv 0, 1 \leq \alpha_k \leq n - 1, K_{\alpha_n}(x, r) \equiv K(r)$), for all $a, x, r \in \mathbb{R}^n$,

$$u(x) \leq \phi(x) + \int_a^x K(r) u(r) dr. \quad (5.4.519)$$

In order to establish the connection between these generalizations and that given in Theorem 5.4.70, we introduce the matrix function $\overline{A}(x; \xi)$ satisfying the equation

$$\overline{A}(x; \xi) = I + (-1)^n \int_{\xi}^{x} \overline{A}(r; \xi) K(r) dr, \quad a \leq x \leq \xi \leq b. \quad (5.4.520)$$

If $K(x)$ is continuous on $[a, b]$, which will be assumed here, it may be shown that Eq. (5.4.520) has a unique solution continuous in x and ξ. When $n = 1$, we see that $\overline{A}(x; \xi)$ is the transpose of a fundamental matrix for the adjoint system. We also note that if $m = 1$, then $\overline{A}(x; \xi)$ is the so-called Riemann function [607] for the hyperbolic equation $u_x(x) = K(x)u(x)$. If $A(x; \xi)$ is the fundamental solution for the equation associated with inequality (5.4.519), then it is possible to establish the following reciprocity relation:

$$A(x, \xi) = \overline{A}(\xi; x), \quad a \leq \xi \leq x \leq b. \quad (5.4.521)$$

We can now use Eq. (5.4.521) to obtain the special form of the estimate (5.4.515) when $u(x)$ satisfies inequality (5.4.519). Using Eqs. (5.4.521), (5.4.520), and the continuity of $\overline{A}(x; \xi)$ in its first variable, we know that for each α_k with $1 \leq k \leq n-1$,

$$A_{\xi_{\alpha_k}}(x; \xi) = (-1)^n \int_{x_{\alpha'_k}}^{\xi_{\alpha'_k}} \overline{A}(w(r, \xi; \alpha_k); x) K(w(r, \xi; \alpha_k)) dr_{\alpha'_k} \quad (5.4.522)$$

and

$$A_\xi(x;\xi) = (-1)^n \overline{A}(\xi;x)K(\xi) \tag{5.4.523}$$

for $a \le x \le \xi \le b$. It follows from Eq. (5.4.522) that $A_{\xi_{\alpha_k}}(x; w(x,\xi;\alpha_k)) = 0$ for each α_k with $1 \le k \le n-1$. Using this fact, Eqs. (5.4.523), and (5.4.521) again, we derive that (5.4.515) now reduces to

$$u(x) < \phi(x) + \int_a^x A(x;r)K(r)\phi(r)dr. \tag{5.4.524}$$

Snow [603, 604] used a different method to obtain two generalizations of the Gronwall's inequality when $n = 2$ and $u(x)$ satisfies an inequality of the form (5.4.519). Since the result given in [603] follows from the one given in [604], we show that the main result in [604] follows from Theorem 5.4.70 above. Changing notation so that $x, y, a, b, r, s \in \mathbb{R}$, we may restate and reprove the following Snow's theorem (i.e., Theorem 5.1.11) by using Theorem 5.4.70.

Theorem 5.4.72 (Snow [603]) *Suppose D is a domain in $\mathbb{R}^2$ and $u, \phi : D \to \mathbb{R}^m$ are continuous on D. Suppose $K(x, y)$ is a continuous symmetric $m \times m$ matrix function having non-negative elements on D. Let $P_0(a, b)$ and $P(x, y)$ be points in D such that $(a, b) \le (x, y)$ and let G be the rectangle having the line joining P_0P as its diagonal. Suppose the matrix $V(r, s; x, y)$ satisfies the characteristic value problem*

$$V_{rs}(r,s;x,y) = K(r,s)V(r,s;x,y), \quad V(x,s;x,y) = V(r,y;x,y) = I. \tag{5.4.525}$$

Let D^+ be the connected sub-domain of D containing P and on which $V(r, s; x, y)$ has non-negative elements. If $G \subset D^+$ and

$$u(x,y) \le \phi(x,y) + \int_a^x \int_b^y K(r,s)u(r,s)dsdr, \tag{5.4.526}$$

then

$$u(x,y) \le \phi(x,y) + \int_a^x \int_b^y V^T(r,s;x,y)K(r,s)\phi(r,s)dsdr \tag{5.4.527}$$

where V^T is the transpose of V.

Proof Let $A(x, y; \xi, \eta)$ be the fundamental solution for the equation associated with (5.4.524). If we assume $G \subset D$, then it follows that under Snow's hypotheses in Theorem 5.4.72, where the estimate (5.4.485) now takes the form given in (5.4.524), it holds

$$u(x,y) \le \phi(x,y) + \int_a^x \int_b^y A(x,y;r,s)K(r,s)\phi(r,s)dsdr. \tag{5.4.528}$$

Integrating the equation in problem (5.4.525) and using the characteristic data, we have

$$V(\xi,\eta;x,y) = I + \int_\xi^x \int_\eta^y K(r,s)V(r,s;x,y)dsdr, \quad (\xi,\eta) \le (x,y). \quad (5.4.529)$$

Comparing Eq. (5.4.529) with Eq. (5.4.520) for $\overline{A}$, using Eq. (5.4.521), and using the symmetry of $K(x,y)$, we can see that $\overline{A}(\xi,\eta;x,y) = V^T(\xi,\eta;x,y) = A(x,y;\xi,\eta)$ and hence we obtain the estimate (5.4.527) by using Theorem 5.4.70. □

We point out that provided $G \subset D$, we obtain the estimate (5.4.528) with no assumption that $A(x,y;\xi,\eta)$ have non-negative elements on a sub-region $D^+ \supset G$. In fact, under the hypothesis that $K(x,y)$ has non-negative elements, it follows from Theorem 5.4.70 that $A(x,y;\xi,\eta)$ has non-negative elements for all $(\xi,\eta) \in G$. We also note that the estimate (5.4.519) is valid without the symmetry assumption on the matrix $K(x,y)$.

Young [677] had generalized Snow's method for a scalar inequality of the form (5.4.519) in n independent variables. Returning to the notation used earlier, we may restate the following Young's extension (see, Theorem 5.4.32).

Theorem 5.4.73 (Young [677]) *Let Ω be an open set in $\mathbb{R}^n$ and let $a, x \in \Omega$ such that $a < x$. Suppose $u(x), \phi(x)$, and $k(x) \ge 0$ are real-valued and continuous on Ω. Let $V(\xi;x)$ be the solution of the characteristic value problem*

$$(-1)^n v_\xi(\xi;x) = k(\xi)v(\xi;x) \quad v(\xi;x) = 1 \quad \text{for} \quad \xi_i = x_i,\ 1 \le i \le n. \quad (5.4.530)$$

Proof Let Ω^+ be the connected sub-domain of Ω containing x such that $v(\xi;x) \ge 0$ for all $\xi \in \Omega^+$. If $[a,x] \subset \Omega^+$ and

$$u(x) \le \phi(x) + \int_a^x k(r)u(r)dr, \quad (5.4.531)$$

then

$$u(x) \le \phi(x) + \int_a^x \phi(r)k(r)v(r;x)dr. \quad (5.4.532)$$

Suppose $[a,x] \subset \Omega$. It follows from problem (5.4.530) that $v(\xi;x)$ is the solution of the equation

$$v(\xi;x) = 1 + (-1)^n \int_x^\xi k(r)v(r;x)dr. \quad (5.4.533)$$

Comparing (5.4.532) with Eq. (5.4.520) and making use of Eq. (5.4.521), we see that $v(\xi;x) = \overline{A}(\xi;x) = A(x;\xi)$, $\xi \le x$. Using this in (5.4.524) with $m = 1$ and $K(x) \equiv k(x)$, we obtain (5.4.532). □

Walter [635] had also given a generalization for a scalar inequality of the form (5.4.532) in n independent variables. Walter's concluding estimate is given in terms of a function $h^*(x, \xi)$ defined as a series of functions determined from an iteration procedure. It may be shown that the function $h^*(x, \xi)$ and the fundamental solution $A(x; \xi)$ for this case are identical and hence the present result is consistent with the result given in [637].

5.4.5 Linear Continuous Abstract Gronwall-Bellman Inequalities

The next result, due to Bainov et al. [41], is to prove the Gronwall-Bellman inequality in the case of a compact metric space.

Let Ω be a compact metric space with a metric ρ and a Borel measure μ, and let for each $x \in \Omega$ the mapping $M : x \longmapsto M_x$ be defined where M_x is a closed subset of Ω. We shall suppose that the mapping M satisfies the following condition:

(A1). For each $\varepsilon > 0$ and each $x \in \Omega$, there exists a number $\delta > 0$ such that for each $y \in \Omega$, with $\rho(x, y) < \delta$, the following inequality holds

$$\mu(\{M_x \setminus M_y\} \cup \{M_y \setminus M_x\}) < \varepsilon.$$

Definition 5.4.6 The mapping M is said to be continuous with respect to the measure μ if it satisfies condition (A1).

Consider the equation

$$\varphi(x) = f(x) + \lambda \int_{M_x} K(x, y)\varphi(y)d\mu(y), \tag{5.4.534}$$

where the kernel $K : \Omega^2 \to \mathbb{C}$ and the function $f : \Omega \to \mathbb{C}$ are continuous, and $\lambda \in \mathbb{C}$.

Denote by $C(\Omega)$ the Banach space of the continuous functions $G : \Omega \to \mathbb{C}$ with a norm $\|g\| = \sup_{x\in\Omega} |g(x)|$ and the linear operator $\mathbf{K}$ by the equality

$$(\mathbf{K}g)(x) := \int_{M_x} K(x, y)g(y)d\mu_y, \ g \in C(\Omega). \tag{5.4.535}$$

The operator $I - \lambda\mathbf{K}, \lambda \in \mathbb{C}, \lambda \neq 0$ is a canonical Fredholm operator. In order to verify the above statement, it is sufficient to prove that the operator $\mathbf{K}$ is compact.

Theorem 5.4.74 (Bainov-Myshkis-Zahariev [41]) *Let the mapping M be continuous with respect to the measure μ. Then the operator $\mathbf{K}$ maps $C(\Omega)$ into $C(\Omega)$ and is compact.*

Proof Since $C(\Omega)$ is a Banach space, then it suffices to show that the image of the unit ball $B \subset C(\Omega)$ is a compact set. For each function $g \in B$, taking into account that μ is a Borel measure, we obtain

$$\|\mathbf{K}g\| = \sup_{x\in\Omega} |\int_{M_x} K(x,y)g(y)d\mu_y| \le A\mu(\Omega),$$

where $A = \sup_{x,y\in\Omega} |K(x,y)|$, i.e., the norms of the functions belonging to the set $\mathbf{K}(B)$ are uniformly bounded. We shall prove that the set $\mathbf{K}(B)$ is equicontinuous.

Let $\varepsilon > 0$ be arbitrary. Then uniform continuity of the kernel $K(x, y)$ implies that there exists a number $\delta = \delta(\varepsilon) > 0$ such that for arbitrary $x, y, z \in \Omega$ if $\rho(x, y) < \delta$, then

$$|K(x,z) - K(y,z)| < \frac{\varepsilon}{2\mu(\Omega)}. \tag{5.4.536}$$

Besides, it can be easily verified that the mapping M is uniformly continuous with respect to the measure μ, and hence there exists a number $\delta^* > 0, \delta^* \le \delta$, such that if $\rho(x, y) < \delta^*$, then

$$\mu(\{M_x \setminus M_y\} \cup \{M_y \setminus M_x\}) < \frac{\varepsilon}{2A\mu(\Omega)}. \tag{5.4.537}$$

Therefore, for each function $g \in B$ and for $x, y \in \Omega$, (5.4.536) and (5.4.537) yield the estimate

$$\begin{aligned} |(\mathbf{K}g(x))(x) - (\mathbf{K}g)(y)| &= |\int_{M_x} K(x,z)g(z)d\mu_z - \int_{M_y} K(y,z)g(z)d\mu_z| \\ &\le |\int_{M_x\cap M_y} [K(x,z) - K(y,z)]g(z)d\mu_z| + |\int_{M_x\setminus M_y} K(x,z)g(z)d\mu_z| \\ &\quad + |\int_{M_y\setminus M_x} K(y,z)g(z)d\mu_z| < \varepsilon \end{aligned}$$

which means that the functions from $\mathbf{K}(B)$ are equicontinuous. Therefore $\mathbf{K}$ maps $C(\Omega)$ into $C(\Omega)$ and it follows from the Ascoli-Arzela theorem that the set $\mathbf{K}(B)$ is compact. □

Theorem 5.4.74 implies that the Fredholm alternative holds for Eq. (5.4.534).

Suppose in addition that the mapping M satisfies the following conditions:

(A2) (Transitinity). For each $x \in \Omega$ and each $y \in M_x$, the inclusion $M_y \subseteq M_x$ holds (in other words $M^2 \subseteq M$).

This enables us to consider, for an arbitrary point $a \in \Omega$, the restriction $\mathbf{K}_a : C(M_a) \to C(M_a)$ of the operator $\mathbf{K}$ where $\mathbf{K}_a$ is defined by equality (5.4.535). In that

case, the restriction φ/M_a of the solution of the equation $\varphi = f + \lambda \mathbf{K}\varphi, \varphi \in C(\Omega)$, is solution of the equation $\tilde{\varphi} = f/M_a + \lambda \mathbf{K}_a \tilde{\varphi},\ \tilde{\varphi} \in C(M_a)$.

(A3) (Semicontinuity from below). For each $x \in \Omega$ and each $\varepsilon > 0$, there exists a $\delta > 0$ such that for each $y \in \Omega$ for which $\rho(x, y) < \delta$, the inclusion $M_x \subseteq U(M_y, \varepsilon)$ holds where $U(M_y, \varepsilon)$ denotes the ε-neighborhood of M_y.

Remark 5.4.27 This condition is very close to condition (A1), and in certain cases is logically related to it. For example, it can easily be verified that if the mapping M satisfies condition (A1), then it is semi-continuous from below for every $x \in \Omega$ such that for each $\varepsilon > 0$, the inequality

$$\inf_{y \in M_x} [\mu(U(y, \varepsilon) \cap M_x)] > 0$$

holds.

(A4). There exists an $x_0 \in \Omega$ such that $\mu(M_{x_0}) = 0$.

Remark 5.4.28 If conditions (A1)–(A4) hold, then Eq. (5.4.506) may be considered as one of the possible generalizations of the Volterra equations. This can be seen, for example, from the fact that if we consider the equation

$$\varphi(x) = f(x) + \int_0^{\psi(x)} K(x, y)\varphi(y)dy, \quad \psi : [0, 1] \to [0, 1],$$

conditions (A1)–(A4) are fulfilled if and only if the function $\psi(x)$ is continuous and $0 \le \psi(x) \le x$ for all $x \in [0, 1]$. It may be some interest to describe the structure of the mapping M satisfying conditions (A1)–(A4) in the general case.

Definition 5.4.7 Condition (A) is said to hold if the conditions (A1)–(A4) are fulfilled and Ω is a connected set.

Theorem 5.4.75 (Bainov-Myshkis-Zahariev [41]) *Let condition (A) hold. Then Eq. (5.4.534) has exactly one solution $\varphi \in C(\Omega)$ for each function $f \in C(\Omega)$.*

Proof Consider the equation

$$\varphi(x) = \lambda \int_{M_x} K(x, y)\varphi(y)d\mu_y,\ \lambda \in \mathbb{C}, \tag{5.4.538}$$

and let $\varphi \in C(\Omega)$ be one of its solutions. Denote by H the set

$$H = \Big\{x | x \in \Omega \textit{ and for each } y \in M_x, \ \textit{ we have } \varphi(y) = 0\Big\}.$$

We shall prove that $H \neq \varnothing$ (the case $M_{x_0} = \varnothing$ is trivial), and therefore condition (A2) implies that for each $x \in M_{x_0}$, we have $\mu(M_x) = 0$. Then (5.4.510) yields $\varphi(x) = 0$ for each $x \in M_{x_0}$ and hence $x_0 \in H$.

The set H is closed. Let us choose an arbitrary fundamental sequence of points $\{x_n\}, x_n \in H, n = 1, 2, \cdots$ and denote by x^* its limit in Ω. We shall prove that $x^* \in H$. For this purpose, it is sufficient to consider only the case when all M_{x^*} and $M_{x^*} \neq \varnothing$. Let $z \in M_{x^*}$ be an arbitrary point and $\varepsilon > 0$ be an arbitrary number. We denote by y_n the point at which the minimum of the distance $\rho(z, x), x \in M_{x_n}$, is reached, i.e., $\rho(z, y_n) = \rho(z, M_{x_n})$ (this minimum is reached because M_{x_n} are closed sets). There exists a number $\delta_1 > 0$ such that if $\rho(z, x) < \delta_1, x \in \Omega$, then $|\varphi(x) - \varphi(z)| < \varepsilon$. Condition (A3) implies that there exists a number $\delta_2 > 0$ such that if $\rho(x_n, x^*) < \delta_2$, then $M_{x^*} \subseteq U(M_{x_n}, \delta_1)$. But since $\lim_{n\to+\infty} \rho(x_n, x^*) = 0$, then there exists a number $N > 0$ such that for all $n > N$, we have, $\rho(x_n, x^*) = 0$, then there exists a number $N > 0$ such that for all $n > N$, we have $\rho(x_n, x^*) < \delta_2$. Therefore for $n > N$, we have $z \in U(M_{x_n}, \delta_1)$. Hence $\rho(z, M_{x_n}) = \rho(z, y_n) < \delta_1$ and thus $|\varphi(z) - \varphi(y_n)| < \varepsilon$. Since $\varphi(y_n) = 0$, then $|\varphi(z)| < \varepsilon$ and hence $\varphi(z) = 0$. Since $z \in M_{x^*}$ is an arbitrary point, then $\varphi(x) = 0$ for each $x \in M_{x^*}$, which implies that $x^* \in H$.

We shall prove that H is an open set as well.

Let $a \in H$ be an arbitrary point and let $\varepsilon > 0$ be such that the inequality $\varepsilon|\lambda A| < \frac{1}{2}$ holds. There exists a number $\delta > 0$ such that for each $x \in \Omega$ for which $\rho(a, x) < \delta$, the following inequality holds

$$\mu(\{M_a \setminus M_x\} \sup\{M_x \setminus M_a\}) < \varepsilon.$$

Let $b \in \Omega$, $\rho(a, b) < \delta$ and consider the set

$$T = \Big\{g | g \in C(M_b), g(x) \equiv 0 \, for \; x \in M_b \cap M_a\Big\}.$$

Condition (A2) implies that the operator $\lambda \mathbf{K}_b$ maps T into T. Let $g_1, g_2 \in T$ be arbitrary functions. Then we have

$$\begin{aligned}
\|\lambda \mathbf{K} g_1 - \lambda \mathbf{K} g_2\|_{M_b} &= |\lambda| \sup_{x\in M_b} |\int_{M_x} K(x,y)(g_1(y) - g_2(y))d\mu_y| \\
&\leq |\lambda|\{|\int_{M_x \setminus M_a} K(x,y)(g_1(y) - g_2(y))d\mu_y| \\
&\quad + |\int_{M_x \cap M_a} K(x,y)(g_1(y) - g_2(y))d\mu_y|\} \\
&\leq |\lambda| A \sup_{x\in M_b} \mu(M_x \setminus M_a)\|g_1 - g_2\|_{M_b} < \frac{1}{2}\|g_1 - g_2\|_{M_b}.
\end{aligned}$$

Therefore the operator $\lambda \mathbf{K}_b$ is contractive on the set T (it is easy to see that the set T is closed) and hence from the uniqueness, we get $g_0(x) \equiv 0$ for all $x \in M_b$. On the other hand, the uniqueness implies that the restriction of $\varphi(x)$ on M_b coincides with $g_0(x)$ and therefore $\varphi(x) \equiv 0$ for all $x \in M_b$, i.e., $b \in H$. Thus we prove that the set

H together with all its points a contains a neighborhood of any of these points as well, i.e., H is open.

Furthermore, taking into account that Ω is connected, we obtain $H = \Omega$ and hence (5.4.538) yields $\varphi(x) \equiv 0$ for all $x \in \Omega$, i.e., Eq. (5.4.538) for each $\lambda \in \mathbb{C}$ has only the trivial solution. The Fredholm alternative implies that for each $\lambda \in \mathbb{C}$, Eq. (5.4.534) has an unique solution $\varphi(x) \in C(\Omega)$ for each function $f(x) \in C(\Omega)$.□

Remark 5.4.29 The spectrum $\sigma(\mathbf{K})$ of the open $\mathbf{K}$ consists of the point $\lambda = 0$ only.

Remark 5.4.30 The spectral radius $\tau(\mathbf{K})$ of $\mathbf{K}$ is zero.

The validity of the above statement follows immediately from Theorems 5.4.74 and 5.4.75 and from [675] (Theorems 3, 4 of Sect. VIII. 2 and Theorems 1, 2 of Sect. X. 5).

Therefore the solution of Eq. (5.4.534) is given by the equality

$$\varphi(x) = ((I - \lambda\mathbf{K})^{-1}f)(x) = \lambda^{-1}(R(\lambda^{-1};\mathbf{K})f)(x), \tag{5.4.539}$$

where the resolvent $R(\lambda^{-1}; K)$ is presented in Neumann series convergent in the operator topology for each $\lambda \in \mathbb{C}$,

$$R(\lambda^{-1};\mathbf{K}) = \lambda I + \lambda^2\mathbf{K} + \lambda^3\mathbf{K}^2 + \cdots$$

or in a more expanded form

$$\lambda^{-1}(R(\lambda^{-1};\mathbf{K})f)(x) = f(x) + \lambda \int_{M_x} K(x,y)f(y)d\mu_y + \cdots \tag{5.4.540}$$

where the series (5.4.540) is uniformly convergent for each $\lambda \in \mathbb{C}$ (see, [675] Theorem 3 of Sect. VII. 2). In particular, if $f(x) \equiv 1$ and $\lambda = 1$, then from (5.4.540) we obtain a special solution $\phi(x)$ of equation :

$$\phi(x) = 1 + \int_{M_x} K(x,y)d\mu_y + \int_{M_x} K(x,y)(\int_{M_y} K(y,y_1)d\mu_{y_1})d\mu_y + \cdots . \tag{5.4.541}$$

Theorem 5.4.76 (Bainov-Myshkis-Zahariev [41]) *Let conditions (A) hold and let the continuous function $\psi : \Omega \to \mathbb{R}$ satisfy the inequality for each $x \in \Omega$,*

$$\psi(x) \le f(x) + \int_{M_x} K(x,y)\psi(y)d\mu_y, \tag{5.4.542}$$

where the function $f : \Omega \to \mathbb{R}$ are continuous, and $K(x,y) \ge 0$ for all $x, y \in \Omega$. Then if $\varphi(x)$ is the solution of the equation

$$\varphi(x) = f(x) + \int_{M_x} K(x,y)\varphi(y)d\mu_y,$$

then for each $x \in \Omega$, there holds the inequality

$$\psi(x) \leq \varphi(x). \tag{5.4.543}$$

Proof Integrating n times the right-hand side of (5.4.542), we obtain

$$\psi(x) \leq f(x) + \int_{M_x} K(x,y)f(y)d\mu_y + \int_{M_x} K(x,y)(\int_{M_y} K(y,y_1)f(y_1)d\mu_{y_1})d\mu_y$$
$$+ \cdots + \int_{M_x} K(x,y)\cdots(\int_{M_{y_n}} K(y_{n-1},y_n)\psi(y_n)d\mu_{y_n})\cdots d\mu_y. \tag{5.4.544}$$

Noting (5.4.539) and (5.4.540) and passing to the limit $n \to +\infty$ in (5.4.544), we obtain

$$\varphi(x) \leq f(x) + \int_{M_x} K(x,y)f(y)d\mu_y + \cdots = \varphi(x). \tag{5.4.545}$$

Thus the proof is now complete. □

Theorems 5.4.75 and 5.4.76 can be naturally applied to the case when Ω is not compact or connected, but the mapping M satisfies certain additional conditions. For example, it is sufficient to require that every non-empty set $M_x, x \in \Omega$, should be compact, connected, have a finite measure (assuming that μ takes the value $+\infty$ as well) and, besides, should contain a point $y = y(x)$ such that $\mu(M_y) = 0$. In that case, all considerations about the operator $\mathbf{K}$ must be referred to its restriction, in particular, the inequality

$$\psi(x) \leq C + \int_{M_x} K(y)\psi(y)d\mu_y, \quad x \in \Omega,$$

where C is an arbitrary constant, $K : \Omega \to \mathbb{R}$ is a continuous function, and $K(y) \geq 0, y \in \Omega$. Then Theorem 5.4.76 and equality (5.4.541) yield, for all $x \in \Omega$,

$$\psi(x) \leq C\phi(x), \tag{5.4.546}$$

where the function $\phi(x)$ is a solution of the equation

$$\phi(x) = 1 + \int_{M_x} K(y)\phi(y)d\mu_y, \tag{5.4.547}$$

which is given in the form

$$\phi(x) = 1 + \int_{M_x} K(y)d\mu_y + \cdots, \tag{5.4.548}$$

where the series (5.4.548) is uniformly convergent.

In fact, (5.4.546) presents an analogue of the Gronwall-Bellman inequality.

If $K(y) \equiv 1$ in (5.4.548), then we have an analogue of the exponent corresponding to the mapping M:

$$\exp_M(x) = 1 + \int_{M_x} d\mu_y + \int_{M_x} (\int_{M_y} d\mu_{y_1}) d\mu_y + \cdots .$$

To illustrate the above results, we shall consider some examples in the case when Ω is finite dimensional.

Example 5.4.2 Let $\Omega = \mathbb{R}_+, M_x = [0, x]$. Then $\exp_M(x) = e^x$ and in inequality (5.4.546), we have

$$\phi(x) = \exp(\int_0^x K(y)dy).$$

Example 5.4.3 Let $\Omega = \mathbb{R}_+, M_x = [0, \psi(x)]$, where $\psi(x)$ is continuously differentiable and $0 \le \psi(x) \le x$. Then $\exp_M(x)$ is a solution of the Cauchy problem

$$\phi'(x) = \psi'(x)\phi(\psi(x)), \quad \phi(0) = 1.$$

If, in particular, $\psi(x) = \frac{x}{2}$, then

$$\exp_m(x) = \sum_{n=0}^{+\infty} \frac{x^n}{2^{n(n+1)/2} n!}.$$

Example 5.4.4 Let $\Omega = \mathbb{R}_+^2, x = (x_1, x_2), x_1, x_2 \ge 0$ and $M_x = \{(y_1, y_2) | 0 \le y_1 \le x_1 + x_2, 0 \le y_2 \le x_1 + x_2 - y_1\}$. Then $\exp_M(x) = \cos h(x_1 + x_2)$.

Example 5.4.5 Let $\Omega = \mathbb{R}_+^n, x = (x_1, x_2, \cdots, x_n), x_i \ge 0, i = 1, 2, \cdots, n$ and $M_x = [0, x_1] \times \cdots \times [0, x_n]$. Then

$$\exp_M(x) = \sum_{i=1}^{+\infty} \frac{(x_1 \cdots x_n)^{i}{}^{n}}{i!}.$$

The following theorem is valid in the general case.

Theorem 5.4.77 (Bainov-Myshkis-Zahariev [41]) *Let $\Omega = \mathbb{R}_+^n$, μ be a Lebesgue measure and let the following conditions hold:*

(1) For each $x \in \Omega$ and each $y \in M_x$, we have $x_i \ge y_i, i = 1, 2, \cdots, n$.
(2) The function $K : \Omega \to \mathbb{R}$ is continuous and $K(y) \ge 0$ for all $y \in \Omega$.

Then the solution $\phi(x)$ of Eq. (5.4.545) satisfies the inequality for all $x \in \Omega$,

$$\phi(x) \le \exp\Big(\int_{M_x} K(y)dy\Big).$$

Proof Let $z \in \mathbb{R}_+^n$ be a point such that the n-dimensional parallelepiped $B_z = [0, z_1] \times \cdots \times [0, z_n]$ contains M_z and let us choose an arbitrary function $\tilde{K} : B_z \to \mathbb{R}_+, \tilde{K} \in C(B_z)$ such that $\tilde{K}(y) \equiv K(y)$ for all $y \in M_k$. Then the function

$$\psi(x) = \exp(\int_{B_x} \tilde{K}(y)dy),\ x \in B_z,$$

satisfies the equation

$$\psi(x) = 1 + \int_{B_x} F(y)\psi(y)dy,$$

where $B_x = [0, x_1] \times \cdots \times [0, x_n]$, while we shall clear up the form of the function F in the case $n = 2$:

$$\begin{aligned}
\psi(x_1, x_2) &= \exp\Big[\int_0^{x_1}(\int_0^{x_2} \tilde{K}(\xi_1, \xi_2 d\xi_2)d\xi_1)\Big] = 1 + \int_0^{x_1}\Big(\frac{\partial}{\partial y_1}\psi(y_1, y_2)\Big)dy_1 \\
&= 1 + \int_0^{x_1}\Big(\exp[\int_0^{y_1}(\int_0^{y_2} \tilde{K}(\xi_1, \xi_2)d\xi_2)d\xi_1] \cdot \int_0^{x_2} \tilde{K}(y_1, \xi_2)d\xi_2\Big)dy_1 \\
&= 1 + \int_0^{x_2}\Big[\frac{\partial}{\partial y_2}\Big(\exp[\int_0^{y_1}(\int_0^{y_2} \tilde{K}(\xi_1, \xi_2)d\xi_2)d\xi_1] \cdot \int_0^{y_2} \tilde{K}(y_1, \xi_2)d\xi_2\Big)dy_1\Big]dy_2 \\
&= 1 + \int_0^{x_2}\Big[\int_0^{x_1} \exp\Big(\int_0^{y_2} \tilde{K}(\xi_1, \xi_2)d\xi_2\Big)d\xi_1\Bigg(\int_0^{y_2} \tilde{K}(\xi_1, y_2)d\xi_1 \cdot \Big(\int_0^{y_2} \tilde{K}(y_1, \xi_2)d\xi_2 \\
&\quad + \tilde{K}(y_1, y_2)\Big)dy_1\Bigg)\Big]dy_2,
\end{aligned} \tag{5.4.549}$$

i.e.,

$$F(x_1, x_2) = \int_0^{x_1} \tilde{K}(y_1, x_2)dy_1 \cdots \int_0^{x_2} \tilde{K}(x_1, y_2)dy_2 + \tilde{K}(x_1, x_2).$$

It is easy to verify that for an arbitrary n, the function F is equal to the sum of $\tilde{K}$ and a polynomial with positive coefficient of integrals of $\tilde{K}$ with multiplicity from 1 to $n - 1$ and with integration bounds from 0 to x_i. Hence $F(x) \ge \tilde{K}(x)$ for all $x \in B_z$ and from Eq. (5.4.547), it follows

$$\psi(x) \ge 1 + \int_{M_x} \tilde{K}(y)\psi(y)dy = 1 + \int_{M_x} K(y)\psi(y)dy.$$

Therefore using Theorem 5.4.76, and noting that $\phi(x)$ is a solution of Eq. (5.4.546), we can obtain the inequality $\phi(x) \le \psi(x)$ which is the desired estimates. □

A similar result was obtained in [216] in the case when $K(y)$ is an integrable function, but the mapping M has a special form.

5.4.6 Linear Multi-Dimensional Continuous Matrix Generalization of the Gronwall-Bellman Inequalities

A variety of linear generalizations of Gronwall's inequality, including multi-variable results of Snow and Young, are subsumed and extended by simple arguments involving the resolvent kernel of the integral operator.

It is well-known that Gronwall's inequality in Theorem 1.1.1 ([239]) is but one example of an inequality for monotone operator $\mathcal{K}$ in which the exact solution of $w = a + \mathcal{K}w$ provides an upper bound on all solutions of $u \le a + \mathcal{K}u$. Nevertheless, this idea is often neglected in deriving new variants of this classical inequality.

The next result generalizes Gronwall's inequalities to systems of m linear inequalities in n variables by arguments involving manipulation of the resolvent kernel equation for $\mathcal{K}$. These results encompass work of Chu and Metcalf [135], Snow [605, 636], Walter [647] (with a restriction noted below), Wendroff [47], and Young [677] as well as providing extensions to kernel having more general form and weaker regularity properties.

Let $G(x)$ and $H(x)$ denote real-valued $m \times m$ matrices and $a(x)$, $u(x)$ denote m-vectors, all of which are continuous functions of $x = (x_1, \ldots, x_m)$. Let x^0 be a fixed n-vector and $\int_{x^0}^{x} \cdot dy$ denote the multiple integral $\int_{x_n^0}^{x_n} \cdots \int_{x_1^0}^{x_1} dy_1 \cdots dy_n$.

Inequalities hold component-wise and I is the identity matrix.

Theorem 5.4.78 (Chandra-Davis [128]) *Let $G(x)$, $H(x)$ be continuous, non-negative matrices for all $x \ge x^0$. If for all $x \ge x^0$,*

$$u(x) \le a(x) + G(x) \int_{x^0}^{x} H(y)u(y)dy, \tag{5.4.550}$$

then for all $x \ge x^0$,

$$u(x) \le a(x) + G(x) \int_{x^0}^{x} V(x,y)H(y)a(y)dy, \tag{5.4.551}$$

where $V(x,y)$ satisfies

$$V(x,y) = I + \int_{y}^{x} H(z)G(z)V(z,y)dz, \quad x^0 \le y \le x. \tag{5.4.552}$$

Proof In the norm in [647, pp. 141–142], the integral operator $\mathcal{K}$ on the right-hand side of (5.4.550) is a contraction on the segment $x^0 \leq x \leq x'$ for any fixed x'. The resulting Neumann series consequently converges uniformly on any such compact set to a resolvent operator $(\varphi - \mathcal{K})^{-1}$, which is monotone because $\mathcal{K}$ is monotone.

A sharp bound on $u(x)$ is therefore the exact solution of $w = a + \mathcal{K}w$. The usual manipulations of the Neumann series, e.g., Yosida [674, pp. 147–149], show that this solution is just the right-hand side of inequality (5.4.551), where $G(y)V(x,y)H(y)$ appears as the resolvent kernel of $G(x)H(y)$. The resolvent equation for $\mathcal{K}$ is (5.4.552) premultiplied by $G(x)$ and postmultiplied by $H(y)$ (see, [677, Eq. (37.9)]). □

Remark 5.4.31 With $G(x) = I$ and other restrictions, Snow [605, 636], Young [680], and Walter [647] have obtained inequalities like (5.4.551). Snow and Young regarded $V(x,y)$ as the Riemann function for the initial value problem equivalent to Eq. (5.4.552). The equivalence of Eq. (5.4.552) and Snow's result [605] for a system of two inequalities follows from Snow's hypothesis that $H(y)$ is self-adjoint, which forces the same property on V. Walter handled a more general region than $x^0 \leq x$, but defined V via the Neumann series for the operator in (5.4.550).

Corollary 5.4.18 (Bainov-Myshkis-Zahariev [41]) *Let* $a(x) \geq 0$ *and* $G(x)$, $H(x) \geq 0$ *for all* $x \geq x^0$. *Define*

$$J(z_1) \equiv \int_{y_2}^{x_2} \cdots \int_{y_n}^{x_n} H(z_1, z_2, \ldots, z_n) G(z_1, z_2, \ldots, z_n) dz_2 \cdots dz_m$$

and suppose that $J(z_1)$ *commutes with* $\exp\left(\int_{y_1}^{z_1} J(s_1)ds_1\right)$ *for all* $z_1 \geq y_1 \geq x_1^0$. *If* $u(x)$ *satisfies (5.4.550), then for all* $x \geq x^0$,

$$u(x) \leq a(x) + G(x) \int_{x^0}^{x} \exp\left(\int_y^x H(z)G(z)dz\right) H(y)a(y)dy. \tag{5.4.553}$$

Proof Let $E(z,y) = \int_y^z H(s)G(s)ds$. Since $\exp E(x,y)$ is increasing in any component of its first argument, we have

$$\begin{aligned}\int_y^x H(z)G(z)\exp E(z,y)dz &\leq \int_{y_1}^{x_1} J(z_1)\exp\left(\int_{y_1}^{z_1} J(s_1)ds_1\right)dz_1 \\ &= \exp E(x,y) - I. \end{aligned}\tag{5.4.554}$$

Consequently, $\exp E(x,y)$ satisfies an integral inequality of which $V(x,y)$ is the exact solution in the case of equality; cf. (5.4.552). The fundamental argument of the theorem (that the solution of the equality provides a bound on all solutions of the corresponding inequality) now gives us $V(x,y) \leq \exp E(x,y)$, and hence (5.4.553) follows from (5.4.551). □

Corollary 5.4.18 extends a two-variable, scalar inequality originally due to Wendroff. In general, (5.4.553) is not sharp unless the inequalities depend only on a single scalar independent variable.

Corollary 5.4.19 (Bainov-Myshkis-Zahariev [41]) *Let the vector $a(t)$ and the non-negative matrices $G(t)$, $H(t)$ be functions of the single scalar variable t for all $t \geq t^0$. Assume that $H(t)G(t)$ and $\int_{t^0}^t H(s)G(s)ds$ commute for all $t \geq t^0$. If (5.4.550) holds (with t, t^0 in place of x, x^0), then for all $t \geq t^0$,*

$$u(t) \leq a(t) + G(t) \int_{t^0}^t \exp\left(\int_s^t H(r)G(r)dr\right) H(s)a(s)ds. \tag{5.4.555}$$

Proof Integration reveals that (5.4.552) is satisfied by

$$V(t, s) = \exp\left(\int_s^t H(r)G(r)dr\right).$$

□

This corollary restates a result of Chu and Metcalf [135], which was obtained by summing a Neumann series, and includes the classical Gronwall inequalities. Willett's technique [647], for treating kernels which are sums of terms like $G(t)H(s)$ could be used to solve (5.4.552) and hence extends Corollary 5.4.19 to kernels of this more general form.

The commutativity assumptions in the preceding corollaries are imposed to permit integration of the matrix exponential function.

In the case of a scalar independent variable, Miller [405], has derived the resolvent kernel equations for a system of Volterra integral equations whose kernels are not necessarily continuous. The obvious extension of these results to several independent variables yields a substantially weakened form of the theorem (the regularity condition given below is not the most general, see [405]).

Theorem 5.4.79 (Alternate Theorem) *Let $G(x)$, $H(x)$ be commuting, non-negative matrices which are merely square integrable on $x^0 \leq x = (x_1, \cdots, x_n) \leq x'$ for each fixed $x' \geq x^0$. If (5.4.550) holds a.e. on $x^0 \leq x$, then (5.4.551) and (5.4.552) hold a.e. on $x^0 \leq x$.*

A differential analysis like that of Snow and Young obviously requires revision if the Riemann function V is defined by a differential equation whose coefficients may not be continuous. The integral equation approach taken here avoids this difficulty by requiring only enough smoothness in G and H to ensure that the resolvent kernel actually provides a solution of the integral equation.

The well-known Gronwall inequality in Theorem 1.1.1 gives us explicit bounds for a continuous function $u(x)$, satisfying

$$0 \leq u(x) \leq a + \int_0^x bu(t)dt,$$

where $a,\ b$ are non-negative constants.

Over the years this has been extended in a variety of ways, in particular by: (1) letting $a = a(x)$ and $b = b(t)$ or $b = b(x,t)$; (2) letting $u = u(x_1, \cdots, x_n)$ and $\int_0^x = \int_0^{x_1} \cdots \int_0^{x_n}$; (3) letting u be a vector, giving rise to a system of inequalities.

We shall next improve the above mentioned extensions, making systematic use of iteration methods, as in [142].

Let $u(x) = (u_1(x), \cdots, u_n(x))^T$, where T denotes the transpose, and we let $K(x,t)$ be the $m \times m$ matrix $k_{ij}(x,t)$, where the u_i and the k_{ij} are all real-valued, continuous, non-negative functions for $0 \leq t \leq x$. By $K \geq M$, we mean $k_{ij} \geq m_{ij}$ for all $i,\ j$, and $u \geq v$, we mean $u_i \geq v_i$ for all i.

Definition 5.4.8 The (matrix) functions K is called a good kernel if each element k_{ij} of K satisfies the following conditions:

$$\begin{cases} (a) \quad k_{ij}(x,t) \geq 0, \\ (b) \quad k_{ij} \textit{ is a continuous function of its } 2n \textit{ variables}, \\ (c) \quad K(x,s)\{\int_t^s K(\xi(\sigma,x)\sigma)d\sigma\} \leq \{K(\xi(\sigma,x),\sigma)d\sigma\}K(x,s) \end{cases} \tag{5.4.556}$$

where $\xi(\sigma;x)$ is a point in $[\sigma,x]$ such that $K(\xi(\sigma,x),\sigma) = \max\{K(t,\sigma) : \ t \in [\sigma,x]\}$.

In the special case where $K(x,s)$ is non-decreasing in x (or non-increasing in x), then $\xi(\sigma,x) = x$ (or $\xi(\sigma,x) = \sigma$). Note, by condition (b), $K(\xi(\sigma,x),\sigma)$ is a continuous function of the $2n$ variables x,σ even though $\xi(\sigma,x)$ need not be.

If K is a good kernel, and if the components of $g = (g_1, \cdots, g_n)^T$ are non-negative and continuous, we define

$$\begin{cases} Tg(x) = T^1 g(x) = \displaystyle\int_0^x K(x,t)g(t)dt, \quad x \geq 0, \\ T^{j+1}g = T(T^j g), \end{cases} \tag{5.4.557}$$

where

$$\begin{cases} K_1(x,t) = \displaystyle\int_t^x K(x,s)ds, \quad x \geq t \geq 0, \\ K_{j+1}(x,t) = \displaystyle\int_t^x K(x,s)K_j(s,t)ds, \\ K_0(x,t) = \textit{Identity matrix}. \end{cases} \tag{5.4.558}$$

Note the K_j are not the usual kernels which appear in the theory of integral equations.

Lemma 5.4.10 (Conlan-Wang [143])

$$T^{j+1}g(x) \le \int_0^x K_j(x,s)K(\xi,s)g(s)ds, \tag{5.4.559}$$

where $\xi = \xi(x,s)$ *is the function defined in condition (c) of Definition 5.4.8.*

Proof For $j = 0$,

$$Tg(x) = \int_0^x K(x,s)g(s)ds \le \int_0^x K_j(x,s)K(\xi(x,s),s)g(s)ds$$

and so (5.4.559) holds for $j = 0$.

Now assume (5.4.559) holds for $j = r$. Then

$$\begin{aligned} T^{r+2}g(x) &= \int_0^x K(x,s)[T^{r+1}g(s)]ds \\ &\le \int_0^x K(x,s)[\int_0^s K_r(s,t)K(\xi(t,s),t)g(t)dt]ds \\ &\le \int_0^x K(x,s)[\int_0^s K_r(s,t)K(\xi(t,x),t)g(t)dt]ds \\ &= \int_0^x [\int_0^s K(x,s)K_r(s,t)ds]K(\xi(t,s),t)g(t)dt \\ &= \int_0^x K_{r+1}(x,t)K(\xi(t,x),t)g(t)dt. \end{aligned}$$

□

The next three lemmas will be useful in the proofs of Lemmas 5.4.13 and 5.4.15.

Lemma 5.4.11 (Conlan-Wang [143]) *Let* A, B *be appropriately differentiable matrix functions of* $s = (s_1, \cdots, s_n)$. *Let all* $D_{s_{j1}} \cdots D_{s_{jk}} C \ge 0$, *where* $C = A$ *or* B, *and* $1 \le k \le n-1$ *and all distinct* $j_1, \cdots, j_k \in \{1, \cdots, n\}$. *Then* $D_{s1} \cdots D_{sn}(AB) \ge (D_{s1} \cdots D_{sn} \cdots D_{sn}A)B + A(D_{s1} \cdots D_{sn}B)$.

Proof The conclusion is true for $n = 1$. Assume it is true for $n = r$, i.e.,

$$D_{s1} \cdots D_{sr}(AB) \ge (D_{s1} \cdots D_{sr}A)B + A(D_{s1} \cdots D_{sr}B).$$

Then

$$\begin{aligned} D_{s1} \cdots D_{s_{r+1}}(AB) &= D_{s_{r+1}} D_{s1} \cdots D_{sr}(AB) \\ &\ge D_{s_{r+1}}[B)] + A(D_{s1} \cdots D_{sr})B) \\ &\ge (D_{s_{r+1}} D_{s1} \cdots D_{sr}A)B + (D_{s1} \cdots D_{sr}A)(D_{s_{r+1}}B) \end{aligned}$$

$$
\begin{aligned}
&+(D_{s_{r+1}}A)(D_{s1}\cdots D_{sr}B)+A(D_{s_{r+1}}D_{s1}\cdots D_{sr}B)\\
\geq\ &(D_s\cdots D_{s_{r+1}}A)B+A(D_{s_1}\cdots D_{s_{r+1}}B).
\end{aligned}
$$

□

Lemma 5.4.12 (Conlan-Wang [143]) *If $M,\ K$ are matrices satisfying $MK \geq KM$, then $M^rK \geq KM^r$.*

Proof If the conclusion is true for $r = j$, then

$$M^{j+1}K = MM^jK \geq MKM^j \geq KM^{j+1}.$$

□

Lemma 5.4.13 (Conlan and Wang [143]) *If $M = \int_t^s K(x,\sigma)d\sigma$, then $D_{s1}\cdots D_{sn}M^r \geq rK(x,s)M^{r-1}$.*

Proof The conclusion is true for $r = 1$, assume it is true for $r = j$. Then

$$
\begin{aligned}
D_{s1}\cdots D_{sn}M^{j+1} &= D_{s1}\cdots D_{sn}(M^jM)\\
&\geq (D_{s1}\cdots D_{sn}M^j)M + M^j(D_{s1}\cdots D_{sn}M)\\
&\geq jKM^{j-1}M + M^jK\\
&\geq (j+1)KM^j.
\end{aligned}
$$

□

Lemma 5.4.14 (Conlan-Wang [143])

$$K_m(x,t) \leq \frac{1}{m!}\Big[\int_t^x K(\xi(s,x),s)ds\Big]^m. \tag{5.4.560}$$

Proof From $m = 1$,

$$K_1(x,t) = \int_t^x K(x,s)ds \leq \int_t^x K(\xi(s,x),s)ds,$$

and so (5.4.560) holds for $m = 1$. Assume it holds for $m = r$. Then

$$
\begin{aligned}
K_{r+1}(x,t) &= \int_t^x K(x,s)K_r(s,t)ds\\
&\leq \frac{1}{r!}\int_t^r K(x,s)\Big(\int_t^s K(\xi(\sigma,s),\sigma)d\sigma\Big)^r ds\\
&\leq \frac{1}{r!}\int_t^r K(x,s)\Big(\int_t^s K(\xi(\sigma,x),\sigma)d\sigma\Big)^r ds
\end{aligned}
$$

$$\leq \frac{1}{(r+1)!}\int_t^x D_{s1}\cdots D_{sn}\Big(\int_t^s K(\xi(\sigma,x),\sigma)d\sigma\Big)^{r+1}ds$$
$$= \frac{1}{(r+1)!}\Big(\int_t^s K(\xi(\sigma,x),\sigma)d\sigma\Big)^{r+1}.$$

□

Lemma 5.4.15 (Conlan-Wang [143])

$$T^{j+1}g(x) \leq \frac{1}{r!}\int_0^x \Big[\int_t^s K(\xi(\sigma,x),\sigma)d\sigma\Big]^n K(\xi(t,x),t)g(t)dt. \tag{5.4.561}$$

Proof This follows directly from (5.4.559) and (5.4.560). □

Theorem 5.4.80 (Conlan-Wang [143]) *Under the previous assumptions, if u satisfies for all $x \geq 0$,*

$$u(x) \leq g(x) + \int_0^x K(x,t)u(t)dt, \tag{5.4.562}$$

then for all $x \geq 0$,

$$u(x) \leq g(x) + \int_0^x \exp\Big(\int_t^x K(x,s)ds\Big)K(x,t)g(t)dt. \tag{5.4.563}$$

Proof From the above lemmas, and the continuity of u on $[0,x]$, it follows that (5.4.560) implies

$$u \leq g + \sum_{j=1}^{r} T^j g + T^{r+1}u \leq g + \sum_{j=1}^{+\infty} T^j g$$
$$\leq g + \int_0^x \sum_{j=1}^{+\infty}\frac{1}{j!}\Big(\int_s^x K(x,t)dt\Big)^j K(s,x)g(s)dx$$
$$= g + \int_0^x \exp\Big(\int_s^x K(x,t)dt\Big)K(s,x)g(s)ds.$$

□

We can extend Theorem 5.4.80 to the case of iterated integrals. If $F(x,s)$ and $G(x,s)$ are $m \times m$ matrices, we define $F * G$ by

$$F * G = \int_s^x F(x,t)G(t,s)dt. \tag{5.4.564}$$

Now for any integer $r > 0$, let

$$\begin{cases} H_1(x,s) = K_{11}(x,s), \\ H_r(x,0) = (((K_{rr}^* K_{r,r-1})^* K_{r,r-2})^* \cdots)^* K_{r,1}, \end{cases} \tag{5.4.565}$$

where the K_{rh} are non-negative, continuous matrices, of order $m \times m$. Note that H_r can be explicitly written as

$$\begin{aligned} H_r(x,s) &= \int_s^x \int_s^{s_{r-1}} \cdots \int_s^{s_2} K_{rr}(x,s_{r-1}) \cdots K_{r1}(s_1,s)ds_1 \cdots ds_{r-1}, \\ Tg(x) &= \int_0^x \Big(H_2(x,s) + H_1(x,s)\Big)g(s)ds \\ &= \int_0^x K_{22}^* K_{21} g(s)ds + \int_0^x H_1(x,s)g(s)ds \\ &= \int_0^x \left(\int_s^x K_{22}(x,t)K_{21}(t,s)dt\right) g(s)ds \\ &\quad + \int_0^x K_{11}(x,s)g(s)ds. \end{aligned}$$

$\square$

Theorem 5.4.81 (Conlan-Wang [143]) *If $\sum_{j=1}^r H_j(x,s)$ is a good kernel, and if $u \le g + Tu$, then*

$$u(x) \le g(x) + \int_0^x \exp\left(\int_s^x \sum_{j=1}^r H_j(x,\sigma)d\sigma\right) \left\{\sum_{j=1}^r H_j(x,s)\right\} g(s)ds.$$

Next we assume that K is a (matrix) product

$$K(x,t) = F(x)H(t), \tag{5.4.566}$$

where F and H are each $m \times m$ matrices, the elements of which are non-negative and continuous. Here condition (c) of Definition 5.4.8 is replaced by

$$K(x,s)\left(\int_t^s K(x,\sigma)d\sigma\right) \le \left(\int_t^s K(x,\sigma)d\sigma\right) K(x,s).$$

We define T^j as in (5.4.557), and let

$$\begin{cases} \tilde{K}_1(x,t) = \displaystyle\int_t^x H(s)F(s)ds, \quad \tilde{K}_{j+1}(x,t) = \int_t^x H(s)F(s)\tilde{K}_j(s,t)ds, \; j = 1, 2, \cdots, \\ \tilde{K}_0(x,t) = \textit{Identity matrix}. \end{cases} \tag{5.4.567}$$

Note that we can not obtain $\tilde{K}_j$ by setting $K = HF$ in (5.4.558).

Lemma 5.4.16 (Conlan-Wang [143])

$$T^{j+1}g(x) = F(x)\int_0^x H(t)g(t)\tilde{K}_j(x,t)dt, \quad j = 0, 1, \cdots \tag{5.4.568}$$

Proof For $j = 0$, (5.4.568) follows from the definition of T^1. Assume (5.4.568) holds for $j = r$. Then

$$\begin{aligned} T^{r+2}g(x) &= F(x)\int_0^x H(s)T^{r+1}g(s)ds \\ &= F(x)\int_0^x H(s)\left[F(s)\int_0^s \tilde{K}_r(s,t)H(t)g(t)dt\right]ds \\ &= F(x)\int_0^x \left[\int_t^s H(s)\tilde{K}_r(s,t)F(s)ds\right]g(t)H(t)dt \\ &= F(x)\int_0^x \tilde{K}_{1+r}(s,t)g(t)H(t)dt. \end{aligned}$$

□

Lemma 5.4.17 (Conlan-Wang [143])

$$\tilde{K}_j(x,t) \le \frac{1}{j!}\Big(\int_t^x H(s)F(s)ds\Big)^j = \frac{1}{j!}\Big(\tilde{K}_1(x,t)\Big)^j; \quad j = 1, 2, \cdots \tag{5.4.569}$$

Proof For $j = 1$, (5.4.569) is trivially true. Assume (5.4.569) holds for $j = r$. Then

$$\begin{aligned} \tilde{K}_{r+1}(x,t) &= \int_t^x H(s)F(s)\tilde{K}_r(s,t)ds \\ &\le \frac{1}{r!}\int_t^x H(s)F(s)\Big(\int_r^x H(\sigma)F(\sigma)d\sigma\Big)^r ds \\ &\le \frac{1}{(r+1)!}\int_t^x D_{s1}\cdots D_{sn}\Big(\int_r^x H(\sigma)F(\sigma)d\sigma\Big)^{r+1} ds \\ &= \frac{1}{(r+1)!}\Big(\int_r^x H(\sigma)F(\sigma)d\sigma\Big)^{r+1}. \end{aligned}$$

□

As an immediate consequence of Lemmas 5.4.14 and 5.4.15, we have the following lemma.

Lemma 5.4.18 (Conlan-Wang [143])

$$T^{j+1}g(x) \leq \frac{1}{j!}\int_0^x [K_1(x,t)]^j H(t)g(t)dt. \tag{5.4.570}$$

Theorem 5.4.82 (Conlan-Wang [143]) *If $u(x)$ satisfies (5.4.562) with K as given by (5.4.566), then*

$$u(x) \leq g(x) + F(x)\int_0^x \exp\left(K_1(x,t)\right) H(t)g(t)dt. \tag{5.4.571}$$

Proof The proof is similar to proof of Theorem 5.4.80. □

We should point out here that Theorem 5.4.82 has previously been obtained by Chandra and Davis [128] and some related results can also be found in Hille [281, 282].

A general method is given whereby a large class of Gronwall-Bellman type inequalities can be reduced to the well-known classical case. The method is applicable to both the continuous case and the discrete case.

Chapter 6
Linear Multi-Dimensional Discrete (Difference) Inequalities

6.1 Linear Two-Dimensional Discrete Gronwall-Bellman Inequalities and Their Generalizations

6.1.1 Linear Two-Dimensional Discrete Gronwall-Bellman Inequalities

In this section, we shall present several new linear discrete inequalities in two independent variables, which are due to Salem and Raslan [565].

Theorem 6.1.1 (Salem-Raslan [565]) *Let $u(m,n), a(m,n), b(m,n)$ be non-negative functions and $a(m,n)$ non-decreasing for all $m, n \in \mathbb{N}_0$. If for all $m, n \in \mathbb{N}_0$,*

$$u(m,n) \leq a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} b(s,t)u(s,t) \tag{6.1.1}$$

then for all $m, n \in \mathbb{N}_0$,

$$u(m,n) \leq a(m,n)\prod_{t=0}^{n-1}\left[1 + \sum_{s=0}^{m-1} b(s,t)\right]. \tag{6.1.2}$$

Proof Define

$$z(m,n) = a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} b(s,t)u(s,t). \tag{6.1.3}$$

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Then, from (6.1.1) and (6.1.3) it follows

$$u(m,n) \le z(m,n). \tag{6.1.4}$$

Since $a(m,n)$ is non-negative for all $m,n \in \mathbb{N}_0$, then from (6.1.3) and (6.1.4), we derive

$$\frac{z(m,n)}{a(m,n)} \le 1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} b(s,t)\frac{z(s,t)}{a(s,t)}. \tag{6.1.5}$$

Define

$$v(m,n) = 1 + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} b(s,t)\frac{z(s,t)}{a(s,t)}, \tag{6.1.6}$$

then, from (6.1.5) and (6.1.6) it follows

$$z(m,n) \le a(m,n)v(m,n). \tag{6.1.7}$$

From (6.1.6), we derive

$$\begin{aligned} v(m+1,n+1) = 1 + b(m,n)\frac{z(m,n)}{a(m,n)} + \sum_{t=0}^{n-1} b(m,t)\frac{z(m,t)}{a(m,t)} \\ + \sum_{s=0}^{m-1} b(s,n)\frac{z(s,n)}{a(s,n)} + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} b(s,t)\frac{z(s,t)}{a(s,t)}, \end{aligned} \tag{6.1.8}$$

then from (6.1.6) and (6.1.8), we deduce

$$\begin{aligned} v(m+1,n+1) - v(m,n) = b(m,n)\frac{z(m,n)}{a(m,n)} + \sum_{t=0}^{n-1} b(m,t)\frac{z(m,t)}{a(m,t)} \\ + \sum_{s=0}^{m-1} b(s,n)\frac{z(s,n)}{a(s,n)}. \end{aligned} \tag{6.1.9}$$

Also from (6.1.7), we have

$$v(m+1,n) - v(m,n) = \sum_{t=0}^{n-1} b(m,t)\frac{z(m,t)}{a(m,t)}, \tag{6.1.10}$$

and

$$v(m,n+1) - v(m,n) = \sum_{s=0}^{m-1} b(s,n) \frac{z(s,n)}{a(s,n)}. \tag{6.1.11}$$

Thus, from (6.1.9)–(6.1.11) it follows

$$[v(m+1,n+1)-v(m,n+1)]-[v(m+1,n)-v(m,n)] \le b(m,n)v(m,n). \tag{6.1.12}$$

Suppose n is fixed, then we derive from (6.1.12)

$$v(m,n+1) \le [1 + \sum_{s=0}^{m-1} b(s,n)]v(m,n)$$

which implies

$$v(m,n) \le \prod_{t=0}^{n-1} \left(1 + \sum_{s=0}^{m-1} b(s,n)\right). \tag{6.1.13}$$

Thus required inequality (6.1.2) follows from (6.1.4), (6.1.7) and (6.1.13). □

The discrete analogues of Theorems 5.1.7 and 5.1.8 are given in the following theorems.

Theorem 6.1.2 (Pachpatte [498]) *Let $u(m,n), a(m,n), b(m,n), c(m,n)$ be non-negative continuous functions defined for all $m, n \in \mathbb{N}_0$.*

(1) If for all $m, n \in \mathbb{N}_0$,

$$u(m,n) \le a(m,n) + b(m,n) \sum_{s=0}^{m-1} \sum_{t=n+1}^{+\infty} c(s,t)u(s,t), \tag{6.1.14}$$

then for all $m, n \in \mathbb{N}_0$,

$$u(m,n) \le a(m,n) + b(m,n)f(m,n) \prod_{s=0}^{m-1} \left[1 + \sum_{t=n+1}^{+\infty} c(s,t)b(s,t)\right], \tag{6.1.15}$$

where for all $m, n \in \mathbb{N}_0$,

$$f(m,n) = \sum_{s=0}^{m-1} \sum_{t=n+1}^{+\infty} c(s,t)a(s,t). \tag{6.1.16}$$

(2) If for all $m, n \in \mathbb{N}_0$,

$$u(m,n) \leq a(m,n) + b(m,n) \sum_{s=m+1}^{+\infty} \sum_{t=n+1}^{+\infty} c(s,t)u(s,t), \tag{6.1.17}$$

then for all $m, n \in \mathbb{N}_0$,

$$u(m,n) \leq a(m,n) + b(m,n)\bar{f}(m,n) \prod_{s=m+1}^{+\infty} \left[1 + \sum_{t=n+1}^{+\infty} c(s,t)b(s,t)\right], \tag{6.1.18}$$

where for all $m, n \in \mathbb{N}_0$,

$$\bar{f}(m,n) = \sum_{s=m+1}^{+\infty} \sum_{t=n+1}^{+\infty} c(s,t)a(s,t). \tag{6.1.19}$$

Proof We only give the proof of (1), the proof of (2) can be completed in the same way.

Define

$$z(m,n) = \sum_{s=0}^{m-1} \sum_{t=n+1}^{+\infty} c(s,t)u(s,t). \tag{6.1.20}$$

Then (6.1.14) can be written as

$$u(m,n) \leq a(m,n) + b(m,n)z(m,n). \tag{6.1.21}$$

Thus, it follows from (6.1.20) and (6.1.21) that

$$\begin{aligned} z(m,n) &\leq \sum_{s=0}^{m-1} \sum_{t=n+1}^{+\infty} c(s,t)[a(s,t) + b(s,t)z(s,t)] \\ &= f(m,n) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{+\infty} c(s,t)b(s,t)z(s,t), \end{aligned} \tag{6.1.22}$$

where $f(m,n)$ is defined by (6.1.16). Clearly, $f(m,n)$ is non-negative, non-decreasing in m and non-increasing in n for all $m, n \in \mathbb{N}_0$. From (6.1.22), we derive

$$\frac{z(m,n)}{f(m,n)} \leq 1 + \sum_{s=0}^{m-1} \sum_{t=n+1}^{+\infty} c(s,t)b(s,t)\frac{z(s,t)}{f(s,t)}. \tag{6.1.23}$$

Define

$$v(m,n) = 1 + \sum_{s=0}^{m-1} \sum_{t=n+1}^{+\infty} c(s,t)b(s,t)\frac{z(s,t)}{f(s,t)}, \tag{6.1.24}$$

then

$$\frac{z(m,n)}{f(m,n)} \le v(m,n) \tag{6.1.25}$$

and

$$\begin{aligned} &[v(m+1,n) - v(m,n)] - [v(m+1,n+1) - v(m,n+1)] \\ &= c(m,n+1)b(m,n+1)\frac{z(m,n+1)}{f(m,n+1)} \\ &\le c(m,n+1)b(m,n+1)v(m,n+1). \end{aligned} \tag{6.1.26}$$

Fixing m in (6.1.26), setting $n = t$ and summing over $t = n, n+1, \cdots, r-1$ ($r \ge n+1$ is arbitrary in $\mathbb{N}_0$), we obtain

$$\frac{[v(m+1,n) - v(m,n)]}{v(m,n)} - \frac{[v(m+1,r) - v(m,r)]}{v(m,r)} \le \sum_{t=n+1}^{r} c(m,t)b(m,t). \tag{6.1.27}$$

Noting that $\lim_{r\to+\infty} v(m,r) = \lim_{r\to+\infty} v(m+1,r) = 1$ and by letting $r \to +\infty$ in (6.1.27), we get

$$\frac{[v(m+1,n) - v(m,n)]}{v(m,n)} \le \sum_{t=n+1}^{+\infty} c(m,t)b(m,t),$$

i.e.,

$$v(m+1,n) \le \left[1 + \sum_{t=n+1}^{+\infty} c(m,t)b(m,t)\right] v(m,n). \tag{6.1.28}$$

Now, by fixing n in (6.1.28) and setting $m = s$ and substituting $s = 0, 1, 2, \cdots, m-1$ successively and using the fact that $v(0,n) = 1$, we get

$$v(m,n) \le \prod_{s=0}^{m-1} \left[1 + \sum_{t=n+1}^{+\infty} c(m,t)b(m,t)\right]. \tag{6.1.29}$$

Using (6.1.29) in $\frac{z(m,n)}{f(m,n)} \le v(m,n)$, we conclude

$$z(m,n) \le f(m,n) \prod_{s=0}^{m-1} \left[1 + \sum_{t=n+1}^{+\infty} c(s,t) b(s,t) \right]. \tag{6.1.30}$$

Therefore, the desired inequality (6.1.15) now follows from (6.1.21) and (6.1.30). If $f(m,n)$ is non-negative, then we carry out the above procedure with $f(m,n) + \varepsilon$ instead of $f(m,n)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (6.1.15). □

The next result is an analogue of Theorems 5.1.9 and 5.1.6.

Theorem 6.1.3 (Pachpatte [499]) *Let $u(m,n), a(m,n), b(m,n)$ be real-valued non-negative functions defined for all $m, n \in \mathbb{N}_0$.*

(a_1) *Let $a(m,n)$ be non-increasing in each variable $m, n \in \mathbb{N}_0$. If for all $m, n \in \mathbb{N}_0$,*

$$u(m,n) \le a(m,n) + \sum_{s=m+1}^{+\infty} \sum_{t=n+1}^{+\infty} b(s,t) u(s,t), \tag{6.1.31}$$

then for all $m, n \in \mathbb{N}_0$,

$$u(m,n) \le a(m,n) \prod_{s=m+1}^{+\infty} \left[1 + \sum_{t=n+1}^{+\infty} b(s,t) \right]. \tag{6.1.32}$$

(a_2) *Let $a(m,n)$ be non-increasing in $m \in \mathbb{N}_0$ and non-increasing in $m \in \mathbb{N}_0$. If for all $m, n \in \mathbb{N}_0$,*

$$u(m,n) \le a(m,n) + \sum_{s=0}^{m-1} \sum_{t=n+1}^{+\infty} b(s,t) u(s,t), \tag{6.1.33}$$

then for all $m, n \in \mathbb{N}_0$,

$$u(m,n) \le a(m,n) \prod_{s=0}^{m-1} \left[1 + \sum_{t=n+1}^{+\infty} b(s,t) \right]. \tag{6.1.34}$$

Proof We only give the details of the proof of (a_1). The proof of (a_2) can be completed similarly with suitable modifications.

(a_1) First, we assume that $a(m,n) > 0$ for all $m,n \in \mathbb{N}_0$. From (6.1.31), it follows

$$\frac{u(m,n)}{a(m,n)} \leq 1 + \sum_{s=m+1}^{+\infty} \sum_{t=n+1}^{+\infty} b(s,t) \frac{u(s,t)}{a(s,t)}. \tag{6.1.35}$$

Define a function $z(m,n)$ by the right-hand side of (6.1.35), then $u(m,n)/a(m,n) \leq z(m,n)$ and

$$\begin{aligned}
&[z(m,n) - z(m+1,n)] - [z(m,n+1) - z(m+1,n+1)] \\
&\quad = b(m+1,n+1) \frac{u(m+1,n+1)}{a(m+1,n+1)} \\
&\quad \leq b(m+1,n+1) z(m+1,n+1).
\end{aligned} \tag{6.1.36}$$

By (6.1.36) and using the facts that $z(m,n) > 0, z(m+1,n+1) \leq z(m+1,n)$ for all $m,n \in \mathbb{N}_0$, we see that

$$\frac{[z(m,n) - z(m+1,n)]}{z(m+1,n)} - \frac{[z(m,n+1) - z(m+1,n+1)]}{z(m+1,n+1)} \leq b(m+1,n+1). \tag{6.1.37}$$

Fixing m in (6.1.37), setting $n = t$ and summing over $t = n, n+1, \ldots, q-1$ ($q \geq n+1$ is arbitrary in $\mathbb{N}_0$), we obtain

$$\frac{z(m,n) - z(m+1,n)}{z(m+1,n)} - \frac{z(m,q) - z(m+1,q)}{z(m+1,q)} \leq \sum_{t=n+1}^{q} b(m+1,t). \tag{6.1.38}$$

Noting that $\lim_{q \to +\infty} z(m,q) = \lim_{q \to +\infty} z(m+1,q) = 1$ and letting $q \to +\infty$ in (6.1.38), we can get

$$\frac{z(m,n) - z(m+1,n)}{z(m+1,n)} \leq \sum_{t=n+1}^{q} b(m+1,t),$$

i.e.,

$$z(m,n) \leq \left[1 + \sum_{t=n+1}^{q} b(m+1,t) \right] z(m+1,n). \tag{6.1.39}$$

Now, keeping n fixed in (6.1.39) and setting $m = s$ and substituting $s = m, m+1, \dots, p-1$, ($p \geq m+1$ is arbitrary in $\mathbb{N}_0$) successively, we obtain

$$z(m,n) \leq z(p,n) \prod_{s=m+1}^{p} \left[1 + \sum_{t=n+1}^{+\infty} b(s,t)\right]. \tag{6.1.40}$$

Noting that $\lim_{p\to+\infty} z(p,n) = 1$, and letting $p \to +\infty$ in (6.1.40), we conclude

$$z(m,n) \leq \prod_{s=m+1}^{+\infty} \left[1 + \sum_{t=n+1}^{+\infty} b(s,t)\right]. \tag{6.1.41}$$

Using (6.1.41) in (6.1.35), we can derive the required inequality (6.1.32). If $a(m,n)$ is non-negative, we carry out the above procedure with $a(m,n) + \varepsilon$ instead of $a(m,n)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to obtain (6.1.32). □

The next theorem extends the above theorem.

Theorem 6.1.4 (Pachpatte [499]) *Let $u(m,n), a(m,n), b(m,n), c(m,n)$ be non-negative continuous functions defined for all $m, n \in \mathbb{N}_0$.*

(1) Assume that $a(m,n)$ is non-increasing in $m \in \mathbb{N}_0$. If for all $m, n \in \mathbb{N}_0$,

$$u(m,n) \leq a(m,n) + \sum_{s=0}^{m-1} b(s,n)u(s,n) + \sum_{s=m+1}^{+\infty} \sum_{t=n+1}^{+\infty} c(s,t)u(s,t), \tag{6.1.42}$$

then for all $m, n \in \mathbb{N}_0$,

$$u(m,n) \leq q(m,n) \left[a(m,n) + G(m,n) \prod_{s=0}^{m-1} \left[1 + \sum_{t=n+1}^{+\infty} c(s,t)q(s,t)\right]\right], \tag{6.1.43}$$

where for all $m, n \in \mathbb{N}_0$,

$$q(m,n) = \sum_{s=0}^{m-1} [1 + b(s,n)], \quad G(m,n) = \sum_{s=0}^{m-1} \sum_{t=n+1}^{+\infty} c(s,t)q(s,t)a(s,t).$$

(2) Assume that $a(m,n)$ is non-increasing in $m \in \mathbb{N}_0$. If for all $m, n \in \mathbb{N}_0$,

$$u(m,n) \leq a(m,n) + \sum_{s=m+1}^{+\infty} b(s,n)u(s,n) + \sum_{s=m+1}^{+\infty} \sum_{t=n+1}^{+\infty} c(s,t)u(s,t), \tag{6.1.44}$$

then for all $m, n \in \mathbb{N}_0$*,*

$$u(m,n) \le \bar{q}(m,n)\left[a(m,n) + \bar{G}(m,n)\prod_{s=m+1}^{+\infty}\left[1 + \sum_{t=n+1}^{+\infty} c(s,t)\bar{q}(s,t)\right]\right], \tag{6.1.45}$$

where for all $m, n \in \mathbb{N}_0$*,*

$$\bar{q}(m,n) = \sum_{s=m+1}^{+\infty}[1 + b(s,n)], \quad \bar{G}(m,n) = \sum_{s=m+1}^{+\infty}\sum_{t=n+1}^{+\infty} c(s,t)\bar{q}(s,t)a(s,t).$$

Proof We only give the proof of (1), the proof of (2) can be completed in the same way.

Define

$$z(m,n) = a(m,n) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{+\infty} c(s,t)u(s,t). \tag{6.1.46}$$

Then (6.1.42) can be restated as

$$u(m,n) \le z(m,n) + \sum_{s=0}^{m-1} b(s,n)u(s,n). \tag{6.1.47}$$

Clearly, $z(m,n)$ is non-negative and non-decreasing in m, $m \in \mathbb{N}_0$. Treating n, $n \in \mathbb{N}_0$ fixed in (6.1.47) and using part (1) of Theorem 6.1.2 to (6.1.47), we obtain

$$u(m,n) \le z(m,n)q(m,n), \tag{6.1.48}$$

where $q(m,n)$ is defined above. From (6.1.48) and (6.1.46), it follows

$$u(m,n) \le q(m,n)[a(m,n) + v(m,n)], \tag{6.1.49}$$

where

$$v(m,n) = \sum_{s=0}^{m-1}\sum_{t=n+1}^{+\infty} c(s,t)u(s,t). \tag{6.1.50}$$

Thus from (6.1.49) and (6.1.50), it follows that

$$v(m,n) \le G(m,n) + \sum_{s=0}^{m-1}\sum_{t=n+1}^{+\infty} c(s,t)q(s,t)v(s,t), \tag{6.1.51}$$

where $G(m,n)$ is as defined above. □

6.1.2 *Linear Two-Dimensional Discrete Generations of Gronwall-Bellman Inequalities*

We shall establish some new discrete inequalities in two independent variables which are due to Pachpatte [489] .

The notation is as follows. The expression $u(0)+\sum_{s=0}^{n-1} b(s)$ represents a solution of the linear difference equation $\Delta u(n) = b(n)$ for all $n \in \mathbb{N}_0$, where Δ is the operator defined by $\Delta u(n) = u(n+1) - u(n)$. The expression $u(0)\prod_{s=0}^{n-1} b(s)$ represents a solution of the linear difference equation $u(n+1) = b(n)u(n)$ for all $n \in \mathbb{N}_0$. We use the convention of writing $\sum_{s\in\varnothing} b(s) = 0$ and $\prod_{s\in\varnothing} = 1$, if $\varnothing$ is the empty set. We also use the following notations of the operators for $m,\ n \in \mathbb{N}_0$,

$$\begin{cases} \Delta_1 u(m,n) = u(m+1,n) - u(m,n), \\ \Delta_2 u(m,n) = u(m,n+1) - u(m,n). \end{cases}$$

We often use the letters m and n to denote the two independent variables which are members of $\mathbb{N}_0$.

For our convenience, we list the following hypotheses:

(H_1) $u(m,n)$ and $h(m,n)$ are real-valued non-negative functions defined for all $m,\ n \in \mathbb{N}_0$.

(H_2) $p_1(m,n),\ p_2(m,n),\ p_3(m,n)$ are real-valued positive functions defined for all $m,\ n \in \mathbb{N}_0$.

(H_3) $a(m,n)$ is real-valued, positive and non-decreasing function in both the variables m and n in $\mathbb{N}_0$.

(H_4) $u(m,n) \geq u_0 \geq 0$, u_0 is a constant, $h(m,n) \geq 0$ are real-valued functions defined for all $m,\ n \in \mathbb{N}_0$.

(H_5) $g(u)$ is continuous, non-decreasing real-valued function defined on an interval $I = [u_0, +\infty)$, $u_0 \geq 0$ is a constant, and $g(u) > 0$ on $(u_0, +\infty)$, $g(u_0) = 0$.

(H_6) $q_1(m,n),\ q_2(m,n),\ q_3(m,n)$ are real-valued positive functions defined for all $m,\ n \in \mathbb{N}_0$.

(H_7) $W(u)$ is continuous, non-decreasing and sub-multiplicative real-valued function defined on an interval I, and $W(u) > 0$ on $(u_0, +\infty)$, $W(u_0) = 0$.

A useful two independent variable discrete inequality is stated in the following theorem.

Theorem 6.1.5 (Pachpatte [489]) *Suppose that* (H_1) *and* (H_2) *hold. If for all* $m,\ n \in \mathbb{N}_0$,

$$u(m,n) \leq c + \sum_{x=0}^{m-1} \frac{1}{p_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s,y)} \sum_{t=0}^{y-1} h(s,t)u(s,t), \tag{6.1.52}$$

where c is a non-negative constant, then for all $m,\ n \in \mathbb{N}_0$,

$$u(m,n) \le c \prod_{x=0}^{m-1} \Big[1 + \frac{1}{p_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s,y)} \sum_{t=0}^{y-1} h(s,t)u(s,t) \Big]. \tag{6.1.53}$$

Proof We first assume that $c > 0$ and define a function $z(m,n)$ by

$$z(m,n) = c + \sum_{x=0}^{m-1} \frac{1}{p_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s,y)} \sum_{t=0}^{y-1} h(s,t)u(s,t). \tag{6.1.54}$$

From (6.1.54) it follows that

$$z(0,n) = z(m,0) = c \tag{6.1.55}$$

and

$$p_1(m,n)\Delta_1 z(m,n) = \sum_{s=0}^{x-1} \frac{1}{p_2(s,y)} \sum_{y=0}^{n-1} \frac{1}{p_3(s,n)} \sum_{t=0}^{y-1} h(s,t)u(s,t), \tag{6.1.56}$$

$$p_2(m,n)\Delta_1[p_1(m,n)\Delta_1 z(m,n)] = \sum_{y=0}^{n-1} \frac{1}{p_3(m,y)} \sum_{t=0}^{y-1} h(m,t)u(m,t), \tag{6.1.57}$$

$$p_3(m,n)\Delta_2[p_2(m,n)\Delta_1[p_1(m,n)\Delta_1 z(m,n)]] = \sum_{t=0}^{y-1} h(m,t)u(m,t), \tag{6.1.58}$$

$$\Delta_2[p_3(m,n)\Delta_2[p_2(m,n)\Delta_1[p_1(m,n)\Delta_1 z(m,n)]]] = h(m,n)u(m,n). \tag{6.1.59}$$

Using the fact that $u(m,n) \le z(m,n)$ in (6.1.59), we have

$$\Delta_2[p_3(m,n)\Delta_2[p_2(m,n)\Delta_1[p_1(m,n)\Delta_1 z(m,n+1)]]] \le h(m,n)z(m,n). \tag{6.1.60}$$

From the definition of $z(m,n)$, we see that $z(m,n) \le z(m,n+1)$ for all $m,\ n \in \mathbb{N}_0$, which, with (6.1.60), gives us

$$\frac{p_3(m,n+1)\Delta_2[p_2(m,n+1)\Delta_1[p_1(m,n+1)\Delta_1 z(m,n+1)]]}{z(m,n+1)} - \frac{p_3(m,n)\Delta_2[p_2(m,n)\Delta_1[p_1(m,n)\Delta_1 z(m,n)]]}{z(m,n)} \le h(m,n). \tag{6.1.61}$$

Now fixing m in (6.1.61), setting $n = t$ and summing over $t = 0,\ 1, \cdots, n-1$ and using the fact that $p_3(m,0)\Delta_2[p_2(m,0)\Delta_1[p_1(m,0)\Delta_1 z(m,0)]] = 0$, from (6.1.58) it follows

$$\frac{p_3(m,n)\Delta_2[p_2(m,n)\Delta_1[p_1(m,n)\Delta_1 z(m,n)]]}{z(m,n)} \le \sum_{t=0}^{n-1} h(m,t). \qquad (6.1.62)$$

From (6.1.62) and in view of the facts that $z(m,n) \le z(m,n+1)$ and $p_2(m,n)\Delta_1[p_1(m,n)\Delta_1 z(m,n)] \ge 0$, we arrive that

$$\frac{p_2(m,n+1)\Delta_1[p_1(m,n+1)\Delta_1 z(m,n+1)]]}{z(m,n+1)} - \frac{p_2(m,n)\Delta_1[p_1(m,n)\Delta_1 z(m,n)]}{z(m,n)}$$
$$\le \frac{1}{p_3(m,n)} \sum_{t=0}^{n-1} h(m,t). \qquad (6.1.63)$$

Fixing m in (6.1.63), setting $n = y$ and summing over $y = 0,\ 1,\ 2,\ \ldots,\ n-1$ and using the fact that $p_2(m,0)\Delta_1[p_1(m,0)\Delta_1 z(m,0)] = 0$, from (6.1.57) it follows

$$\frac{p_2(m,n)\Delta_1[p_1(m,n)\Delta_1 z(m,n)]}{z(m,n)} \le \sum_{y=0}^{n-1} \frac{1}{p_3(m,y)} \sum_{t=0}^{y-1} h(m,t). \qquad (6.1.64)$$

From (6.1.64) and in view of the facts that $z(m,n) \le z(m+1,n)$ and $p_1(m,n)\Delta_1 z(m,n) \ge 0$, from (6.1.56), we deduce

$$\frac{p_1(m+1,n)\Delta_1 z(m+1,n)}{z(m+1,n)} - \frac{p_1(m,n)\Delta_1 z(m,n)}{z(m,n)} \le \frac{1}{p_2(m,n)} \sum_{y=0}^{n-1} \frac{1}{p_3(m,y)} \sum_{t=0}^{y-1} h(m,t). \qquad (6.1.65)$$

Now fixing n in (6.1.65), setting $m = s$ and summing over $s = 0,\ 1,\ 2, \ldots, m-1$ and using the fact that $p_1(0,n)\Delta_1 z(0,n) = 0$, then from (6.1.56), we derive

$$\frac{\Delta_1 z(m,n)}{z(m,n)} \le \frac{1}{p_1(m,n)} \sum_{s=0}^{m-1} \frac{1}{p_2(m,y)} \sum_{y=0}^{n-1} \frac{1}{p_3(s,n)} \sum_{t=0}^{y-1} h(s,t), \qquad (6.1.66)$$

which, together with (6.1.66), gives us

$$z(m+1,n) \le z(m,n)[1 + \frac{1}{p_1(m,n)} \sum_{s=0}^{m-1} \frac{1}{p_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s,n)} \sum_{t=0}^{y-1} h(s,t)]. \qquad (6.1.67)$$

Now fixing n in (6.1.67), setting $n = x$ and substituting $x = 0, 1, 2, \ldots, m-1$ successively and using the fact that $z(0, n) = c$, then, from (6.1.55) it follows

$$z(m,n) \leq c \prod_{x=0}^{m-1} [1 + \frac{1}{p_1(x,n)} \sum_{s=0}^{m-1} \frac{1}{p_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s,y)} \sum_{t=0}^{y-1} h(s,t)].$$

Substituting this bound on $z(m,n)$ on the right-hand side of (6.1.52), we can obtain the required inequality (6.1.53).

Now suppose $c = 0$. Then, from (6.1.52) we derive

$$u(m,n) \leq \epsilon + \sum_{x=0}^{m-1} \frac{1}{p_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s,y)} \sum_{t=0}^{y-1} h(s,t)u(s,t)$$

for every arbitrary positive number ϵ and $m, n \in \mathbb{N}_0$, which, by the above argument, yields

$$u(m,n) \leq \epsilon \prod_{x=0}^{m-1} [1 + \frac{1}{p_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s,y)} \sum_{t=0}^{y-1} h(s,t)]. \quad (6.1.68)$$

Since $u(m,n) \geq 0$ and $\epsilon > 0$ is arbitrarily independent of m, n, then, from (6.1.68) it follows that $u(m,n) = 0$. This thus completes the proof . □

A slightly different version of Theorem 6.1.5 is now given in the following theorem.

Theorem 6.1.6 (Pachpatte [489]) *Suppose that* (H_1), (H_2) *and* (H_3) *hold. If for all* $m, n \in \mathbb{N}_0$,

$$u(m,n) \leq a(m,n) + \sum_{x=0}^{m-1} \frac{1}{p_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s,y)} \sum_{t=0}^{y-1} h(s,t)u(s,t), \quad (6.1.69)$$

then for all $m, n \in \mathbb{N}_0$,

$$u(m,n) \leq a(m,n) \prod_{x=0}^{m-1} [1 + \frac{1}{p_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s,y)} \sum_{t=0}^{y-1} h(s,t)u(s,t)]. \quad (6.1.70)$$

Proof Since $a(m,n)$ is positive and non-decreasing, we derive from (6.1.69) that

$$\frac{u(m,n)}{a(m,n)} \leq 1 + \sum_{x=0}^{m-1} \frac{1}{p_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{p_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{p_3(s,y)} \sum_{t=0}^{y-1} h(s,t) \frac{u(s,t)}{a(s,t)}.$$

Now applying of Theorem 6.1.5 yields the required bound (6.1.70) and the proof is now complete. □

The next result, due to Blandzi, Popenda and Agarwal [378], concerns an inequality with one continuous variable and one discrete variable.

Theorem 6.1.7 (Blandzi-Popenda-Agarwal [378]) *Let $u : \mathbb{N} \times J \to \mathbb{R}$, $\alpha : \mathbb{N} \times J \to \mathbb{R}_0 \equiv (0, +\infty)$, $c \in \mathbb{R}_0$ and $u(n,\cdot)$, $\alpha(n,\cdot)$ be continuous functions with respect to the second variable on J for every $n \in \mathbb{N}$, where $J = [0,\alpha)$ or $(J = \mathbb{R}_+)$. If the following inequality holds for all $x \in J$, $\quad n \in \mathbb{N}$,*

$$u(n,x) \leq c + \int_0^x \sum_{j=1}^{n} \alpha(j,s) u(j,s)\, ds, \tag{6.1.71}$$

then the following estimate holds for all $n > 1, x \in J$,

$$u(n,x) \leq c \left(1 + \int_0^x r_{n-1}(t) e^{\int_0^t [\alpha_{n-1,s} - \alpha_{n,s}]\, ds}\, dt \right) e^{\int_0^x \alpha_{n,t}\, dt}, \tag{6.1.72}$$

where $\{r_n(x)\}_{n=1}^{\infty}$ is the solution of

$$\begin{cases} r_{n+1}(x) = r_n(x) e^{\int_0^x [\alpha(n,t) - \alpha(n+1,t)]\, dt} + \alpha(n+1,x) \displaystyle\int_0^x r_n(t) e^{\int_0^t [\alpha(n,s) - \alpha(n+1,s)]\, ds}\, dt \\ \qquad\qquad + \alpha(n+1,s), \quad n \in \mathbb{N}, x \in J, \\ r_1(x) = \alpha(1,x). \end{cases} \tag{6.1.73}$$

Furthermore, if we take $r_0(x) \equiv 0$, $\alpha(0,x) \equiv \alpha(1,x), x \in J$, then the estimate (6.1.72) is also true for $n = 1$.

Remark 6.1.1 Another estimate of the inequality (6.1.71), which is more useful in applications, has the following form for all $x \in J, n \in \mathbb{N}$,

$$u(n,x) \leq c \exp \left\{ \int_0^x \sum_{j=1}^{n} \alpha(j,t)\, dt \right\}.$$

To show how this estimate can take various forms, following [378], we give an example from graph theory. Let us denote by P_n, the set of all subsets of the set $\{1, 2, \cdots, n\}$ such that for arbitrary $p_k \in P_n$, there is $\{n\} \subset p_k$. By G_n, we denote

the connected graph having no cycles, no loops, and no multiple edges, with the set $V(G_n)$ of vertices equal to P_n and the set $E(Gn)$ of edges such that the edge $[u, v] \in E(Gn)$ if $u \neq \{n\}$ and $u = p_k$, $v = p \setminus \{\min p_k\}$, where $\min p_k$ denotes the smallest number contained in the set p_k.

By $T(p_k)$, we denote the trail beginning with the vertex $\{n\}$ and ending with the vertex p_k. Furthermore, we suppose that all vertices are labeled by the function $\lambda(p_k) = \min p_k$. Let $\phi(p_k)$ denote the cardinal of the set p_k. Furthermore, let $q(p_k) = q_1, q_2, \cdots, q_{\phi(p_k)}$ be a strictly decreasing sequence of elements of the set p_k.

Theorem 6.1.8 (Blandzi-Popenda-Agarwal [378]) *Let $u, b : \mathbb{N} \to \mathbb{R}$, $f^i : \mathbb{N}^2 \to \mathbb{R}$, $i = 1, \cdots, t$. Further, let the inequality hold for all $n \in \mathbb{N}$,*

$$u_n \le b_n + \sum_{j_1=1}^{n-1} f^i(n, j_1) \sum_{j_2=1}^{j_1-1} f^2(j_1, j_2) \cdots \sum_{j_t=1}^{j_{t-1}-1} f^t(j_{t-1}, j_t) u_{jt}. \quad (6.1.74)$$

Then, the following inequality holds

$$u_n \le b_n + \sum_{T(p_k) p_k \in V(G_n)} b_{\lambda(p_k)} \prod_{j=0}^{\bar{s}} \prod_{i=1}^{t} f^i(q_{jt+i}, q_{jt+i+1}), \quad n \in \mathbb{N}, \quad (6.1.75)$$

where

$$\bar{s} = \frac{\phi(p_k) - 1}{t} - 1$$

and the summation is taken over all trails $T(p_k)$, $p_k \in V(G_n)$ such that $\bar{s}$ is a non-negative integer. If such trail does not exist, then the sum is equal to zero.

We also note that the above result provides the best possible estimate. Indeed, the equality in (6.1.74) implies the equality in (6.1.75). As an example, let $t = 3$, so that the inequality (6.1.74) gives us

$$u_1 \le b_1, u_2 \le b_2, u_3 \le b_3,$$
$$u_4 \le b_4 + f^1(4,3) f^2(3,2) f^3(2,1) u_1$$

whence,

$$u_4 \le b_4 + f^1(4,3) f^2(3,2) f^3(2,1) b_1.$$

Similarly, we get

$$\begin{aligned} u_5 \le b_5 &+ f^1(5,3) f^2(3,2) f^3(2,1) b_1 + f^1(5,4) f^2(4,2) f^3(2,1) b_1 \\ &+ f^1(5,4) f^2(4,3) f^3(3,1) b_1 + f^1(5,4) f^2(4,3) f^3(3,2) b_2. \end{aligned} \quad (6.1.76)$$

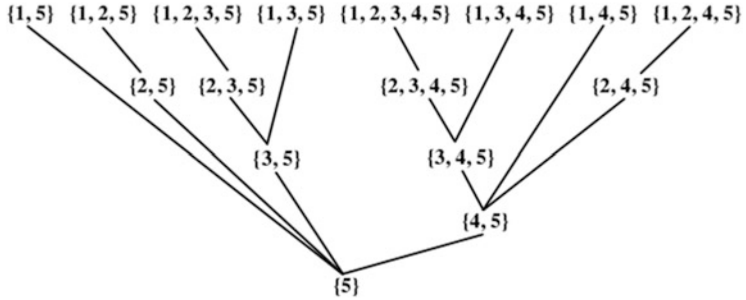

Fig. 6.1 Graph of G5

To clarify the bounds given in Theorem 6.1.8, we draw the graph of G_5 as shown in Fig. 6.1. There exist 15 trails from the vertex $\{5\}$. If card $(p_k) = 2$, then $\bar{s} = -(2/3)$. The trail $T(p_k)$ is not taken under consideration (for example, $T(\{4,5\})$). Similarly, if card $(p_k) = 3$, then $\bar{s} = -(1/3)$ (for example, $T(\{1,2,5\})$), and if card $(p_k) = 5$, then $\bar{s} = (1/3)$ (for example, $T(\{1,2,3,4,5\})$). For the remaining trails $(T(\{1,2,3,5\})$, $T(\{1,2,4,5\})$, $T(\{1,3,4,5\})$, $T(\{2,3,4,5\})$, we obtain the same estimate as in the inequality (6.1.76). In this case, $\bar{s} = 0$.

6.2 Linear Three-Dimensional Discrete Gronwall-Bellman Inequalities and Their Generalizations

6.2.1 Linear Three-Dimensional Linear Discrete Gronwall-Bellman Inequalities

We first collect a few of the basic notions and definitions from [448, 468]. We also use the following notions of the operators $\Delta u_x(x,y,z) = u(x+1,y,z) - u(x,y,z)$, $\Delta u_y(x,y,z) = u(x,y+1,z) - u(x,y,z)$, $\Delta u_z(x,y,z+1) = u(x+1,y,z) - u(x,y,z)$ and $\Delta u^2_{xy}(x,y,z) = \Delta u_x(x,y+1,z) - \Delta u_x(x,y,z)$ and so on. We often use the letters x, y, z to denote the three independent variables which are the members of $\mathbb{N}_0$. For $x, y, z \in \mathbb{N}_0$, and functions a, b, c with domain $\mathbb{N}_0$, and p with domain $\mathbb{N}_0^3$, set

$$\phi(x,y,z;a,b,c;p) = [a(0) + b(y) + c(z)] \prod_{s=0}^{x-1} \times \left[1 + \frac{\Delta a(s)}{a(s) + b(0) + c(z)} + \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} p(s,t,r) \right]. \quad (6.2.1)$$

Theorem 6.2.1 (Pachpatte-Singare [511]) *Let $u(x,y,z)$, and $p(x,y,z)$ be real-valued non-negative functions defined for all $(x,y,z) \in \mathbb{N}_0^3$ such that the following inequality holds for all $(x,y,z) \in \mathbb{N}_0^3$,*

$$u(x,y,z) \leq a(x) + b(y) + c(z) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} p(s,t,r)u(s,t,r), \tag{6.2.2}$$

where $a(x), b(y), c(z) > 0$; $\Delta a(x), \Delta b(y), \Delta c(z) > 0$ are real-valued functions defined on $\mathbb{N}_0$. Then for all $(x,y,z) \in \mathbb{N}_0^3$,

$$u(x,y,z) \leq \phi(x,y,z;a,b,c;p). \tag{6.2.3}$$

Proof Define a function $m(x,y,z)$ by

$$m(x,y,z) = a(x) + b(y) + c(z) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} p(s,t,r)u(s,t,r),$$

so that, by definition,

$$\begin{cases} m(0,y,z) = a(0) + b(y) + c(z), \\ m(x,0,z) = a(x) + b(0) + c(z), \\ m(x,y,0) = a(x) + b(y) + c(0). \end{cases}$$

Then

$$\Delta m_x(x,y,z) = \Delta a(x) + \sum_{t=0}^{y-1}\sum_{r=0}^{z-1} p(x,t,r)u(x,t,r), \tag{6.2.4}$$

and from (6.2.4) it follows

$$\Delta m_x(x,y+1,z) - \Delta m_x(x,y,z) = \sum_{r=0}^{z-1} p(x,y,r)u(x,y,r), \tag{6.2.5}$$

$$\Delta m_x(x,y+1,z+1) - \Delta m_x(x,y,z+1) = \sum_{r=0}^{z} p(x,y,r)u(x,y,r). \tag{6.2.6}$$

On the one hand, from (6.2.5) and (6.2.6) we derive

$$\Delta^2 m_{xy}(x,y,z+1) - \Delta^2 m_{xy}(x,y,z) = p(x,y,z)u(x,y,z)$$

which, in view of (6.2.2), implies

$$\Delta^2 m_{xy}(x,y,z+1) - \Delta^2 m_{xy}(x,y,z) \le p(x,y,z)m(x,y,z). \tag{6.2.7}$$

On the other hand, from the definition of $m(x,y,z)$, we see that $m(x,y,z) \le m(x,y,z+1)$, for all $(x,y,z) \in \mathbb{N}_0^3$. Using this fact in (6.2.7), we have

$$\frac{\Delta^2 m_{xy}(x,y,z+1)}{m(x,y,z+1)} - \frac{\Delta^2 m_{xy}(x,y,z)}{m(x,y,z+1)} \le p(x,y,z). \tag{6.2.8}$$

Thus from (6.2.8), we derive

$$\frac{\Delta^2 m_{xy}(x,y,z+1)}{m(x,y,z+1)} - \frac{\Delta^2 m_{xy}(x,y,z)}{m(x,y,z)} \le p(x,y,z). \tag{6.2.9}$$

Now fixing x, y in (6.2.9), setting $z = r$ and summing over $r = 0, 1, \cdots, z-1$, we can obtain

$$\frac{\Delta^2 m_{xy}(x,y,z)}{m(x,y,z)} \le \sum_{r=0}^{z-1} p(x,y,r). \tag{6.2.10}$$

From (6.2.10), in view of the fact that $m(x,y,z) \le m(x,y+1,z)$, we derive

$$\frac{\Delta m_x(x,y+1,z)}{m_(x,y+1,z)} - \frac{\Delta m_x(x,y,z)}{m(x,y,z)} \le \sum_{r=0}^{z-1} p(x,y,r). \tag{6.2.11}$$

Fixing x, z in (6.2.11), setting $y = t$ and summing over $t = 0, 1, \cdots, y$, we obtain

$$m(x+1,y,z) \le m(x,y,z)\left[1 + \frac{\Delta a(x)}{a(x)+b(0)+c(z)} + \sum_{t=0}^{y-1}\sum_{r=0}^{z-1} p(x,t,r)\right]. \tag{6.2.12}$$

Now keeping y, z fixed in (6.2.12), setting $x = s$ and substituting $s = 0, 1, \cdots, x-1$ successively in (6.2.12), we may obtain

$$\begin{aligned} m(x,y,z) &\le [a(0)+b(y)+c(z)]\sum_{s=0}^{x-1}\left[1 + \frac{\Delta a(s)}{a(x)+b(0)+c(z)} + \sum_{t=0}^{y-1}\sum_{r=0}^{z-1} p(s,t,r)\right] \\ &= \phi(x,y,z;a,b,c;p). \end{aligned}$$

Substituting this bound for $m(x,y,z)$ in (6.2.2), we can obtain the desired bound (6.2.3). □

Remark 6.2.1 We note that for the method of proof in Theorem 6.2.1 and all other theorems given below, the following must be satisfied:

$$\begin{cases} m(x,y,z+1) \geq m(x,y,z) > 0; \ \Delta^2 m_{xy}(x,y,z+1) \geq 0; \\ \Delta^2 m_{xy}(x,y,0) = 0, \quad \Delta m_x(x,y+1,z) \geq 0, \Delta m_y(x,y,z) \geq 0, \end{cases}$$

x, y, z and as well as in a, b, c. Hence there are $3! = 6$ different conclusions, we can state in Theorem 6.2.1 corresponding to the 6 permutations of (x, y, z) and corresponding permutations of (a, b, c). For example, in Theorem 6.2.1, we can conclude, in addition to (6.2.3), that

$$u(x,y,z) \leq \phi(z,x,y;c,a,b;p) \tag{6.2.13}$$

where, by (6.2.1) above, the right-hand side of (6.2.13) is

$$[c(0)+a(x)+b(y)] \sum_{s=0}^{z-1} \left[1 + \frac{\Delta c(s)}{c(s)+a(0)+b(y)} + \sum_{t=0}^{x-1} \sum_{r=0}^{y-1} p(s,t,r) \right].$$

Similarly, we can use $\phi(y, x, z; b, a, c; p)$, etc. We also note that a similar permutation applies to the conclusion of Theorem 6.2.2 given below, which deals with the three independent variable generalization of the discrete inequality established in Theorem 1 by Pachpatte [446], which, in turn, is a discrete analogue of the integral inequality established in Theorem 1 by Pachpatte [511].

Theorem 6.2.2 (Pachpatte-Singare [511]) *Let $u(x,y,z), p(x,y,z)$, and $q(x,y,z)$ be real-valued non-negative functions defined for all $(x,y,z) \in \mathbb{N}_0^3$ such that the inequality holds for all $(x,y,z) \in \mathbb{N}_0^3$,*

$$u(x,y,z) \leq a(x)+b(y)+c(z) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} p(s,t,r)\left[u(s,t,r) + \sum_{k=0}^{s-1} \sum_{l=0}^{t-1} \sum_{n=0}^{r-1} q(k,l,n)u(k,l,n)\right], \tag{6.2.14}$$

where $a(x), b(y), c(z) > 0$, $\Delta a(x), \Delta b(y), \Delta c(z) \leq 0$, are real-valued functions defined on $\mathbb{N}_0$. Then for all $(x,y,z) \in \mathbb{N}_0^3$,

$$u(x,y,z) \leq [a(0)+b(y)+c(z)] + \sum_{s=0}^{x-1} \left[\Delta a(s) + \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} p(s,t,r)R(s,t,r)\right], \tag{6.2.15}$$

where for all $(x, y, z) \in \mathbb{N}_0^3$,

$$R(x, y, z) = \phi(x, y, z; a, b, c; p + q). \tag{6.2.16}$$

Proof Define a function $m(x, y, z)$ by

$$m(x, y, z) = a(x) + b(y) + c(z) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} p(s, t, r)[u(s, t, r) + \sum_{k=0}^{s-1}\sum_{l=0}^{t-1}\sum_{n=0}^{r-1} q(k, l, n)u(k, l, n)],$$

so that, by definition,

$$\begin{cases} m(0, y, z) & = a(0) + b(y) + c(z), \\ m(x, 0, z) & = a(x) + b(0) + c(z), \\ m(x, y, 0) & = a(x) + b(y) + c(0). \end{cases}$$

Then following the same steps as in the proof of Theorem 6.2.1, we can derive

$$\Delta^2 m_{xy}(x, y, z + 1) - \Delta^2 m_{xy}(x, y, z) = p(x, y, z)[u(x, y, z) + \sum_{k=0}^{x-1}\sum_{l=0}^{y-1}\sum_{n=0}^{z-1} q(k, l, n)u(k, l, n)]$$

which, in view of the definition of $m(x, y, z)$, implies

$$\Delta^2 m_{xy}(x, y, z + 1) - \Delta^2 m_{xy}(x, y, z) = p(x, y, z)[m(x, y, z) + \sum_{k=0}^{x-1}\sum_{l=0}^{y-1}\sum_{n=0}^{z-1} q(k, l, n)m(k, l, n)]. \tag{6.2.17}$$

If we put

$$v(x, y, z) = m(x, y, z) + \sum_{k=0}^{x-1}\sum_{l=0}^{y-1}\sum_{n=0}^{z-1} q(k, l, n)m(k, l, n), \tag{6.2.18}$$

such that

$$\begin{cases} v(0,y,z) &= a(0) + b(y) + c(z), \\ v(x,0,z) &= a(x) + b(0) + c(z), \\ v(x,y,0) &= a(x) + b(y) + c(0), \end{cases}$$

then following the same argument as in the proof of Theorem 6.2.1, we can obtain

$$\Delta^2 v_{xy}(x,y,z+1) - \Delta^2 m_{xy}(x,y,z) = \Delta^2 m_{xy}(x,y,z+1) - \Delta^2 m_{xy}(x,y,z) \\ + q(x,y,z)m(x,y,z). \tag{6.2.19}$$

Using the facts that $\Delta^2 m_{xy}(x,y,z+1) - \Delta^2 m_{xy}(x,y,z) \le p(x,y,z)m(x,y,z)$ from (6.2.17) and $m(x,y,z) \le v(x,y,z)$ from (6.2.18) in (6.2.19), we have

$$\Delta^2 v_{xy}(x,y,z+1) - \Delta^2 v_{xy}(x,y,z) \le [p(x,y,z) + q(x,y,z)]v(x,y,z).$$

Now following the same argument as in the proof of Theorem 6.2.1, we obtain

$$v(x,y,z) \le [a(0) + b(y) + c(z)] \prod_{s=0}^{x-1} [1 + \frac{\Delta a(s)}{a(s) + b(0) + c(z)} \\ + \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} [p(s,t,r) + q(s,t,r)]] = R(x,y,z).$$

Substituting this bound for $v(x,y,z)$ in (6.2.17) and following the last argument as in the proof of Theorem 6.2.1, we can obtain

$$m(x,y,z) \le [a(0) + b(y) + c(z)] + \sum_{s=0}^{x-1} [\Delta a(s) + \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} p(s,t,r)R(s,t,r)].$$

Substituting this bound for $m(x,y,z)$ in (6.2.14), we can obtain the desired bound (6.2.15). □

Remark 6.2.2 We note that, if (6.2.14) holds, then from the definitions of $m(x,y,z)$ and $v(x,y,z)$, we have

$$u(x,y,z) \le R(x,y,z), \tag{6.2.20}$$

on $\mathbb{N}_0^3$, where $R(x,y,z)$ is defined by (6.2.16). Certainly, (6.2.20) is less work to compute in any given case. On the other hand, in the special case that a, b, c are

constants (> 0), and $p \equiv p_0, q \equiv q_0$ are also constants (< 0), then we have

$$R(x, y, z) = (a + b + c)[1 + (p_0 + q_0)yz]^x,$$

while the bound in (6.2.15) is, say,

$$\begin{aligned}
\overline{R}(x, y, z) &= (a + b + c)\Big\{1 + \sum_{t=0}^{y-1}\sum_{r=0}^{z-1}\sum_{s=0}^{x-1} p_0[1 + (p_0 + q_0)tr]^s\Big\} \\
&\leq (a + b + c)\Big\{1 + p_0 yz \sum_{s=0}^{x-1}[1 + (p_0 + q_0)yz]^s\Big\} \\
&= (a + b + c)\Big\{1 + \frac{p_0}{p_0 + q_0}([1 + (p_0 + q_0)yz]^x - 1)\Big\} \\
&\leq R(x, y, z).
\end{aligned}$$

Thus, in the case (6.2.20) gives us the simpler, but not necessarily smaller bound than (6.2.15).

Remark 6.2.3 We note that the bounds obtained in (6.2.13) and (6.2.15) are independent of the unknown function $u(x, y, z)$. The estimates (6.2.13) and (6.2.15) have interesting applications to uniqueness, boundedness, continuous dependence and other problems in the analysis of a class of finite difference equations involving three independent variables.

Next, we shall introduce some discrete inequalities, due to Singare and Pachpatte [598], involving three independent variables which can be used in the study of discrete versions of partial differential and integral equations involving three independent variables.

Let $\mathbb{N}_{n_0}$ be the set of points $n_0 + k$ ($k = 0, 1, 2, \cdots$), where $n_0 \geq 0$ is a given integer. The expression $u(n_0) + \sum_{s=n_0}^{n-1} b(s)$ represents a solution of the linear difference equation $\Delta u(n) = b(n)$ for all $n \in \mathbb{N}_{n_0}$, where Δ is the difference operator by $\Delta u(n) = u(n + 1) - u(n)$.

It is supposed that $\sum_{s=n_0}^{n_0-1} b(s) = 0$. The expression $u(n_0)\Pi_{s=n_0}^{n-1} c(s)$ represents a solution of the linear difference equation $u(n + 1) = c(n)u(n)$ for all $n \in \mathbb{N}_{n_0}$. It is supposed that $\Pi_{s=n_0}^{n_0-1} c(s) = 1$.

We also use the following notions of the operators,

$$\begin{cases}
\Delta_x[u(x, y, z)] = \Delta u_x(x, y, z) = u(x + 1, y, z) - u(x, y, z), \\
\Delta_y[u(x, y, z)] = \Delta u_y(x, y, z) = u(x, y + 1, z) - u(x, y, z), \\
\Delta_z[u(x, y, z)] = \Delta u_z(x, y, z) = u(x, y, z + 1) - u(x, y, z), \\
\Delta_y[\Delta u_x(x, y, z)] = \Delta^2 u_{xy}(x, y, z) = \Delta u_x(x, y + 1, z) - \Delta u_x(x, y, z),
\end{cases}$$

and so on.

We often use the letters x, y and z to denote the three independent variables which are the members of $\mathbb{N}_{n_0}$.

Theorem 6.2.3 (Singare-Pachpatte [598]) *Let $u(x, y, z)$ and $b(x, y, z)$ be real-valued non-negative functions defined for all $x \geq 0, y \geq 0, z \geq 0$, and let $a(x, y, z)$ be positive non-decreasing in all three variables, and defined for all $x \geq 0, y \geq 0, z \geq 0$, such that the inequality holds for all $x \geq 0, y \geq 0, z \geq 0$,*

$$u(x, y, z) \leq a(x, y, z) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(s, t, r)u(s, t, r). \tag{6.2.21}$$

Then for all $x \geq 0, y \geq 0, z \geq 0$,

$$u(x, y, z) \leq a(x, y, z)\prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(s, t, r)\right]. \tag{6.2.22}$$

Proof Since $a(x, y, z)$ is positive, non-decreasing, we derive from (6.2.21) that

$$\begin{aligned}\frac{u(x, y, z)}{a(x, y, z)} &\leq 1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(s, t, r)\frac{u(s, t, r)}{a(x, y, z)}\\ &\leq 1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(s, t, r)\frac{u(s, t, r)}{a(s, t, r)}.\end{aligned} \tag{6.2.23}$$

Define

$$\begin{aligned}m(x, y, z) &= 1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(s, t, r)\frac{u(s, t, r)}{a(s, t, r)},\\ m(0, y, z) &= m(x, 0, z) = m(x, y, 0) = 1,\end{aligned}$$

then

$$\Delta m_x(x, y, z) = \sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(x, t, r)\frac{u(x, t, r)}{a(x, t, r)} \tag{6.2.24}$$

which yields

$$\begin{cases} \Delta m_x(x, y+1, z) - \Delta m_x(x, y, z) = \sum_{r=0}^{z-1} b(x, y, r) \dfrac{u(x, y, r)}{a(x, y, r)}, & (6.2.25) \\ \Delta m_x(x, y+1, z+1) - \Delta m_x(x, y, z+1) = \sum_{r=0}^{z} b(x, y, r) \dfrac{u(x, y, r)}{a(x, y, r)}. & (6.2.26) \end{cases}$$

Thus it follows from (6.2.25) and (6.2.26) that

$$\Delta^2 m_{xy}(x, y, z+1) - \Delta^2 m_{xy}(x, y, z) = b(x, y, z) \frac{u(x, y, z)}{a(x, y, z)}$$

which, combined with (6.2.23), implies

$$\Delta^2 m_{xy}(x, y, z+1) - \Delta^2 m_{xy}(x, y, z) \le b(x, y, z) m(x, y, z). \tag{6.2.27}$$

From the definition of $m(x, y, z)$, we deduce that $m(x, y, z) \le m(x, y, z+1)$, for all $x \ge 0, y \ge 0, z \ge 0$. Using this fact in (6.2.27), we get

$$\Delta^2 m_{xy}(x, y, z+1) - \Delta^2 m_{xy}(x, y, z) \le b(x, y, z) m(x, y, z+1),$$

i.e.,

$$\frac{\Delta^2 m_{xy}(x, y, z+1)}{m(x, y, z+1)} - \frac{\Delta^2 m_{xy}(x, y, z)}{m(x, y, z+1)} \le b(x, y, z). \tag{6.2.28}$$

From (6.2.28), we derive

$$\frac{\Delta^2 m_{xy}(x, y, z+1)}{m(x, y, z+1)} - \frac{\Delta^2 m_{xy}(x, y, z)}{m(x, y, z)} \le b(x, y, z). \tag{6.2.29}$$

Now keeping x, y fixed in (6.2.29), setting $z = r$ and substituting $r = 0, 1, 2, \cdots, z-1$, we obtain

$$\frac{\Delta^2 m_{xy}(x, y, z)}{m(x, y, z)} \le \sum_{r=0}^{z-1} b(x, y, r). \tag{6.2.30}$$

Then from (6.2.30) and in view of the fact that $m(x, y, z) \le m(x, y+1, z)$, we obtain

$$\frac{\Delta m_x(x, y+1, z)}{m(x, y+1, z)} - \frac{\Delta m_x(x, y, z)}{m(x, y, z)} \le \sum_{r=0}^{z-1} b(x, y, r). \tag{6.2.31}$$

Keeping x, z fixed in (6.2.31), setting $y = t$ and substituting $t = 0, 1, 2, \cdots, y-1$, we obtain

$$\frac{\Delta m_x(x, y, z)}{m(x, y, z)} \leq \sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(x, t, r). \tag{6.2.32}$$

We derive from (6.2.32)

$$m(x+1, y, z) \leq m(x, y, z)\left[1 + \sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(x, t, r)\right]. \tag{6.2.33}$$

Again keeping y, z fixed in (6.2.33), setting $x = s$ and substituting $s = 0, 1, 2, \cdots, x-1$, we conclude

$$m(x, y, z) \leq \prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(x, t, r)\right]$$

which, substituted in (6.2.23), gives us the desired bound (6.2.22). □

We can also apply Theorem 6.2.3 to establish the following discrete inequality in three independent variables.

Theorem 6.2.4 (Singare-Pachpatte [598]) *Let $u(x, y, z), b(x, y, z)$, and $c(x, y, z)$ be real-valued non-negative functions defined for all $x \geq 0, y \geq 0, z \geq 0$, and let $W(u)$ be continuous, positive strictly increasing function on $I = [u_0, +\infty), u_0 > 0$, and suppose further that the inequality holds for all $x \geq 0, y \geq 0, z \geq 0$,*

$$u(x, y, z) \leq M + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(s, t, r)u(s, t, r) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} c(s, t, r)W(u(s, t, r)) \tag{6.2.34}$$

where $M > 0$ is a constant. Then, for all $0 \leq x \leq x_1, 0 \leq y \leq y_1, 0 \leq z \leq z_1$,

$$u(x, y, z) \leq \Omega^{-1}\Big[\Omega(M) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} c(s, t, r)W(P(s, t, r))\Big]P(x, y, z) \tag{6.2.35}$$

where

$$\begin{cases} P(x,y,z) &= \prod_{s=0}^{x-1}\Big[1+\sum_{t=0}^{y-1}\sum_{r=0}^{z-1}b(s,t,r)\Big], & (6.2.36)\\ \Omega(r) &= \int_{r_0}^{r}\frac{ds}{W(s)},\ r\ge r_0>0 & (6.2.37)\end{cases}$$

and Ω^{-1} *is the inverse function of* Ω *and*

$$\Omega(M)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1}c(s,t,r)W(P(s,t,r))\in Dom\ (\Omega^{-1})$$

for all x, y, z *lying in the subintervals* $0\le x\le x_1, 0\le y\le y_1, 0\le z\le z_1$ *of* $\mathbb{N}_0$.

Proof Define

$$\begin{cases} a(x,y,z) &= M+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1}c(s,t,r)W(u(s,t,r)),\\ a(0,y,z) &= a(x,0,z)=a(x,y,0)=M, \end{cases} \tag{6.2.38}$$

then (6.2.34) can be rewritten as

$$u(x,y,z)\le a(x,y,z)+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1}b(s,t,r)u(s,t,r). \tag{6.2.39}$$

Since $a(x,y,z)$ is positive, non-decreasing, we derive from Theorem 6.2.3 that

$$u(x,y,z)\le a(x,y,z)\prod_{s=0}^{x-1}\left[1+\sum_{t=0}^{y-1}\sum_{r=0}^{z-1}b(s,t,r)\right]=a(x,y,z)P(x,y,z). \tag{6.2.40}$$

Furthermore, since W is sub-multiplicative, we have

$$W(u(x,y,z))\le W(a(x,y,z))W(P(x,y,z)).$$

Hence,

$$\frac{c(x,y,z)W(u(x,y,z))}{W(a(x,y,z))}\le c(x,y,z)W(P(x,y,z))$$

which, due to (6.2.38), reduces to

$$\frac{\Delta^2 a_{xy}(x,y,z+1) - \Delta^2 a_{xy}(x,y,z)}{W(a(x,y,z))} \le c(x,y,z)W(P(x,y,z)). \tag{6.2.41}$$

Then from (6.2.41) we derive that

$$\frac{\Delta^2 a_{xy}(x,y,z+1)}{W(a(x,y,z+1))} - \frac{\Delta^2 a_{xy}(x,y,z)}{W(a(x,y,z))} \le c(x,y,z)W(P(x,y,z)). \tag{6.2.42}$$

Now keeping x, y fixed in (6.2.42), setting $z = r$ and substituting $r = 0, 1, 2, \cdots, z-1$ in (6.2.42), we can obtain

$$\frac{\Delta^2 a_{xy}(x,y,z)}{W(a(x,y,z))} \le \sum_{r=0}^{z-1} c(x,y,r)W(P(x,y,r)). \tag{6.2.43}$$

Now from (6.2.43) it follows that

$$\frac{\Delta a_x(x,y+1,z)}{W(a(x,y+1,z))} - \frac{\Delta a_x(x,y,z)}{W(a(x,y,z))} \le \sum_{r=0}^{z-1} c(x,y,r)W(P(x,y,r)). \tag{6.2.44}$$

Keeping x, z fixed in (6.2.44) and setting $y = t$ and substituting $t = 0, 1, 2, \cdots, t-1$ in (6.2.44), we can derive

$$\frac{\Delta a_x(x,y,z)}{W(a(x,y,z))} \le \sum_{t=0}^{y-1}\sum_{r=0}^{z-1} c(x,y,r)W(P(x,y,r)). \tag{6.2.45}$$

Thus, we can infer from (6.2.37) and (6.2.45) that

$$\Omega(a(x+1,y,z)) - \Omega(a(x,y,z) = \int_{a(x,y,z)}^{a(x+1,y,z)} \frac{ds}{W(s)}$$
$$\le \frac{\Delta a_x(x,y,z)}{W(a(x,y,z))} \le \sum_{t=0}^{y-1}\sum_{r=0}^{z-1} c(x,y,r)W(P(x,y,r)).$$

Again keeping y, z fixed in the above inequality and setting $x = s$ and substituting $s = 0, 1, 2, \cdots, x-1$, we conclude

$$\Omega(a(x,y,z)) - \Omega(M) \le \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} c(s,t,r)W(P(s,t,r)). \tag{6.2.46}$$

Thus the desired bound (6.2.35) now follows by substituting the bound on $a(x, y, z)$ from (6.2.46) in (6.2.40). The subintervals of $\mathbb{N}_0$ for x, y, and z are obvious. $\square$

We next the following three independent variable generalization of the discrete inequality established by Pachpatte [447].

Theorem 6.2.5 (Pachpatte [447]) *Let $u(x, y, z), b(x, y, z)$, and $c(x, y, z)$ be as defined in Theorem 6.2.4; and let $a(x, y, z)$ be as defined in Theorem 6.2.3, such that the following inequality holds for all $x \geq 0, y \geq 0, z \geq 0$,*

$$u(x, y, z) \leq a(x, y, z) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(s, t, r)\left[u(s, t, r) + \sum_{k=0}^{s-1}\sum_{l=0}^{t-1}\sum_{n=0}^{r-1} c(k, l, n)W(u(k, l, n))\right]. \tag{6.2.47}$$

Then for all $x \geq 0, y \geq 0, z \geq 0$,

$$u(x, y, z) \leq a(x, y, z)\left[1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(s, t, r)Q(s, t, r)\right], \tag{6.2.48}$$

where for all $x \geq 0, y \geq 0, z \geq 0$,

$$Q(x, y, z) = \prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1}\sum_{r=0}^{z-1}[b(s, t, r) + c(s, t, r)]\right]. \tag{6.2.49}$$

Proof Since $a(x, y, z)$ is positive, non-decreasing, we may derive from (6.2.47)

$$\frac{u(x, y, z)}{a(x, y, z)} \leq 1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(s, t, r)\left[\frac{u(s, t, r)}{a(s, t, r)} + \sum_{k=0}^{s-1}\sum_{l=0}^{t-1}\sum_{n=0}^{r-1} c(k, l, n)\frac{u(k, l, n)}{a(k, l, n)}\right]. \tag{6.2.50}$$

Define

$$\begin{cases} m(x, y, z) = 1 + \displaystyle\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(s, t, r)\left[\frac{u(s, t, r)}{a(s, t, r)} + \sum_{k=0}^{s-1}\sum_{l=0}^{t-1}\sum_{n=0}^{r-1} c(k, l, n)\frac{u(k, l, n)}{a(k, l, n)}\right], \\ m(0, y, z) = m(0, x, z) = m(x, y, 0) = 1. \end{cases}$$

Then following the similar argument as in the proof of Theorem 6.2.3, we have

$$\Delta^2 m_{xy}(x,y,z+1) - \Delta^2 m_{xy}(x,y,z)$$
$$\leq b(x,y,z)\left[m(x,y,z) + \sum_{k=0}^{x-1}\sum_{l=0}^{y-1}\sum_{n=0}^{z-1} c(k,l,n)m(k,l,n)\right]. \quad (6.2.51)$$

If we put

$$\begin{cases} \nu(x,y,z) = m(x,y,z) + \displaystyle\sum_{k=0}^{x-1}\sum_{l=0}^{y-1}\sum_{n=0}^{z-1} c(k,l,n)m(k,l,n), & (6.2.52) \\ \nu(0,y,z) = \nu(x,0,z) = \nu(x,y,0) = 1, & (6.2.53) \end{cases}$$

then again following the similar argument as in the proof of Theorem 6.2.3 and using (6.2.51) and the fact that $m(x,y,z) \leq \nu(x,y,z)$ from (6.2.52), we derive

$$\Delta^2 \nu_{xy}(x,y,z+1) - \Delta^2 \nu_{xy}(x,y,z) \leq [b(x,y,z) + c(x,y,z)]\nu(x,y,z)$$

which, by following the same technique as in the proof of Theorem 6.2.3, yields

$$\nu(x,y,z) \leq \prod_{s=0}^{x-1}\left[1 + \sum_{t=0}^{y-1}\sum_{r=0}^{z-1}[b(s,t,r) + c(s,t,r)]\right] = Q(x,y,z).$$

Substituting this bound on $\nu(x,y,z)$ in (6.2.51) and once again following the last argument as in the proof of Theorem 6.2.3, we conclude

$$m(x,y,z) \leq 1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(s,t,r)Q(s,t,r)$$

which, substituted in (6.2.50), gives us the desired bound (6.2.48). □

We now apply Theorem 6.2.5 to establish the following more general inequality, due to Singare and Pachpatte [598] .

Theorem 6.2.6 (Singare-Pachpatte [598]) *Let $u(x,y,z), b(x,y,z), c(x,y,z)$ and $p(x,y,z)$ be real-valued non-negative functions defined for all $x \geq 0, y \geq 0, z \geq 0$, and let $W(u)$ be as defined in Theorem 6.2.4, and suppose further that the following*

inequality holds for all $x \geq 0, y \geq 0, z \geq 0$,

$$u(x,y,z) \leq M + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(s,t,r)\Big[u(s,t,r) + \sum_{k=0}^{s-1}\sum_{l=0}^{t-1}\sum_{n=0}^{r-1} c(k,l,n)u(k,l,n)\Big]$$
$$+\sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} p(s,t,r)W(u(s,t,r)) \tag{6.2.54}$$

where $M > 0$ *is a constant. Then for* $0 \leq x \leq x_2, 0 \leq y \leq y_2, 0 \leq z \leq z_2$,

$$u(x,y,z) \leq \Omega^{-1}\left[\Omega(M) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} p(s,t,r)W(R(s,t,r))\right] R(x,y,z) \tag{6.2.55}$$

where Ω, Ω^{-1} *are as defined in Theorem 6.2.4, and*

$$R(x,y,z) \quad = 1 + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} b(s,t,r)\prod_{k=1}^{s-1}\left[1 + \sum_{l=0}^{t-1}\sum_{n=0}^{r-1}[b(k,l,n) + c(k,l,n)]\right], \tag{6.2.56}$$

and

$$\Omega(M) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} p(s,t,r)W(R(s,t,r)) \in Dom\ (\Omega^{-1})$$

for all x, y, z *lying in the subintervals* $0 \leq x \leq x_2, 0 \leq y \leq y_2, 0 \leq z \leq z_2$ *of* $\mathbb{N}_0$.

Proof The proof follows from similar argument as in the proof of Theorem 6.2.4, and using Theorem 6.2.5. We omit the details. □

Remark 6.2.4 We note that discrete inequalities established in Theorems 6.2.3–6.2.6 can be extended easily to the case of n independent variables.

6.3 Linear Multi-Dimensional Discrete Gronwall-Bellman Inequalities

6.3.1 Linear Multi-Dimensional Discrete Gronwall-Bellman Inequalities and Their Generalizations

It is well-known that the discrete inequalities of the Gronwall type play a vital role in the theory of finite difference equations and numerical analysis (see,

[75, 96, 119, 176, 305, 393, 475, 535, 614, 648, 656] and the references therein). We shall introduce some discrete inequalities of the Gronwall type in n-independent variables, which are due to Yang [660], generalizes all of the known theorems obtained by Pachpatte and Singare [511] for the case of $n = 3$.

We shall use the conventions of writing

$$\sum_{j\in\mathbb{Z}} b_j \equiv 0, \quad \prod_{j\in\mathbb{Z}} c_j \equiv 1,$$

if $\mathbb{Z}$ is not the empty set. For simplicity, in the sequel, we shall denote $(x_1, x_2, \ldots, x_n) \in \mathbb{N}_0^n$ by x, and $(x_1, x_2, \ldots, x_j)$, $(x_j, x_{j+1}, \ldots, x_n)$ and $(x_i, x_{i+1}, \ldots, x_k)$ by x_j, $\tilde{x}_j$ and $x_{i,k}$ respectively, here i, k are integers from 1, 2, $\ldots, n$ with $i < k$. Furthermore, we denote the multiple-summation symbol

$$\sum_{y_j=0}^{x_j-1} \sum_{y_j=0}^{x_{j+1}-1} \cdots \sum_{y_k=0}^{x_k-1} \quad \text{by} \quad \sum_{y,j}^{x,k}$$

where x_i, $x_j \in \mathbb{N}_0$, $j \le i \le k$, and $1 \le j \le k \le n$. Moreover, we define

$$\Delta u_{x_j}(x) = u(\underline{x}_j - 1, x_j + 1, \tilde{x}_{k+1}) - u(x),$$

$$\Delta^2 u_{x_j x_k}(x) = \Delta u_{x_j}(\underline{x}_j - 1, x_k + 1, \tilde{x}_{k+1}) - \Delta u_{x_j}(x),$$

$$\cdots\cdots\cdots\cdots$$

and so on, where x_j, x_k, $\ldots$ are numbers from $\mathbb{N}_0$, and i, k, $\ldots$are integers from 1, 2, $\ldots, n$. We write also here that

$$\Delta^{(r)} L(x) := \Delta L^r_{x_1 x_2 \ldots x_r}(x)$$

for any real-valued function $L(x)$ on $\mathbb{N}_0^n$, here $1 \le r \le n$, $x \in \mathbb{N}_0^n$. In addition, we shall define a class of functions on $\mathbb{N}_0^n$ by

$$K = \{f : f(x) \ge 0,\ \Delta f(x) \ge 0,\ r = 1, \ldots, n-1, \text{and } \Delta^{(n)} f(x) \le 0\}.$$

It is obvious that the following properties are true:

(1) if $f(x) \in K$ and $c \ge 0$ is a real number, then $cf(x) \in K$.
(2) if $f(x)$, $g(x) \in K$, then $f(x) + g(x) \in K$.
(3) all functions of the form

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \ldots x_{i_{k-1}}^{r_{k-1}} \quad (k = 1, 2, \ldots, n)$$

are in the class K, here $r_h \ge 0$ and $i_h \quad (h = 1, \ldots, k-1)$ are integers with $1 < r_1 < i_2 < \ldots < i_{k-1} \le n$.

Theorem 6.3.1 (Yang [660]) *Let $u(x)$ and $p(x)$ are real-valued non-negative functions defined for all $x \in \mathbb{N}_0$, and let $f(x)$ be a real-valued positive and non-decreasing function in K. Suppose further that the following discrete inequality holds for all $x \in \mathbb{N}_0$,*

$$u(x) \le f(x) + \sum_{y,1}^{x,n} p(y)u(y). \tag{6.3.1}$$

Then we have, for all $x \in \mathbb{N}_0^n$,

$$u(x) \le f(0, \tilde{x}_2) \prod_{y_1=0}^{x_1-1} [1 + G(y_1, \tilde{x}_2) + \sum_{y,2}^{x,n} p(y)], \tag{6.3.2}$$

where

$$\begin{aligned} G(x) = {} & g_1(x_1, 0, \tilde{x}_3) + \sum_{y,2}^{x,2} g_2(x_1, y_2, 0, \tilde{x}_4) + \ldots \\ & + \sum_{y,2}^{x,n-2} g_{n-2}(x_1, \overline{y}_{2,n-2}, 0, x_n) + \sum_{y,2}^{x,n-1} g_{n-1}(x_1, \overline{y}_{2,n-1}, 0), \end{aligned} \tag{6.3.3}$$

and for $1 \le k \le n-1$, $x \in \mathbb{N}_0^n$,

$$g_k(\underline{x}_k, 0, \tilde{x}_{k+2}) = \frac{\Delta^{(k)} f(\underline{x}_k, 0, \tilde{x}_{k+2})}{f(\underline{x}_k, 0, \tilde{x}_{k+2})}. \tag{6.3.4}$$

Proof We define a function $U(x)$ on $\mathbb{N}_0^n$ by the right-hand of inequality (6.3.1), so that by definition

$$\begin{cases} U(\underline{x}_{k-1}, 0, \tilde{x}_{k+1}) = f(\underline{x}_{k-1}, 0, \tilde{x}_{k+1}) > 0, \\ U(\underline{x}_{k-1}, x_k + 1, \tilde{x}_{k+1}) \ge U(x) > 0, \ x_k \in \mathbb{N}_0, \ 1 \le k \le n, \end{cases} \tag{6.3.5}$$

since $f(x)$ is non-decreasing.
Furthermore, we can obtain from the definition of $U(x)$ that,

$$\begin{aligned} \Delta^{(n)} U(x) &= \Delta^{(n)} f(x) + p(x)u(x) \\ &\le p(x)U(\underline{x}_{n-1}, x_n + 1), \end{aligned} \tag{6.3.6}$$

since $\Delta^{(n)}f(x) \le 0$, $p(x) \ge 0$, (6.3.1) and (6.3.5). Note that

$$\Delta^{(n)}U(x) = \Delta^{(n)}f(x) + \sum_{y,k+1}^{x,n} p(\underline{x}_k, \tilde{y}_{k+1})u(\underline{x}_k, \tilde{y}_{k+1}) \ge 0, \tag{6.3.7}$$

is valid for $1 \le k \le n-1$ and $x \in \mathbb{N}_0^n$. Applying (6.3.5) and (6.3.7), we derive from (6.3.6)

$$\frac{\Delta^{(k)}U(\underline{x}_{n-1}, x_n + 1)}{U(\underline{x}_{n-1}, x_n + 1)} - \frac{\Delta^{(n-1)}U(x)}{U(x)} \le p(x). \tag{6.3.8}$$

Keeping $\underline{x}_{n-1}$ fixed in (6.3.8), setting $x_n = y_n$ and summing over $y_n = 0, 1, 2, \ldots, x_n - 1$, we obtain

$$\frac{\Delta^{(n-1)}U(x)}{U(x)} \le g_{n-1}(\underline{x}_{n-1}, 0) + \sum_{y,n}^{x,n} p(\underline{x}_{n-1}, y_n), \tag{6.3.9}$$

where g_{n-1} is given by (6.3.4). In fact, we may rewrite (6.3.9) as

$$\frac{\Delta^{(n-2)}U(\underline{x}_{n-2}, x_{n-1} + 1, x_n)}{U(\underline{x}_{n-2}, x_{n-1} + 1, x_n)} - \frac{\Delta^{(n-2)}U(x)}{U(x)} \le g_{n-1}(\underline{x}_{n-1}, 0) + \sum_{y,n}^{x,n} p(\underline{x}_{n-1}, y_n), \tag{6.3.10}$$

since (6.3.5) and (6.3.7). Keeping now $\underline{x}_{n-2}$ and x_n fixed in (6.3.10), setting $x_{n-1} = y_{n-1}$ and summing over $y_{n-1} = 0, 1, 2, \ldots, x_{n-1} - 1$, we get

$$\frac{\Delta^{(n-2)}U(x)}{U(x)} \le g_{n-2}(\underline{x}_{n-2}, 0, x_n) + \sum_{y,n-1}^{x,n-1} g_{n-1}(\underline{x}_{n-2}, y_{n-1}, 0) + \sum_{y,n-1}^{x,n} p(\underline{x}_{n-2}, y_{n-1}), \tag{6.3.11}$$

where g_{n-2} is given by (6.3.4). If $n - 2 > 1$, then using a similar argument as used above for (6.3.9)–(6.3.11), we can obtain

$$\frac{\Delta^{(n-3)}U(x)}{U(x)} \le g_{n-3}(\underline{x}_{n-3}, 0, \tilde{x}_{n-1}) + \sum_{y,n-2}^{x,n-2} g_{n-2}(\underline{x}_{n-3}, y_{n-2}, 0, x_n)$$
$$+ \sum_{y,n-2}^{x,n-1} g_{n-1}(\underline{x}_{n-3}, \overline{y}_{n-2,n-1}, 0) + \sum_{y,n-2}^{x,n} p(\underline{x}_{n-3}, \tilde{y}_{n-2}).$$

Continuing in this way, we obtain

$$\frac{\Delta^{(1)}U(x)}{U(x)} \le G(x) + \sum_{y,2}^{x,n} p(x_1, \tilde{y}_2),$$

where $G(x)$ is defined by (6.3.3). Obviously, the above inequality can be rewritten as

$$\frac{U(x_1+1, x_2)}{U(x)} \le 1 + G(x) + \sum_{y,2}^{x,n} p(x_1, \tilde{y}_2). \tag{6.3.12}$$

Keeping $\tilde{x}_2$ fixed in (6.3.12), setting $x_1 = y_1$ and then substituting $y_1 = 0, 1, 2, \ldots, x_1 - 1$ successively in (6.3.12), we can conclude

$$\frac{U(x)}{U(0, \tilde{x}_2)} \le \prod_{y_1=0}^{x_1-1} [1 + G(y_1, \tilde{x}_2) + \sum_{y,2}^{x,n} p(y)]. \tag{6.3.13}$$

Thus the desired bound for $u(x)$ in (6.3.2) follows from (6.3.1), (6.3.5) and (6.3.13) immediately. □

Example 6.3.1 Suppose that the discrete inequality

$$v(x_1, x_2, x_3) \le a + x_2x_3 + x_1^4x_2 + \sum_{y,1}^{x,3} Q(y_1, y_2, y_3)v(y_1, y_2, y_3) \tag{6.3.14}$$

holds for all $(x_1, x_2, x_3) \in \mathbb{N}_0^3$, where $a > 0$ is a constant, v and Q are real-valued non-negative functions defined on $\mathbb{N}_0^3$. Then, by Theorem 6.3.1, we have the non-decreasing function

$$f(x_1, x_2, x_3) = a + x_2x_3 + x_1^4x_2 \quad (> 0) \in K,$$

since the following conditions hold for all $(x_1, x_2, x_3) \in \mathbb{N}_0^3$,

$$\begin{cases} \Delta^{(1)}f(x_1, x_2, x_3) = x_2(1 + 4x_1 + 6x_1^2 + 4x_1^3) \ge 0, \\ \Delta^{(2)}f(x_1, x_2, x_3) = 1 + 4x_1 + 6x_1^2 + 4x_1^3 > 0, \\ \Delta^{(3)}f(x_1, x_2, x_3) = 0. \end{cases}$$

Therefore,

$$\begin{cases} g_1(x_1,0,x_3) = \dfrac{\Delta^{(1)}f(x_1,x_2,x_3)}{f(x_1,0,x_3)} = 0, \\ g_2(x_1,x_2,0) = \dfrac{\Delta^{(2)}f(x_1,x_2,x_3)}{f(x_1,x_2,0)} = \dfrac{1+4x_1+6x_1^2+4x_1^3}{a+x_1^4x_2}. \end{cases}$$

Hence we can derive the desired bound on v from (6.3.14) such that for all $(x_1,x_2,x_3) \in \mathbb{N}_0^3$,

$$v(x_1,x_2,x_3) \le (a+x_2x_3)\prod_{y_1=0}^{x_1-1}\left\{1+\sum_{y,2}^{x,2}\frac{1+4y_1+6y_1^2+4y_1^3}{a+y_1^4y_2}+\sum_{y,2}^{x,3}Q(y_1,y_2,y_3)\right\}.$$

We note here that the above inequality (6.3.14) can not be treated by means of the known results established in [511].

Theorem 6.3.2 (Yang [660]) *Let $u(x)$, $f(x)$, $p(x)$ be the same as in Theorem 6.3.1, and let $q(x)$ be a real-valued non-negative functions defined for all $x \in \mathbb{N}_0$. Suppose that the following inequality holds for all $x \in \mathbb{N}_0^n$,*

$$u(x) \le f(x) + \sum_{y,1}^{x,n} p(y)[u(y) + \sum_{z,1}^{y,n} q(z)u(z)]. \tag{6.3.15}$$

Then for all $x \in \mathbb{N}_0^n$, we have

$$u(x) \le F(x) + \sum_{y,1}^{x,n} p(y)f(0,\tilde{y}_2)\prod_{z_1=0}^{y_1-1}[1+G(z_1,\tilde{y}_2)+\sum_{z,2}^{y,n}(p(z)+q(z))], \tag{6.3.16}$$

where G is the same as in Theorem 6.3.1, and $F(x)$ is given by

$$F(x) = f(0,\tilde{x}_2) + \sum_{y,1}^{x,1}\Delta^{(1)}f(y_1,0,\tilde{x}_3) + \sum_{y,1}^{x,2}\Delta^{(2)}f(\underline{y}_2,0,\tilde{x}_4)\ldots$$
$$+ \sum_{y,1}^{x,n-2}\Delta^{(n-2)}f(\underline{y}_{n-2},0,x_n) + \sum_{y,1}^{x,n-1}\Delta^{n-1}f(\underline{y}_1,0,0). \tag{6.3.17}$$

Proof We define a function $V(x)$ and $W(x)$ on $\mathbb{N}_0^n$ by the right-hand side of (6.3.15) and the following equality

$$W(x) = V(x) + \sum_{y,1}^{x,n} q(y)V(y) \tag{6.3.18}$$

respectively, so that by the definitions, we know that $f(x)$ is non-decreasing, and

$$W(\underline{x}_{j-1}, 0, \tilde{x}_{j+1}) = V(\underline{x}_{j-1}, 0, \tilde{x}_{j+1}) = f(\underline{x}_{j-1}, 0, \tilde{x}_{j+1}) > 0, \tag{6.3.19}$$

$$V(\underline{x}_{j-1}, x_j + 1, \tilde{x}_{j+1}) \geq V(x) \geq 0, \tag{6.3.20}$$

$$W(\underline{x}_{j-1}, x_j + 1, \tilde{x}_{j+1}) \geq W(x) > 0, \tag{6.3.21}$$

where $x_j \in \mathbb{N}_0^n$, $j = 1, 2, \ldots, n$. In addition, we obtain

$$\Delta^{(r)}V(x) = \Delta^{(r)}f(x) + \sum_{y,r+1}^{x,n} p(\underline{x}_r, \tilde{y}_{r+1})[u(\underline{x}_r, \tilde{y}_{r+1}) + \sum_{z,1}^{x,r}\sum_{z,r+1}^{y,n} q(z)u(z)], \tag{6.3.22}$$

and

$$\Delta^{(r)}W(x) = \Delta^{(r)}V(x) + \sum_{y,r+1}^{x,n} q(\underline{x}_r, \tilde{y}_{r+1})V(\underline{x}_r, \tilde{y}_{r+1}), \quad \text{for} \quad 1 \leq r \leq n, \; x \in \mathbb{N}_0^n. \tag{6.3.23}$$

Letting $r = n$ in the above (6.3.22) and using $\Delta^{(n)}f(x) \leq 0$, we then derive for all $x \in \mathbb{N}_0^n$,

$$\Delta^{(n)}V(x) \leq p(x)W(x), \tag{6.3.24}$$

since $p(x)$, $q(x)$ are non-negative and $u(x) \leq V(x) \leq W(x)$. Now by (6.3.23), we obtain for all $x \in \mathbb{N}_0^n$,

$$\Delta^{(n)}W(x) \leq p(x)W(x) + q(x)V(x) \leq [p(x) + q(x)]W(x), \tag{6.3.25}$$

since $q(x) \geq 0$ and $V(x) \leq W(x)$. Thus it follows from (6.3.19) that,

$$\frac{\Delta^{(r)}W(\underline{x}_r, 0, \tilde{x}_{r+2})}{W(\underline{x}_r, 0, \tilde{x}_{r+2})} = \frac{\Delta^{(r)}V(\underline{x}_r, 0, \tilde{x}_{r+2})}{W(\underline{x}_r, 0, \tilde{x}_{r+2})} = g_r(\underline{x}_r, 0, \underline{x}_{r+2}), \tag{6.3.26}$$

where $1 \leq r \leq n-1$, $x \in \mathbb{N}_0^n$, $j = 1, 2, \ldots, n$ and g_r is given by (6.3.4).

Now following the same argument as used in the proof of Theorem 6.3.1, and using (6.3.19), (6.3.26), we derive from (6.3.25) for all $x \in \mathbb{N}_0^n$,

$$W(x) \leq f(0, \tilde{x}_2) \prod_{y_1=0}^{x_1-1} [1 + G(y_1, \tilde{x}_2) + \sum_{y,2}^{x,n} (p(z) + q(z))]. \tag{6.3.27}$$

Substituting this bound for $W(x)$ in (6.3.26), we can rewrite it as

$$\Delta^{(n-1)} V(\underline{x}_{n-1}, x_n + 1) - \Delta^{(n-1)} V(x) \leq h(x). \tag{6.3.28}$$

where the function $h(x)$ is defined by

$$h(x) = p(x) f(0, \tilde{x}_2) \prod_{y_1=0}^{x_1-1} [1 + G(y_1, \tilde{x}_2) + \sum_{y,2}^{x,n} (p(z) + q(z))].$$

Keeping $\underline{x}_{n-1}$ fixed in the above (6.3.28), setting $x_n = y_n$ and summing over $y_n = 0,\ 1,\ 2,\ \ldots, x_n - 1$, we may get

$$\begin{aligned} \Delta^{(n-1)} V(x) &\leq \Delta^{(n-1)} V(\underline{x}_{n-1}, 0) + \sum_{y,n}^{x,n} h(\underline{x}_{n-1}, y_n) \\ &= \Delta^{(n-1)} f(\underline{x}_{n-1}, 0) + \sum_{y,n}^{x,n} h(\underline{x}_{n-1}, y_n). \end{aligned}$$

Keeping now $\underline{x}_{n-2}$, x_n fixed in the last inequality, setting $x_{n-1} = y_{n-1}$ and summing over $y_{n-1} = 0,\ 1,\ 2, \ldots, x_{n-1} - 1$, we obtain

$$\begin{aligned} \Delta^{(n-2)} V(x) \leq \Delta^{(n-2)} f(\underline{x}_{n-2}, 0, x_n) &+ \sum_{y,n-1}^{x,n-1} \Delta^{(n-1)} f(\underline{x}_{n-2}, y_{n-1}, 0) \\ &+ \sum_{y,n-1}^{x,n} h(\underline{x}_{n-2}, \tilde{y}_{n-1}). \end{aligned}$$

Continuing in this way, we finally conclude

$$\begin{aligned} \Delta^{(1)} V(x) &\equiv V(x_1 + 1, \tilde{x}_2) - V(x) \\ &\leq \Delta^{(1)} f(x_1, 0, \tilde{x}_3) + \sum_{y,2}^{x,2} \Delta^2 f(x_1, y_2, 0, \tilde{x}_4) + \ldots \\ &\quad + \sum_{y,2}^{x,n-1} \Delta^{(n-1)} f(x_1, \overline{y}_{2,n-1}, 0) + \sum_{y,2}^{x,n} h(x_1, \tilde{y}_2). \end{aligned} \tag{6.3.29}$$

Keeping $\tilde{x}_2$ fixed in (6.3.29), setting $x_1 = y_1$ and substituting $y_1 = 0, 1, 2, \ldots, x_1 - 1$ successively in (6.3.27) to derive the bound for $V(x)$ such that

$$V(x) \le F(x) + \sum_{y,2}^{x,n} h(x), \tag{6.3.30}$$

since $V(0, \tilde{x}_2) = f(0, \tilde{x}_2)$, where $F(x)$ is defined in (6.3.18). Hence the desired bound (6.3.16) follows from (6.3.15), (6.3.30), and the definition of $V(x)$ and $h(x)$ immediately. □

Remark 6.3.1 If $n = 3$ and $f(x) = a_1(x_1) + a_2(x_2) + a_3(x_3)$ in Theorems 6.3.1 and 6.3.2, where $a_j : \mathbb{N}_0 \to (0, +\infty)$, $\Delta a_j(z) \ge 0$ for all $z \in \mathbb{N}_0$, $j = 1, 2, 3$, then we can derive Theorems 6.3.1 and 6.3.2 of [511] respectively.

The discrete inequalities play an important role not only in the field of finite difference equations and numerical analysis, but also in certain areas of engineering, technology, economics and biological sciences. One of the most used result in this direction is the discrete analogue of the celebrated Gronwall-Bellman-Reid inequality [595, 611] and its variants [2, 14, 17, 271, 299, 305, 449, 465, 471, 611, 648]. The two and more independent variable generalizations of this inequality were established in [511, 595, 597].

In the next Theorems 6.3.3–6.3.6, we shall discuss some discrete inequalities in n independent variables which are further generalizations of some results obtained in [17] for $n = 1$. Some unified results are also presented which covers several results of Pachpatte and Singare [511, 595, 597] (see, Theorem 6.2.1, etc.).

A point $(x_1^i, \ldots, x_n^i)$ in $\mathbb{N}_0^n$ is denoted by x^i. The first difference with respect to the variable x_i of the function on $u(x_1, \ldots, x_n)$ is defined as $\Delta u_{x_i}(x_1, \ldots, x_n) = u(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_n) - u(x_1, \ldots, x_n)$. The second difference with respect to the variables x_i, x_j is defined as $\Delta^2 u_{x_i x_j}(x_1, \ldots, x_n) = \Delta u_{x_i}(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_n) - \Delta u_{x_i}(x_1, \ldots, x_n) = u(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) - u(x_1, \ldots, x_{i-1}, x_i + 1, x_{i+1}, \ldots, x_n) - u(x_1, \ldots, x_{j-1}, x_j + 1, x_{j+1}, \ldots, x_n) + u(x_1, \ldots, x_n)$. The higher order differences are defined analogously. The functions which appears in the inequalities are assumed to be real-valued, non-negative and defined in $\mathbb{N}_0^n$.

Theorem 6.3.3 (Agarwal-Thandapani [19]) *Let the following inequality hold for all $x \in \mathbb{N}_0^n$,*

$$u(x) \le \sum_{i=1}^{n} a_i(x_i) + \sum_{r=1}^{m} E^r(x, u) \tag{6.3.31}$$

where

$$E^r(x,u) = \sum_{x^1=0}^{x-1} f_{r1}(x^1) \sum_{x^2=0}^{x^1-1} f_{r2}(x^2) \ldots \sum_{x^r=0}^{x^{r-1}-1} f_{rr}(x^r)u(x^r)$$

where $a_i(x_i) > 0,\ \Delta a_i(x_i) \geq 0$. *Then for all* $x \in \mathbb{N}_0^n$,

$$u(x) \leq [a_1(0) + \sum_{i=2}^{n} a_i(x_i)] \prod_{s_1=0}^{x_1-1} [1 + \frac{\Delta a_1(s_1)}{a_1(s_1) + a_2(0) + \sum_{i=3}^{n} a_i(x_i)} + \sum_{r=1}^{m} \Delta E_{s_1}^r(s_1, x_2, \ldots, x_n, 1)]. \tag{6.3.32}$$

Proof Let $\phi(x)$ be the right-hand side of inequality (6.3.31). Then

$$\Delta\phi_{x_1}(x) = \Delta a_1 x_1 + \sum_{r=1}^{m} \Delta E_{x_1}^r(x,u)$$

and

$$\Delta^n \phi_x(x) = \sum_{r=1}^{m} \Delta^n E_x^r(x,u) \tag{6.3.33}$$

where

$$\phi(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = a_i(0) + \sum_{j=1, j\neq i}^{n} a_j(x_j). \tag{6.3.34}$$

Since $u(x) \leq \phi(x)$ and $\phi(x)$ is non-decreasing in x, from (6.3.33) it follows

$$\Delta^n \phi_x(x) \leq \sum_{r=1}^{m} \Delta^n E_x^r(x,\phi) \leq \sum_{r=1}^{m} \Delta^n E_x^r(x,1)\phi(x). \tag{6.3.35}$$

On the one hand, by (6.3.35) and using the fact $\phi(x_1, \ldots, x_{n-1}, x_n + 1) \geq \phi(x)$, we obtain

$$\frac{\Delta^{n-1}\phi_{x_1 \ldots x_{n-1}}(x_1, \ldots, x_{n-1}, x_n + 1)}{\phi(x_1, \ldots, x_{n-1}, x_n + 1)} - \frac{\Delta^{n-1}\phi_{x_1 \ldots x_{n-1}}(x)}{\phi(x)} \leq \sum_{r=1}^{m} \Delta^n E_x^r(x,1).$$

Now keeping $x_1, \dots, x_{n-1}$ fixed and setting $x_n = s_n$ and summing over $s_n = 0, 1, \dots, x_n - 1$ in the above inequality, we find

$$\frac{\Delta^{n-1}\phi_{x_1 \dots x_{n-1}}(x)}{\phi(x)} \leq \sum_{s_n=0}^{x_n-1} \sum_{r=1}^{m} \Delta^n E^n_{x_1 \dots x_{n-1} s_n}(x_1, \dots, x_{n-1}, s_n, 1) = \sum_{r=1}^{m} \Delta^{n-1} E^r_{x_1 \dots x_{n-1}}(x, 1).$$

On the other hand, repeating the above arguments successively, we obtain

$$\frac{\Delta\phi_{x_i}(x)}{\phi(x)} \leq \frac{\Delta a_1(x_1)}{a_1(s_1) + a_2(0) + \sum_{i=3}^{n} a_i(x_i)} + \sum_{r=1}^{m} \Delta E^r_{x_1}(x, 1). \tag{6.3.36}$$

From (6.3.36), we derive

$$\phi(x_1 + 1, x_2, \dots, x_n) \leq \left[1 + \frac{\Delta a_1(x_1)}{a_1(s_1) + a_2(0) + \sum_{i=3}^{n} a_i(x_i)} + \sum_{r=1}^{m} \Delta E^r_{x_1}(x, 1)\right] \phi(x).$$

Now keeping $x_2, \dots, x_n$ fixed and setting $x_1 = s_1$ and summing over $s_1 = 0, 1, \dots, x_1 - 1$ in the above inequality, we thus deduce from (6.3.34) that

$$\phi(x) \leq [a_1(0) + \sum_{i=2}^{n} a_i(x_i)] \prod_{s_1=0}^{x_1-1} [1 + \frac{\Delta a_1(x_1)}{a_1(s_1) + a_2(0) + \sum_{i=3}^{n} a_i(x_i)} + \sum_{r=1}^{m} \Delta E^r_{x_1}(x, 1)].$$

Thus (6.3.32) now follows from $u(x) \leq \phi(x)$. □

Remark 6.3.2 There are $n!$ different conclusions possible for Theorem 6.3.3, corresponding to n permutations of $(x_1, \dots, x_n)$ and corresponding permutations of $a_1, \dots, a_n$.

Remark 6.3.3 For $n = 3$, $m = 1$, the estimate (6.3.32) is same as that in Theorem 1 of [511]. For $n = 3$, $m = 2$, $f_{11} = f_{21}$, the estimate (6.3.32) is not comparable to that obtained in Theorem 2 of [511]. For $n = 2$ and m up to 2, some results are given in [571].

Theorem 6.3.4 (Agarwal-Thandapani [19]) *Let the following inequality hold for all $x \in \mathbb{N}_0^n$,*

$$u(x) \leq a(x) + b(x) \sum_{r=1}^{m} E^r(x, u) \tag{6.3.37}$$

where: (i) $a(x) > 0$ and non-decreasing, (ii) $b(x) \geq 1$. Then for all $x \in \mathbb{N}_0^n$,

$$u(x) \leq a(x)b(x) \prod_{s_1=0}^{x_1-1} [1 + \sum_{r=1}^{m} \Delta E^r_{s_1}(s_1, x_2, \dots, x_n, b)]. \tag{6.3.38}$$

Proof From the assumptions on a and b, inequality (6.3.37) can be written as

$$v(x) \leq 1 + \sum_{r=1}^{m} E^r(x, bv)$$

where $v = u/ab$. The rest of the proof is the same as that of Theorem 6.3.3. □

Remark 6.3.4 For inequality (6.3.31) under the assumptions of Theorem 6.3.3, we have from Theorem 6.3.4

$$u(x) \leq \sum_{i=1}^{n} a_i(x_i) \prod_{s_1=0}^{x_1-1} [1 + \sum_{r=1}^{m} \Delta E^r_{s_1}(s_1, x_2, \ldots, x_n, 1)]. \tag{6.3.39}$$

Remark 6.3.5 There are $n!$ different conclusions possible for Theorem 6.3.4 and *(6.3.39)*.

Remark 6.3.6 If $a_1 = k$ (constant), then (6.3.32) and (6.3.39) are same. In the general case, (6.3.32) and (6.3.39) are not comparable. In applications, (6.3.39) require less work to compute the estimate than (6.3.32).

Theorem 6.3.5 (Agarwal-Thandapani [19]) *Let the inequality (6.3.31) hold where $a_i(x_i)$ is the same as in Theorem 6.3.3 and*

$$f_{ii}(x) = f_i(x), \quad 1 \leq i \leq m, \quad f_{i+1,i}(x) = f_{i+2,i}(x) = \ldots = f_{m,1}(x) = g_i(x), \tag{6.3.40}$$

for all $x \in \mathbb{N}_0^n$ and $1 \leq i \leq m-1$. Then for all $x \in \mathbb{N}_0^n$,

$$u(x) \leq P_i(x), \quad i = 1, 2, \tag{6.3.41}$$

where

$$\begin{cases} P_1(x) = [a_1(0) + \sum_{i=2}^{n} a_i(x_i)] \prod_{s_1=0}^{x_1-1} [1 + \dfrac{\Delta a_1(s_1)}{a_1(s_1) + a_2(0) + \sum_{i=3}^{n} a_i(x_i)} \\ \qquad + \sum_{s_2=0}^{n} \cdots \sum_{s_n=0}^{x_n-1} (\sum_{r=1}^{m} f_r(s) \bigcup_{i=1}^{m-1} g_i(s))], & (6.3.42) \\ P_2(x) = \sum_{i=1}^{n} a_i(x_i) + \sum_{s=0}^{x-1} (f_1(s) \bigcup g_i(s)) P_1(s). & (6.3.43) \end{cases}$$

In (6.3.42) and (6.3.43), the term $\sum_{r=1}^{r_1} f_r(x) \bigcup_{i=1}^{r_2} g_i(x)$ represents the sum of all functions except when $f_k(x) = g_l(x)$ for some $1 \leq k \leq r_1$, $1 \leq l \leq r_2$, then $g_l(x)$ is taken to be zero, also $\bigcup_{i=1}^{0} = 0$.

Proof Indeed, inequality (6.3.31) with (6.3.40) is equivalent to the following system

$$\begin{cases} u_1(x) \le \sum_{i=1}^{n} a_i(x_i) + \sum_{s=0}^{x-1}[f_1(s)u_1(s) + g_1(s)u_2(s)], \\ u_{j-1}(x) = \sum_{s=0}^{x-1}[f_{j-1}(s)u_1(s) + g_{j-1}(s)u_j(s)] \quad 3 \le j \le m, \\ u_m(x) = \sum_{s=0}^{x-1} f_m(s)u_1(s). \end{cases} \tag{6.3.44}$$

Define $\phi_1(x)$, $\phi_{j-1}(x)$ $(3 \le j \le m)$, $\phi_m(x)$ as the right-hand side of (6.3.44), respectively. Then we find

$$\begin{cases} \Delta^n \phi_{1x}(x) \le f_1(x)\phi_1(x) + g_1(x)\phi_2(x)], \\ \Delta^n \phi_{j-1x}(x) = f_{j-1}(x)\phi_1(x) + g_{j-1}(x)\phi_j(x), \quad 3 \le j \le m, \\ \Delta^n \phi_{mx}(x) \le f_m(x)\phi_1(x), \end{cases} \tag{6.3.45}$$

where $\phi_1(x)$ satisfies (6.3.34) and $\phi_j(x)$ $(2 \le j \le m)$, together with all mixed differences up to order $n-1$ are zero at $x_i = 0$ $(1 \le i \le n)$. Adding (6.3.45), we obtain

$$\sum_{r=1}^{m} \Delta^n \phi_{rx}(x) \le \sum_{r=1}^{m} f_r(x)\phi_1(x) + \sum_{r=1}^{m-1} g_r(x)\phi_{r+1}(x)$$

whence

$$\sum_{r=1}^{m} \Delta^n \phi_{rx}(x) \le (\sum_{r=1}^{m} f_r(x) \bigcup_{i=1}^{-1} g_i(x))(\sum_{r=1}^{m} \phi_r(x)).$$

Now following the proof of Theorem 6.3.3, we obtain $\sum_{r=1}^{m} \phi_r(x) \le P_1(x)$. Using this in (6.3.45), we get

$$\Delta^n \phi(x) \le (f_i(x) \bigcup g_1(x))P_1(x)$$

and similarly Theorem 6.3.3, we can get $\phi_1(x) \le P_2(x)$. Since $u(x) = u_1(x) \le \phi_1(x) \le \sum_{r=1}^{m} \phi_r(x)$, (6.3.41) follows. $\square$

Remark 6.3.7 As in Theorem 6.3.3, there are $n!$ different conclusions possible for Theorem 6.3.5.

Remark 6.3.8 For $n = 1$, $m = 2$, Theorem 6.3.5 reduces to Theorem 1 of [511]. For $n = 3$, $m = 2$, $f_1 = g_1$, Theorem 6.3.5 reduce to Theorem 2 of [511]. This also covers some results given in [571] for $n = 2, m$ up to 2.

Remark 6.3.9 $P_1(x)$ and $P_2(x)$ cannot be compared.

The next result is a discrete analogue of Willett's inequality [634] as discussed in [17] for $n = 1$.

Theorem 6.3.6 (Agarwal-Thandapani [19]) *Let the following inequality hold for all* $x \in \mathbb{N}_0^n$,

$$u(x) \le a(x) + \sum_{i=1}^{m} g_i(x) \sum_{s=0}^{x-1} h_i(s)u(s) \tag{6.3.46}$$

where: (i) $a(x) > 0$ *and non-decreasing, (ii)* $g_i(x) > 1$ *for* $1 \le i \le m$ *and non-decreasing for* $2 \le i \le m$. *Then for all* $x \in \mathbb{N}_0^n$,

$$u(x) \le F_m a(x) \tag{6.3.47}$$

where

$$F_0 w = w, \quad F_k w = w(F_{k-1}g_k) \prod_{s_1=0}^{x_1-1} [1 + \sum_{s_2=0}^{x_2-1} \dots \sum_{s_n=0}^{x_n-1} h_k(s) F_{k-1} g_k(s)] \tag{6.3.48}$$

for $k = 1, \cdots, m$.

Proof The proof is by finite induction. For $m = 1$, we derive from Theorem 6.3.4 that

$$u(x) \le a(x)g_1(x) \prod_{s_1=0}^{x_1-1} [1 + \sum_{s_2=0}^{x_2-1} \dots \sum_{s_n=0}^{x_n-1} h_1(s)g_1(s)] = F_1 a(x).$$

Now, assume that the result is true for some k such that $1 \le k \le m - 1$, then for $k + 1$, we have

$$u(x) \le a(x) + g_{k+1}(x) \sum_{k+1} (s)u(s) + \sum_{i=1}^{k} g_i(x) \sum_{s=0}^{x-1} h_i(s)u(s)$$

and

$$u(x) \le F_k a^*(x)$$

where

$$a^*(x) = a(x) + g_{k+1}(x) \sum_{s=0}^{x-1} h_{k+1}(s)u(s).$$

Thus we find

$$\frac{u}{aF_k g_{k+1}} \leq 1 + \sum_{s=0}^{x-1} h_{k+1} F_k g_{k+1} \frac{u}{aF_k g_{k+1}}.$$

Now applying Theorem 6.3.4 yields

$$u(x) \leq a(x)F_k g_{k+1}(x) \prod_{s_1=0}^{x_1-1} \left[1 + \sum_{s_2=0}^{x_2-1} \cdots \sum_{s_n=0}^{x_n-1} h_{k+1}(s)F_k g_{k+1}(s) \right] = F_{k+1}a(x).$$

Hence (6.3.47) follows for all m.

Corollary 6.3.1 (Agarwal-Thandapani [19]) *Let the inequality (6.3.46) hold for all $x \in \mathbb{N}_0^n$, where: (i) $a(x) > 0$ and non-decreasing, (ii) $g_i(x) \geq 1$ for all $1 \leq i \leq m$. Then for all $x \in \mathbb{N}_0^n$,*

$$u(x) \leq a(x) \prod_{i=1}^{m} g_i(x) \prod_{s_1=0}^{x_1-1} \left[1 + \sum_{s_2=0}^{x_1-1} \cdots \sum_{s_n=0}^{x_n-1} \sum_{r=1}^{m} h_r(s) \prod_{i=1}^{m} g_i(s) \right].$$

Proof In fact, inequality (6.3.46) can be rewritten as

$$u(x) \leq a(x) + \prod_{i=1}^{m} g_i(x) \sum_{s=0}^{x-1} (\sum_{i=1}^{m} h_i(s))u(s).$$

Thus, the desired result follows from Theorem 6.3.4 immediately. □

We shall establish an essentially new generalization of linear Gronwall discrete inequalities in several independent variables, which are due to [534]. We shall use the following notations: $\mathbb{R}, \mathbb{R}_0$, and $\mathbb{R}_+$ the set of real, non-negative real, and positive real numbers, respectively. $\mathbb{N}_\xi = \{\xi, \xi + 1, \cdots\}$ where ξ is a non-negative integer. Let $\nu = (\nu_1, \cdots, \nu_m)$, then $\mathbb{N}_\nu = \mathbb{N}_{\nu_1} \times \cdots \times \mathbb{N}_{\nu_m}$ (the Cartesian product). Let $\alpha = (\alpha_1, \cdots, \alpha_m)$ where $\alpha_i \in \mathbb{N}_0$, then $|\alpha| = \sum_{i=1}^{m} \alpha_i$ we shall call α a multi-index or m-index.

We shall need the following operators which can be defined both for sequences of integers as well as multi-indices. For $n = (n_1, \cdots, n_m) \in \mathbb{N}_\nu$, we define $E_{/i}^{j} n = (n_1, \cdots, n_{i-1}, n_i + j, n_{i+1}, \cdots, n_m)$, shift operators (acting here on arguments). In particular, $E_{/i} = (n_1, \cdots, n_{i-1}, n_i + 1, n_{i+1}, \cdots, n_m)$,

$P(n; n_\mu = \nu) = (n_1, \cdots \nu_{\mu_1}, n_{\mu_1+1}, \cdots, \nu_{\mu_k}, \cdots, n_m)$, projection operators, in short $P_{\mu/\nu}n$, where $\mu = \{\mu_1, \cdots, \mu_k\}$ is any subsequence of the sequence $\{1, \ldots, m\}$. For example, if $n = (n_1, n_2, n_3, n_4), \mu = (2, 4), \nu = (1, 7, 5, 3)$, then $P(n; n_{(2,4)} = \nu) = (n_1, 7, n_3, 3)$. In particular, $P(n; n_i = a) = P_{i/a}n = (n_1, \cdots, n_{i-1}, a, n_{i+l}, \cdots, n_m)$. $R(\alpha, k) = (\alpha_1, \cdots, \alpha_{m-k})$ for $0 \le k \le m$ reduction operator. It is clear that $P_{\mu/\nu}n$ can be presented as the compositions $P(n; n_\mu = \nu) = P_{\mu_1/\nu_1} \cdots P_{\mu_k/\nu_k}n$. The difference operators on any function $w : \mathbb{N}_\nu \to \mathbb{R}$ are defined as follows:

$$\Delta_\alpha^{|\alpha|}\omega(n) = \Delta_{/1}^{\alpha_1}\left(\Delta_{/2}^{\alpha_2}(\cdots(\Delta_{/m}^{\alpha_m}\omega(n)))\right)$$

where, for all $k \ge 1$,

$$\Delta_{/i}^k\omega(n) = \sum_{j=0}^k C_k^j(-1)^{k-j}\omega(n_1, \cdots, n_i + j, n_{i+1}, \cdots, n_m),$$

and on using shift operators

$$\Delta_{/i}^k\omega(n) = \sum_{j=0}^k C_k^j(-1)^{k-j}\omega(E_{/i}^j n).$$

It is supposed that $\Delta_{/i}^0\omega(n) = \omega(n)$ so that if in the multi-index α, some of $\alpha_i = 0$, then in the definition of $\Delta_\alpha^{|\alpha|}\omega(n)$ suitable partial differences $\Delta_{/i}^{\alpha_i}$ should be omitted. For a sequence $\sigma = (\sigma_1, \cdots, \sigma_j)$, not necessarily of different elements $\sigma_i \in \{1, \cdots, m\}$, we shall use

$$\Delta_{/\sigma}^j\omega(n) = \Delta_{/\sigma_1}\left(\Delta_{/\sigma_2}(\cdots(\Delta_{/\sigma_j}\omega(n)))\right).$$

Let us note the difference between Δ_β^* and $\Delta_{/\beta}^*$. For this, let $\beta = (1, 2, 1)$, then according to the definitions, $\Delta_\beta^{|\beta|}\omega(n) = \Delta_{/1}^1(\Delta_{/2}^2(\Delta_{/3}^3\omega(n)))$, while $\Delta_{/\beta}^3\omega(n) = \Delta_{/1}^1(\Delta_{/2}^2(\Delta_{/3}^3\omega(n)))$ (here all the differences are of the first order, and β_i denotes the order of the difference with respect to the i-th variable to which variable the difference has to be applied). For the multiple summation operators, we denote

$$S_\alpha(n, \nu; \omega) = \sum_{j_{1,1}=\nu_1}^{n_1-1} \cdots \sum_{j_{1,\alpha_1}=\nu_1}^{j_{1,\alpha_1-1}-1} \cdots \sum_{j_{m,1}=\nu_m}^{n_m-1} \cdots \sum_{j_{m,\alpha_m}=\nu_m}^{j_{m,\alpha_{m-1}}-1} \omega(j_{1,\alpha_1}, \cdots j_{m,\alpha_m}).$$

It is obvious that suitable summations have to be omitted if some of $\alpha_i = 0$. In particular, if $\alpha = (0, \cdots, 0, \alpha_i, 0, \cdots, 0)$, then

$$S_\alpha(n, \nu; \omega) = S_{\alpha_i}(n, \nu; \omega) = \sum_{j_{i,1}=\nu_i}^{n_i-1} \cdots \sum_{j_{i,\alpha_i}=\nu_i}^{j_{i,\alpha_{i-1}}-1} \omega(n_1, \cdots, n_{i-1}, j_{i,\alpha_i}, n_{i+1}, \cdots, n_m),$$

while

$$\begin{aligned} S_\alpha(n, \nu; \omega(P_{i/\nu_i}n)) &= \sum_{j_{i,1}=\nu_i}^{n_i-1} \cdots \sum_{j_{i,\alpha_i}=\nu_i}^{j_{i,\alpha_i-1}-1} \omega(n_1, \cdots, n_{i-1}, \nu_i, n_{i-1}, \cdots, n_m) \\ &= \omega(n_1, \cdots, n_{i-1}, \nu_i, n_{i-1}, \cdots, n_m) \sum_{j_{i,1}=\nu_i}^{n_i-1} \cdots \sum_{j_{i,\alpha_i}=\nu_i}^{j_{i,\alpha_i-1}-1} \mathbf{1} \\ &= \omega(n_1, \cdots, n_{i-1}, \nu_i, n_{i-1}, \cdots, n_m) C_{n-\nu_i}^{\alpha_i}. \end{aligned}$$

We shall follow the standard convention that the empty sums are zero. Therefore, if for some i we have $n_i < \nu_i + \alpha_i$, then $S_\alpha(n, \nu; \omega) = 0$.

If $\alpha = (\alpha_1, \cdots, \alpha_m)$ and $\beta = (\beta_1, \cdots, \beta_m)$ are two multi-indices, then $\alpha - \beta = (\alpha_1 - \beta_1, \cdots, \alpha_m - \beta_m)$ and if $\alpha_i \geq \beta_i$, for all $i \in \{1, \cdots, m\}$,

$$\Delta_\beta^{|\beta|} S_\alpha(n, \nu; \omega) = S_{\alpha-\beta}(n, \nu; \omega).$$

Moreover,

$$\Delta_{/i}^{\alpha_i} S_{\alpha_i}(n, \nu; \omega) = \omega(n), \quad \Delta_\alpha^{|\alpha|} S_\alpha(n, \nu; \omega) = \omega(n).$$

If $\omega : \mathbb{N}_\nu \mapsto \mathbb{R}_+$, then from the above $\Delta_\beta^{|\beta|} S_\alpha(n, \nu; \omega) \geq 0$ and $\Delta_{/i}^{j} S_\alpha(n, \nu; \omega) \geq 0$ for $j \leq \alpha_i$. On the other hand,

$$\Delta_{/i}^{k} S_\alpha(n, \nu; \omega) = \sum_{j=0}^{k} C_k^j (-1)^{k-j} S_\alpha(E_{/i}^j n, \nu; \omega) = 0 \ \ if \ \ n_i + k < \nu_i + \alpha_i.$$

If $\beta_i > \alpha_i$, then

$$\Delta_{/i}^{\beta_i} S_{\alpha_i}(n, \nu; \omega) = S_{\alpha_i - \beta_i}(n, \nu; \omega) = \Delta_{/i}^{\beta_i - \alpha_i} \omega(n).$$

It is clear that for some values of n, we have $S_\alpha(n,\nu;\omega) = 0$, while $\Delta_\beta^{|\beta|} S_\alpha(n,\nu;\omega) \neq 0$. For example, let $\omega : \mathbb{N}_{(\nu_1,\nu_2)} \to \mathbb{R}$ and $\alpha = (3,2)$, then

$$S_\alpha(n,\nu;\omega) = \sum_{j_{1,1}=\nu_1}^{n_1-1} \sum_{j_{1,2}=\nu_1}^{j_{1,1}-1} \sum_{j_{1,3}=\nu_1}^{j_{1,2}-1} \sum_{j_{2,1}=\nu_2}^{n_2-1} \sum_{j_{2,2}=\nu_2}^{j_{2,1}-1} \omega(j_{1,3}, j_{2,2}),$$

$$\Delta_{/1} S_\alpha(n,(\nu_1,\nu_2);\omega) = \sum_{j_{1,2}=\nu_1}^{n_1-1} \sum_{j_{1,3}=\nu_1}^{j_{1,2}-1} \sum_{j_{2,1}=\nu_2}^{n_2-1} \sum_{j_{2,2}=\nu_2}^{j_{2,1}-1} \omega(j_{1,3}, j_{2,2}),$$

$$\begin{aligned}\Delta^2_{/(1,1)} S_\alpha(n,(\nu_1,\nu_2);\omega) &\equiv \Delta^2_{/1} S_\alpha(n,(\nu_1,\nu_2);\omega) \equiv \Delta^2_{(2,0)} S_\alpha(n,(\nu_1,\nu-2);\omega) \\ &= \sum_{j_{1,3}=\nu_1}^{n_1-1} \sum_{j_{2,1}=\nu_2}^{n_2-1} \sum_{j_{2,2}=\nu_2}^{j_{2,1}-1} \omega(j_{1,3}, j_{2,2})\end{aligned}$$

and $S_\alpha((\nu_1+2,\nu_2+2),(\nu_1,\nu_2);\omega) = 0$, in fact, $S_\alpha((\nu_1+i,\nu_2+j),(\nu_1,\nu_2);\omega) = 0$, if $i < 3$ or $j < 2$, while $\Delta_{/1} S_\alpha((\nu_1+2,\nu_2+2),(\nu_1,\nu_2);\omega) = \omega(\nu_1,\nu_2)$.

For a given multi-index $\alpha = (\alpha_1,\cdots,\alpha_m)$, we can construct the set Ξ_α of $|\alpha|!\prod_{i=1}^m 1/(\alpha_i!)$ sequences $\Xi_\alpha = \{\sigma : (\sigma_1,\cdots,\sigma_{|\alpha|})\}$ such that $\sigma_j = i$ for some $i \in \{1,\cdots,m\}$ and all $j \in \{1,\cdots,|\alpha|\}$ and card $\{\sigma_j : \sigma_j = \alpha\} = \alpha_i$. Here by card $\{A\}$, we shall denote the cardinal of the set A. For example, let $\alpha = (3,2)$, then $\Xi_\alpha = \{(1,1,1,2,2),(1,2,1,1,2),(2,1,1,2,1),\cdots\}$, and card $\Xi_\alpha = 5!/(2!3!)$.

We shall say the function f belongs to the class $M(\beta)$ if $f : \mathbb{N}_\nu \to \mathbb{R}_+$ and

(i) $\Delta^s_{/(\beta_1,\cdots,\beta_s)} f(n) \geq 0$ *for* $s = 1, \cdots, r-1$ *and all* $n \in \mathbb{N}_\nu$, and ,
(ii) $\Delta^1_{/\beta_j} f(n) \geq 0$ *for* $j = 1,\cdots,r$ *and all* $n \in \mathbb{N}_\nu$, where $\beta = (\beta_1,\cdots,\beta_r)$ and $\beta_i \in \{1,\cdots,m\}$ for all $i \in \{1,\cdots,r\}$.

It is clear that if $f_1, f_2 \in M(\beta)$ and $a > 0$, then $f_1 + f_2 \in M(\beta)$ and $af_1 \in M(\beta)$. Moreover, if $f \in M(\beta)$ is such that $\Delta^r_{/(\beta_1,\cdots,\beta_r)} f(n) = 0$ for all $n \in \mathbb{N}_\nu$, then $f \in M(\gamma)$ for any $\gamma = (\gamma_1,\cdots,\gamma_r,\cdots,\gamma_k)$ such that $\gamma_i = \beta_i$ for all $i \in \{1,\cdots,r\}$. For example, let $\beta = (3,1,2,1)$, then $f \in M(\beta)$ if from condition (i),

$$\begin{cases} \Delta^3_{/(3,1,2)} f(n) \equiv \Delta_{/3}(\Delta_{/1}(\Delta_{/2} f(n))) \geq 0, \\ \Delta^2_{/(3,1)} f(n) \equiv \Delta_{/3}(\Delta_{/1} f(n)) \geq 0, \Delta^1_{/3} f(n) \geq 0, \end{cases}$$

and by condition (ii),

$$\Delta^1_{/1} f(n) \equiv \Delta_{/1} f(n) \equiv \Delta^1_{/1} f(n) \geq 0, \quad \Delta^1_{/2} f(n) \geq 0, \quad \Delta^1_{/3} f(n) \geq 0.$$

Theorem 6.3.7 (Popenda-Agarwal [534]) *Let $u, b, c : \mathbb{N}_\nu \to \mathbb{R}_+$ and there exists a sequence $\sigma \in \Xi_\alpha$ such that $c \in M(\sigma)$. Then for every solution u of the inequality*

$$u(n) \leq c(n) + S_\alpha(n,\nu;bu), \quad \forall n \in \mathbb{N}_\nu, \tag{6.3.49}$$

the following holds for all $n \in \mathbb{N}_\nu$,

$$u(n) \le min_{\sigma \in \Xi_\sigma : c \in M(\sigma)} c(P(n; n_{\sigma_1} = \nu_{\sigma_1})) \prod_{j_{|\alpha|}=\nu_{\sigma_1}}^{n_{\sigma_1}-1} \left\{1 + \Phi_{|\alpha|}(P(n; n_{\sigma_1} = j_{|\alpha|}))\right\}, \tag{6.3.50}$$

where

$$\Phi_1(n) = \frac{\max\{0, \Delta_\alpha^{|\alpha|} c(n)\}}{c(n)} + b(n)$$

and

$$\Phi_{k+1}(n) = \frac{\Delta_{/R(\sigma,k)}^{|\alpha|-k} c(P(n; n_{\sigma_{|\alpha|-k+1}} = \nu_{\sigma_{|\alpha|-k+1}}))}{c(P(n; n_{\sigma_{|\alpha|-k+1}} = \nu_{\sigma_{|\alpha|-k+1}}))} + \sum_{j_k=\nu_{\sigma_{|\alpha|-k+1}}}^{n_{\sigma_{|\alpha|-k+1}}} \Phi_k(P(n; n_{\sigma_{|\alpha|-k+1}} = j_k)), \quad k = 1, \cdots, |\alpha| - 1.$$

Proof Let $\sigma \in \Xi_\alpha$ be such that $c \in M(\sigma)$ and let

$$z(n) = c(n) + S_\alpha(n, \nu; bu), \quad n \in \mathbb{N}_\nu. \tag{6.3.51}$$

Then inequality (6.3.49) reduces to

$$u(n) \le z(n). \tag{6.3.52}$$

Therefore, for all $n \in \mathbb{N}_\nu$, from (6.3.51) it follows

$$\Delta_{/\sigma}^{|\alpha|} z(n) = \Delta_{/\sigma}^{|\alpha|} c(n) + b(n)u(n) \le \Delta_{/\sigma}^{|\alpha|} c(n) + b(n)z(n) \le \max(0, \Delta_{/\sigma}^{|\alpha|} c(n)) + b(n)z(n).$$

Since $c \in M(\sigma)$ and $z(n) \ge c(n) > 0$, we get

$$\frac{\Delta_{/R(\sigma,0)}^{|\alpha|} z(n)}{z(n)} = \frac{\Delta_{/\sigma}^{|\alpha|} z(n)}{z(n)} \le \frac{\max(0, \Delta_{/\sigma}^{|\alpha|} c(n))}{c(n)} + b(n) = \Phi_1(n).$$

Hence,

$$\frac{\Delta_{/R(\sigma,1)}^{|\alpha|-1} z(E_{/\sigma_{|\alpha|}} n) - \Delta_{/R(\sigma,1)}^{|\alpha|-1} z(n)}{z(n)} \le \Phi_1(n). \tag{6.3.53}$$

Notice that

$$\Delta_{/\sigma_j} z(n) = \Delta_{/\sigma_j} c(n) + \Delta_{/\sigma_j} S_\alpha(n, \nu; bu), \quad j = 1, \cdots, |\alpha|,$$

and by condition (ii) of the definition $M(\sigma)$,

$$\Delta_{/\sigma_j} S_\alpha(n, \nu; bu) \geq 0, \quad \Delta_{/\sigma_j} c(n) \geq 0.$$

Thus it follows that for all $j = 1, \cdots, |\alpha|$ and $n \in \mathbb{N}_\nu$,

$$z(E_{/\sigma_j} n) \geq z(n).$$

Moreover, by condition (i),

$$\Delta^{|\alpha|-1}_{/R(\sigma,1)} z(n) = \Delta^{|\alpha|-1}_{/R(\sigma,1)} c(n) + \Delta^{|\alpha|-1}_{/R(\sigma,1)} S_\alpha(n, \nu; bu) \geq 0.$$

Hence, from (6.3.53) we may derive

$$\frac{\Delta^{|\alpha|-1}_{/R(\sigma,1)} z(E_{/\sigma_{|\alpha|}} n)}{z(E_{/\sigma_{|\alpha|}} n)} - \frac{\Delta^{|\alpha|-1}_{/R(\sigma,1)} z(n)}{z(n)} \leq \Phi_1(n). \tag{6.3.54}$$

Now substituting in (6.3.54), $n = P(n; n_{\sigma_{|\alpha|}} = j_1)$ and summing with respect to j_1 from $\nu_{\sigma_{|\alpha|}}$ to $n_{\sigma_{|\alpha|}} - 1$, we can get

$$\frac{\Delta^{|\alpha|-1}_{/R(\sigma,1)} z(n)}{z(n)} - \frac{\Delta^{|\alpha|-1}_{/R(\sigma,1)} z(P(n; n_{\sigma_{|\alpha|}} = \nu_{\sigma_{|\alpha|}}))}{z(P(n; n_{\sigma_{|\alpha|}} = \nu_{\sigma_{|\alpha|}}))} \leq \sum_{j_1=\nu_{\sigma_{|\alpha|}}}^{n_{\sigma_{|\alpha|}}-1} \Phi_1(P(n; n_{\sigma_{|\alpha|}} = j_1)). \tag{6.3.55}$$

Let $\sigma_{|\alpha|} = \xi \in \{1, \cdots, m\}$, then card $\{\sigma_i : \sigma_i = \xi \;\; and \;\; \sigma_i \in R(\sigma, 1)\} = \alpha_\xi - 1$; furthermore, let $\tau = (\tau_1, \cdots, \tau_{|\alpha|})$ where $\tau_i = \sigma_i$ if $\sigma_i \neq \xi$, $\tau_i = 0$ if $\sigma_i = \xi$, then for all $n \in \mathbb{N}_{\nu_1} \times \cdots \times \mathbb{N}_{\nu_{\xi-1}} \times \nu_\xi \times \mathbb{N}_{\nu_{\xi+1}} \times \cdots \times \mathbb{N}_{\nu_m}$,

$$\Delta^{|\alpha|-1}_{/R(\sigma,1)} S_\alpha(P(n; n_\xi = \nu_\xi), \nu; bu) = \Delta^{|\alpha|-\alpha_\xi}_{/\tau} (\Delta^{\alpha_\xi - 1}_{/\xi} S_\alpha(P(n; n_\xi = \nu_\xi), \nu; bu)) = 0.$$

Therefore,

$$\begin{aligned}
\Delta^{|\alpha|-1}_{/R(\sigma,1)} z(P(n; n_{\sigma_{|\alpha|}} = \nu_{\sigma_{|\alpha|}})) &= \Delta^{|\alpha|-1}_{/R(\sigma,1)} c(P(n; n_\xi = \nu_\xi)) \\
&\quad + \Delta^{|\alpha|-1}_{/R(\sigma,1)} S_\alpha(P(n; n_\xi = \nu_\xi), \nu; bu) \\
&= \Delta^{|\alpha|-1}_{/R(\sigma,1)} c(P(n; n_{\sigma_{|\alpha|}} = \nu_{\sigma_{|\alpha|}}))
\end{aligned}$$

and noting

$$z(P(n; n_{\sigma_{|\alpha|}} = \nu_{\sigma_{|\alpha|}})) = c(P(n; n_{\sigma_{|\alpha|}} = \nu_{\sigma_{|\alpha|}})) + S_\alpha(P(n; n_{\sigma_{|\alpha|}} = \nu_{\sigma_{|\alpha|}}), \nu; bu)$$
$$= c(P(n; n_{\sigma_{|\alpha|}} = \nu_{\sigma_{|\alpha|}})),$$

from (6.3.55) we can derive

$$\frac{\Delta^{|\alpha|-1}_{/R(\sigma,1)} z(n)}{z(n)} \le \frac{\Delta^{|\alpha|-1}_{/R(\sigma,1)} c(P(n; n_{\sigma_{|\alpha|}} = \nu_{\sigma_{|\alpha|}}))}{c(P(n; n_{\sigma_{|\alpha|}} = \nu_{\sigma_{|\alpha|}}))} + \sum_{j_1 = \nu_{\sigma_{|\alpha|}}}^{n_{\sigma_{|\alpha|}}-1} \Phi_1(P(n; n_{\sigma_{|\alpha|}} = j_1)) = \Phi_2(n),$$

which thus yields

$$\frac{\Delta^{|\alpha|-2}_{/R(\sigma,2)} z(E_{/\sigma_{|\alpha|-1}} n) - \Delta^{|\alpha|-2}_{/R(\sigma,2)} z(n)}{z(n)} \le \Phi_2(n),$$

i.e., an inequality similar to that of (6.3.53). Now following the same reasoning and inductive hypotheses, we can conclude

$$\frac{\Delta^1_{/\sigma_1} z(n)}{z(n)} \le \Phi_{|\alpha|}(n)$$

which gives us readily

$$z(E_{/\sigma_1} n) \le \{1 + \Phi_{|\alpha|}(n)\} z(n),$$
$$z(n) \le z(P(n; n_{\sigma_{|\alpha|}} = \nu_{\sigma_{|\alpha|}})) \prod_{j_{|\alpha|}=\nu_{\sigma_1}}^{n_{\sigma_1}-1} \{1 + \Phi_{|\alpha|}(P(n; n_{\sigma_1} = j_{|\alpha|}))\}.$$

Noting

$$z(P(n; n_{\sigma_1} = \nu_{\sigma_1})) = c(P(n; n_{\sigma_1} = \nu_{\sigma_1}))$$

and using (6.3.53), we conclude

$$u(n) \le z(n) \le c(P(n; n_{\sigma_1} = \nu_{\sigma_1})) \prod_{j_{|\alpha|}=\nu_{\sigma_1}}^{n_{\sigma_1}-1} \left(1 + \Phi_{|\alpha|}(P(n; n_{\sigma_1} = j_{|\alpha|}))\right).$$

An similar estimate can be obtained for each $\sigma \in \Xi_\alpha$, such that $c \in M(\sigma)$. Therefore, (6.3.50) follows immediately. □

Remark 6.3.10 The estimate (6.3.50) can be rearranged as follows for all $n \in \mathbb{N}_\nu$,

$$u(n) \leq \min_{\sigma\in\Xi_\alpha : c\in M(\sigma)} c(P(n; n_{\sigma_1} = \nu_{\sigma_1})) \prod_{j_{|\alpha|}=\nu_{\sigma_1}}^{n_{\sigma_1}-1} \big(1 + \Phi_{|\alpha|}(P(n; n_{\sigma_1} = j_{|\alpha|}))\big)$$
$$+ S_{E^{-1}_{/\sigma_1}\alpha}((P(n; n_{\sigma_1} = j_{|\alpha|}), \nu; b),$$

where

$$\begin{cases} \Psi_1(n) = \dfrac{max(0, \Delta_\alpha^{|\alpha|} c(n))}{c(n)}, \\[2ex] \Psi_{k+1}(n) = \dfrac{\Delta^{|\alpha|-1}_{/R(\sigma,k)} c(P(n; n_{\sigma_{|\alpha|}-k+1} = \nu_{\sigma_{|\alpha|}-k+1}))}{c(P(n; n_{\sigma_{|\alpha|}-k+1} = \nu_{\sigma_{|\alpha|}-k+1}))} \\[2ex] \qquad + \Sigma_{j_k=\nu_{\sigma_{|\alpha|}-k+1}}^{n_{\sigma_{|\alpha|-1}-k+1}-1} \Psi_k(P(n; n_{\sigma_{|\alpha|-1}-k+1} - 1 = j_k)), \quad k = 1, \cdots, |\alpha| - 1. \end{cases}$$

Remark 6.3.11 The method used in Theorem 6.3.7 can be applied (with slight modifications) to general type of inequalities such as for all $n \in \mathbb{N}_\nu$,

$$u(n) \leq c(n) + \sum_{i=1}^{k} S_{\alpha^i}(n, \nu; b_i u),$$

where $\alpha^i = (\alpha_1^i, \cdots, \alpha_m^i)$, $i = 1, \cdots, k$ and c belongs to a suitable class M. In fact, to obtain such a bound, it first suffices to obtain some linear inequality of the type

$$\Delta_\psi^k z(n) \leq \Lambda(n) z(n) + Y(n) \tag{6.3.56}$$

and then to follow the method of Theorem 6.3.7. To illustrate this, we present the following examples.

Example 6.3.2 Consider the inequality

$$u(n_1, n_2) \leq c(n_1, n_2) + \sum_{j_{1,1}=1}^{n_1-1} b_1(j_{1,1}, n_2)\Big[u(j_{1,1}, n_2)$$
$$+ \sum_{j_{1,2}=1}^{j_{1,1}-1} \sum_{j_{1,3}=1}^{j_{1,2}-1} \sum_{j_{2,1}=4}^{n_2-1} b_2(j_{1,3}, j_{2,1}) u(j_{1,3}, j_{2,1})\Big]. \tag{6.3.57}$$

Denoting the right-hand side of (6.3.57) by $z(n_1, n_2)$, we get

$$\begin{aligned}\Delta_{/1} z(n) &= \Delta_{/1} c(n) + b_1(n) \left[u(n) + \sum_{j_{1,2}=1}^{j_{1,1}-1} \sum_{j_{1,3}=1}^{j_{1,2}-1} \sum_{j_{2,1}=4}^{n_2-1} b_2(j_{1,3}, j_{2,1}) u(j_{1,3}, j_{2,1}) \right] \\ &\leq \Delta_{/1} c(n) + b_1(n) \left[z(n) + \sum_{j_{1,2}=1}^{j_{1,1}-1} \sum_{j_{1,3}=1}^{j_{1,2}-1} \sum_{j_{2,1}=4}^{n_2-1} b_2(j_{1,3}, j_{2,1}) u(j_{1,3}, j_{2,1}) \right].\end{aligned} \tag{6.3.58}$$

Let $b_1 \geq 0, b_2 \geq 0, u \geq 0$, if $\Delta_{/1} c \geq 0$, then $\Delta_{/1} z \geq 0$, if, moreover, $\Delta_{/2} c \geq 0$, $\Delta_{/2} b_1 \geq 0$, and we need to estimate u such that $\Delta_{/1} u \geq 0$, then by the definition of $z(n_1, n_2)$, we have

$$\begin{aligned}&\Delta_{/2} z(n) \\ &= \Delta_{/2} c(n) + \sum_{j_{1,1}=1}^{n_1-1} b_1(j_{1,1}, n_2+1) \left[\Delta_{/2} u_1(j_{1,1}, n_2) + \sum_{j1,2=1}^{j_{1,1}-1} \sum_{j_{1,3}=1}^{j_{1,2}-1} b_2(j_{1,3}, n_2) u(j_{1,3}, n_2) \right] \\ &+ \sum_{j_{1,1}=1}^{n_1-1} \left[u(j_{1,1}, n_2) + \sum_{j_{1,2}=1}^{j_{1,1}-1} \sum_{j_{1,3}=1}^{j_{1,2}-1} \sum_{j_{2,1}=4}^{n_2-1} b_2(j_{1,3}, j_{2,1}) u(j_{1,3}, j_{2,1}) \right] \Delta_{/2} b_1(j_{1,1}, n_2) \geq 0.\end{aligned}$$

Let

$$\omega(n) = z(n) + \sum_{j1,2=1}^{n_1-1} \sum_{j_{1,3}=1}^{j_{1,2}-1} \sum_{j_{2,1}=4}^{n_2-1} b_2(j_{1,3}, j_{2,1}) z(j_{1,3}, j_{2,1}).$$

Hence it follows

$$\begin{aligned}\Delta_{/1} \omega(n) &= \Delta_{/1} z(n) + \sum_{j_{1,2}=1}^{n_1-1} \sum_{j_{1,3}=4}^{n_2-1} b_2(j_{1,3}, j_{2,1}) z(j_{1,3}, j_{2,1}) \\ &\leq \Delta_{/1} c(n) + b_1(n) \omega(n) + z(n_1-1, n_2-1) \sum_{j_{1,2}=1}^{n_1-1} \sum_{j_{1,3}=4}^{n_2-1} b_2(j_{1,3}, j_{2,1}) \\ &\leq \Delta_{/1} c(n) + \left[b_1(n) + \sum_{j_{1,2}=1}^{n_1-1} \sum_{j_{1,3}=4}^{n_2-1} b_2(j_{1,3}, j_{2,1}) \right] \omega(n),\end{aligned} \tag{6.3.59}$$

which is of the form (6.3.56). Now we can apply the method of Theorem 6.3.7. The obtained bound for $\omega(n)$ is then used in the inequality $\Delta_{/1} z(n) \leq \Delta_{/1} c(n) + b_1(n) \omega(n)$, which, in turn after suitable summations, leads to the bound for $z(n)$,

and consequently, the bound for $u(n)$. We can get another inequality for $\Delta_{/1}z$, which follows directly from (6.3.58), namely,

$$\Delta_{/1}z(n) \leq \Delta_{/1}c(n) + b_1(n)\left[1 + \sum_{j_{1,2}=1}^{n_1-1}\sum_{j_{1,3}=1}^{j_{1,2}-1}\sum_{j_{2,1}=4}^{n_2-1} b_2(j_{1,3}, j_{2,1})\right] z(n)$$

which is also of the form (6.3.56), consequently, the method of Theorem 6.3.7 is applicable.

In Example 6.3.1, we have three summations with respect to the first variable, and one with respect to the second. In fact, we do not suppose that all α_i in α are the same.

Example 6.3.3 Consider the inequality

$$u(n) \leq c(n) + S_\alpha(n, \nu; b_1(j)(u(j) + S_\beta(j, \nu; b_2(i)u(i)))),$$

where $\beta \leq \alpha$, that is, $\beta_k \leq \alpha_k$ for all $k = 1, \cdots, m$. Let $c \in M(\sigma)$ and $\sigma \in \Xi - \beta$, $\Delta_\beta^{|\beta|}c \geq 0$. Let

$$z(n) = c(n) + S_\alpha(n, \nu; b_1(u + S_\beta(j, \nu; b_2 u))),$$

then

$$\Delta_\beta^{|\beta|}z(n) = \Delta_\beta^{|\beta|}c(n) + S_{\alpha-\beta}(n, \nu; b_1(u + S_\beta(j, \nu; b_2 u))) \leq \Delta_\beta^{|\beta|}c(n) + S_{\alpha-\beta}(n, \nu; b_1\omega),$$

where

$$(n) = z(n) + S_\beta(n, \nu; b_2 z).$$

Hence,

$$\begin{aligned}\Delta_\beta^{|\beta|}\omega(n) &= \Delta_\beta^{|\beta|}z(n) + b_2(n)z(n) \leq \Delta_\beta^{|\beta|}c(n) + S_{\alpha-\beta}(n, \nu; b_1\omega) + b_2(n)z(n)\\ &\leq \Delta_\beta^{|\beta|}c(n) + [S_{\alpha-\beta}(n, \nu; b_1) + b_2(n)]\omega(n).\end{aligned}$$

Thus, the resulting inequality is of the form (6.3.56), and so the method used in Theorem 6.3.7 allows us to get an estimate on w, and consequently, after suitable summations on z and then on u. If $\beta = \alpha$ as in [489] and $\Delta_\beta^{|\beta|}c(n) = 0$, then $\Delta_\beta^{|\beta|}\omega(n) \leq [b_1(n) + b_2(n)]\omega(n)$.

The next result, due to Yeh [671], is concerned with some discrete Bellman-Gronwall inequality.

We give the notation used in next result as follows:

For $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n), \hat{1} = (1, \cdots, 1), \hat{0} = (0, \cdots, 0) \in \mathbb{N}_0^n$ and $u : \mathbb{N}_0^n \to \mathbb{R}_+$, we define

(a) $x := (x_1, \tilde{x})$, where $\tilde{x} = (x_2, \cdots, x_n)$,
(b) $x \leq y$ if and only if $x_i \leq y_i$ for $i = 1, \cdots, n$,
(c) $\sum_{y=0}^{x-\hat{1}} u(y) := \sum_{y_1=0}^{x_1-1} \cdots \sum_{y_n=0}^{x_n-1} u(y_1, \cdots, y_n)$, $\sum_{y=0}^{x-\hat{1}} u(y) := 0$ for some $x_i = 0$,
(d) $\prod_{y=0}^{x-\hat{1}} u(y) := \prod_{y_1=0}^{x_1-1} \cdots \prod_{y_n=0}^{x_n-1} u(y_1, \cdots, y_n)$, $\prod_{y=0}^{x-\hat{1}} u(y) := 1$ for some $x_i = 0$,
(e) $\triangle u_{x_1}(x) := u(x_1 + 1, \tilde{x}) - u(x)$, $\Delta u_{x_2}(x) := u(x_1, x_2 + 1, x_3, \cdots, x_n) - u(x), \cdots$, $\Delta u_{x_n}(x) := u(x_1, \cdots, x_{n-1}, x_n + 1) - u(x)$, and so on.

Theorem 6.3.8 (Yeh [671]) *Let $u(x), k(x) : \mathbb{N}_0^n \to \mathbb{R}_+$ and $f(x; s) : \mathbb{N}_0^{2n} \to \mathbb{R}_+$ with $s \leq x$. If for all $x \in \mathbb{N}_0^n$,*

$$u(x) \leq k(x) + \sum_{s=0}^{x-\hat{1}} f(x; s)u(s), \tag{6.3.60}$$

then for all $x \in \mathbb{N}_0^n$,

$$u(x) \leq K(x) \prod_{s_1=0}^{x_1-1} \left[1 + \sum_{\tilde{s}=\tilde{0}}^{\tilde{x}-\tilde{1}} F(x; s_1, \tilde{s}) \right], \tag{6.3.61}$$

where

$$K(x) := \sup\{k(s) : \hat{0} \leq s \leq x\}, \quad F(x; s) := \sup\{f(t; s) : \hat{0} \leq t \leq x\}. \tag{6.3.62}$$

Proof For any fixed point Y on $\mathbb{N}_0^n$, it follows from (6.3.60) and (6.3.62) that for all $\hat{0} \leq x \leq Y$,

$$u(x) \leq K(Y) + \sum_{s=\hat{0}}^{x-\hat{1}} F(Y; s)u(s).$$

Setting for all $\varepsilon > 0$,

$$V(Y; x) := K(Y) + \sum_{s=\hat{0}}^{x-\hat{1}} F(Y; s)u(s) + \varepsilon, \tag{6.3.63}$$

so, by (6.3.63),

$$u(x) \leq V(Y; x)$$

and

$$\triangle^n V_x(Y;x) = F(Y;x)u(x) \leq F(Y;x)V(Y;x). \tag{6.3.64}$$

Hence it follows from (6.3.64) that

$$\frac{\triangle^{n-1} V_{x_1\cdots x_{n-1}}(Y;x_1,\cdots,x_{n-1},x_n+1)}{V(Y;x_1,\cdots,x_{n-1},x_n+1)} - \frac{\triangle^{n-1} V_{x_1\cdots x_{n-1}(Y;x)}}{V(Y;x)} \leq F(Y;x).$$

Keeping $x_1,\cdots,x_{n-1}$ fixed in the above inequality, setting $x_n = s_n$, and summing over $s_n = 0, 1, \cdots, Y_n - 1$, we have

$$\frac{\triangle^{n-1} V_{x_1\cdots x_{n-1}}(Y;x_1,\cdots,x_{n-1},Y_n)}{V(Y;x_1,\cdots,x_{n-1},Y_n)} \leq \sum_{s_n=0}^{Y_n-1} F(Y;x_1,\cdots,x_{n-1},s_n).$$

Continuing in this way and using the method described in [670] , we have

$$\frac{V_{x_1}(Y;x_1,\tilde{Y})}{V(Y;x_1,\tilde{Y})} = \frac{V(Y;x_1+1,\tilde{Y})}{V(Y;x_1,\tilde{Y})} - 1 \leq \sum_{\tilde{s}=\tilde{0}}^{\tilde{Y}-\tilde{1}} F(Y;x_1,\tilde{s}).$$

Keeping $\tilde{Y}$ fixed in this inequality, setting $x_1 = s_1$, and taking the product over $s_1 = 0, 1, \cdots, Y_1 - 1$, we have

$$u(Y) \leq V(Y;Y) \leq (K(Y)+\varepsilon) \prod_{s_1=0}^{Y_1-1} \left[1 + \sum_{\tilde{s}=\tilde{0}}^{\tilde{Y}-\tilde{1}} F(Y;s_1,\tilde{s}) \right].$$

Letting $\varepsilon \downarrow 0$ and replacing Y by x, we can obtain the desired result (6.3.61). □

Remark 6.3.12 For $n = 1$, the continuous analogue of Theorem 6.3.8 is due to Butler and Rogers [118].

As an application of Theorem 6.3.8, we can prove the following theorem.

Theorem 6.3.9 (Yeh [671]) *Let u, k, f, K, F be defined as in Theorem 6.3.8. Let $g(x;s) : \mathbb{N}_0^{2n} \to \mathbb{R}_+$ with $s \leq x$. If for all $x \in \mathbb{N}_0^n$,*

$$u(x) \leq k(x) + \sum_{s=\hat{0}}^{x-\hat{1}} f(x;s)[u(s) + \sum_{t=\hat{0}}^{s-\hat{1}} g(s;t)u(t)], \tag{6.3.65}$$

then for all $x \in \mathbb{N}_0^n$,

$$u(x) \leq K(x) \prod_{s_1=0}^{x_1-1} \left\{ 1 + \sum_{\tilde{s}=\tilde{0}}^{\tilde{x}-\tilde{1}} [F(x; s_1, \tilde{s}) + G(x; s_1, \tilde{s})] \right\} \tag{6.3.66}$$

or

$$u(x) \leq k(x) + \sum_{s=\hat{0}}^{x-\hat{1}} f(x; s) K(s) \prod_{t_1=0}^{s_1-1} \left\{ 1 + \sum_{\tilde{t}=\tilde{0}}^{\tilde{s}-\tilde{1}} [F(s; t_1, \tilde{t}) + G(s; t_1, \tilde{t})] \right\}, \tag{6.3.67}$$

where $G(x; s) := \sup\{g(t; s) : \hat{0} \leq t \leq x\}$.

Proof Let

$$\omega(x) := u(x) + \sum_{t=\hat{0}}^{x-\hat{1}} g(x; t) u(t).$$

Then

$$u(x) \leq \omega(x) \tag{6.3.68}$$

and , by (6.3.65),

$$u(x) = \omega(x) - \sum_{s=\hat{0}}^{x-\hat{1}} g(x; s) u(s) \leq k(x) + \sum_{s=\hat{0}}^{x-\hat{1}} f(x; s) \omega(s). \tag{6.3.69}$$

Thus

$$\omega(x) \leq k(x) + \sum_{s=\hat{0}}^{x-\hat{1}} [f(x; s) + g(x; s)] \omega(s).$$

Applying Theorem 6.3.8 to the above inequality, we have

$$\omega(x) \leq K(x) \prod_{s_1=0}^{x_1-1} \left\{ 1 + \sum_{\tilde{s}=\tilde{0}}^{\tilde{x}-\tilde{1}} [F(x; s_1, \tilde{s}) + G(x; s_1, \tilde{s})] \right\}. \tag{6.3.70}$$

Therefore, from (6.3.68) and (6.3.70), we can obtain (6.3.66) and from (6.3.69) and (6.3.70), we have (6.3.67). □

Remark 6.3.13 The discrete inequalities established in Theorem 6.3.9 can also be extended either to nonlinear cases as shown in [670], or continuous analogues, or both; we omit the details.

We note that a continuous analogue of inequality (6.3.66) was established by Beesack [57].
Now we may use Corollary 1 of [670] to give another brief proof of Theorem 6.3.8.

Proof For any fixed point Y of $\mathbb{N}_0^n$, it follows from (6.3.60) and (6.3.62) that for all $\hat{0} \le x \le Y$,

$$u(x) \le K(Y) + \sum_{s=\hat{0}}^{x-\hat{1}} F(Y; s)u(s).$$

By Corollary 1 in [670], we have for all $\hat{0} \le x \le Y$,

$$u(x) \le K(Y) \prod_{s_1=0}^{x_1-1} [1 + \sum_{\tilde{s}=\tilde{0}}^{\hat{x}-\hat{1}} F(Y; s_1, \tilde{s})]. \tag{6.3.71}$$

In particular, (6.3.71) holds for $x = Y$. Replacing Y by x in (6.3.71) gives us (6.3.61). □

6.4 Difference Inequalities in Several Independent Variables

In this section, we shall introduce some difference inequalities in several independent variables.

6.4.1 Discrete Riemann's Function

In this section, we begin with the established discrete analogue of Riemann's function. The function is repeatedly used to study linear Gronwall-Bellman type inequalities. Next we shall provide an upper estimate on the Riemann's function which is quite adequate in practical applications and provides Wendroff's type estimates. Inequalities involving higher order differences in two independent variables are also directly considered. For this, the relevant Taylor's formula in multi-dimensional linear discrete inequalities, and wherever possible provide upper bounds in terms of discrete resolvent function.

First, we first introduce the notation as follows. Let the product $\mathbb{N}_0 \times \cdots \times \mathbb{N}_0$ (m times) be denoted by $\mathbb{N}_0^m$. A point $(x_1, \cdots, x_m)$ in $\mathbb{N}_0^m$ is denoted by x, whereas $\bar{x}_i$ represents $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_m)$, and $(\bar{x}_i, \cdot)$ stands for

$(x_1, \cdots, x_{i-1}, \cdot, x_{i+1}, \cdots, x_m)$, also for all $s, x \in \mathbb{N}_0^m$, $0 \le s \le x$ represents $0 \le s_i \le x_i$, $1 \le i \le m$. For a given function $u(x)$ on $\mathbb{N}_0^m$, the first order difference with respect to the variable x_i is defined as $\Delta_{x_i} u(x) = u(\bar{x}_i, \bar{x}_i + 1) - u(x)$, and the second order difference with respect to the variables x_i and x_j is defined as $\Delta_{x_i}\Delta_{x_j} u(x) = \Delta_{x_i} u(\bar{x}_j, \bar{x}_j + 1) - \Delta_{x_i} u(x) = u(x_1, \cdots, x_{i-1}, x_i + 1, x_{i+1}, \cdots, x_{j-1}, x_j + 1, \cdots, x_{j+1}, \cdots, x_m) - u(\bar{x}_i, \bar{x}_i + 1) - u(\bar{x}_j, \bar{x}_j + 1) + u(x)$. The higher order differences are defined analogously. The $S_{\ell=s}^{x-1} u(\ell)$ represents the m fold sum $\sum_{\ell_1=s_1}^{x_1-1} \cdots \sum_{\ell_m=s_m}^{x_m-1} u(\ell_1, \cdots, \ell_m)$, and $\Delta_x^m u(x)$ denotes $\Delta_{x_1} \cdots \Delta_{x_m} u(x_1, \cdots, x_m)$.

Lemma 6.4.1 (Agarwal [9, 10]) *Let $g(x)$ be defined on $\mathbb{N}_0^m$, then the function $V(s; x)$, $s \le x - 1$, $(s; x) \in \mathbb{N}_0^m \times \mathbb{N}_0^m$ is a solution of*

$$\begin{cases} (-1)^m \Delta_s^m V(s; x) \;=\; g(s) V(s + 1, x), & (6.4.1) \\ \quad V(\bar{s}_i, x_i; x) \;=\; 1, 1 \le i \le m & (6.4.2) \end{cases}$$

if and only if

$$V(s; x) = 1 + S_{\ell=s}^{x-1} g(\ell) V(\ell + 1; x). \tag{6.4.3}$$

Proof Obviously, from (6.4.1) it follows

$$(-1)^m \Delta_{\bar{\ell}_m}^{m-1} [V(\bar{\ell}_m, \ell_m + 1; x) - V(\ell; x)] = g(\ell) V(\ell + 1; x),$$

and hence by summing the above equality from $\ell_m = s_m$ to $\ell_m = x_m - 1$, we get

$$(-1)^m \Delta_{\bar{\ell}_m}^{m-1} \left[V(\bar{\ell}_m, \ell_m; x) \Big|_{\ell_m = s_m}^{x_m} \right] = \sum_{\ell_m = s_m}^{x_m - 1} g(\ell) V(\ell + 1; x),$$

which , together with (6.4.2), is the same as

$$(-1)^{m+1} \Delta_{\bar{\ell}_m}^{m-1} V(\bar{\ell}_m, s_m; x) = \sum_{\ell_m = s_m}^{x_m - 1} g(\ell) V(\ell + 1; x).$$

Continuing in this way, we obtain

$$(-1)^{m+m+1} \Delta_{\bar{\ell}_1} V(\bar{s}_1, \ell_1; x) = \sum_{\bar{\ell}_1 = \bar{s}_1}^{\bar{x}_1 - 1} g(\ell) V(\ell + 1; x)$$

and hence by summing the above equality from $\ell_1 = s_1$ to $\ell_1 = x_1 - 1$, we have

$$(-1)^{2m+1} \left[V(\bar{s}_1, \ell_1; x) \Big|_{\ell_1 = s_1}^{x_1} \right] = S_{\ell=s}^{x-1} g(\ell) V(\ell + 1; x),$$

which , combined with (6.4.2), is the same as

$$-1 + V(s;x) = S_{\ell=s}^{x-1} g(\ell)V(\ell+1;x).$$

Thus the proof is now complete $\square$

Lemma 6.4.2 (Agarwal [10]) *The problem (6.4.1) and (6.4.2) or equivalently (6.4.3), has a solution $V(s;x)$. Furthermore, if $g(x) \geq 0$ on $\mathbb{N}_0^m$, then $V(s;x) \geq 1$ on $\mathbb{N}^m \times \mathbb{N}_0^m$.*

Proof Obviously, for the following iteration

$$\begin{cases} V_0(s;x) = 1, \\ V_{n+1}(s;x) = 1 + S_{\ell=s}^{x-1} g(\ell)V(\ell+1;x), \quad n = 0, 1, \cdots, \end{cases} \tag{6.4.4}$$

an easy introduction gives us

$$|V_n(s;x) - V_{n-1}(s;x)| \leq G^n \frac{1}{(n!)^m} \prod_{i=1}^{m} (x_i - s_i)^n,$$

where $G = \max_{0 \leq \ell \leq x-1} |g(\ell)|$.

Therefore, for all $(s;x) \in \mathbb{N}_0^m \times \mathbb{N}_0^m$, we have

$$\begin{aligned} &|V_0(s;x)| + \sum_{k=1}^{n} |V_k(s;x) - V_{k-1}(s;x)| \\ &\qquad \leq 1 + \sum_{k=1}^{n} G^k \frac{1}{k!} \left[\prod_{i=1}^{m} (x_i - s_i) \right]^k \leq \exp\left[G \prod_{i=1}^{m} (x_i - s_i) \right] \end{aligned}$$

whence the sequence $\{V_n(s;x)\}$ converges to a solution $V(s;x)$ of Eq. (6.4.3). The uniqueness of $V(s;x)$ and the inequality $V(s;x) \geq 1$ on $\mathbb{N}_0^m \times \mathbb{N}_0^m$ (when $g(x) \geq 0$ on $\mathbb{N}_0^m$) follow easily from (6.4.4). $\square$

Lemma 6.4.3 (Agarwal [10]) *Let $g(x) \geq 0$ and $h(x)$ be defined on $\mathbb{N}_0^m$ and the following inequality holds for all $x \in \mathbb{N}_0^m$,*

$$\Delta_x^m u(x) \leq g(x)u(x) + h(x), \tag{6.4.5}$$

where

$$u(\bar{x}_i, 0) = 0, \quad 1 \leq i \leq m. \tag{6.4.6}$$

Then for all $x \in \mathbb{N}_0^m$,

$$u(x) \le S_{s=0}^{x-1} h(s) V(s+1;x), \tag{6.4.7}$$

where $V(s;x)$ *is the solution of problem (6.4.1)–(6.4.2).*

Proof Clearly, from (6.4.1) and (6.4.5) it follows that

$$S_{s=0}^{x-1} V(s+1;x)\Delta_s^m u(s) - S_{s=0}^{x-1}(-1)^m \Delta_s^m V(s;x)u(s) \le S_{s=0}^{x-1} h(s)V(s+1;x). \tag{6.4.8}$$

An application of (6.4.5) easily implies

$$S_{s=0}^{x-1}(-1)^m u(s)\Delta_s^m V(s;x) = (-1)^m S_{\bar{s}_m=0}^{\bar{x}_m-1}\Big[u(s)\Delta_{\bar{s}_m}^{m-1} V(s;x)\Big|_{s_m=0}^{x_m} - \sum_{s_m=0}^{x_m-1} \Delta_{s_m} u(s)\Delta_{\bar{s}_m}^{m-1} V(\bar{s}_m, s_m+1;x)\Big]. \tag{6.4.9}$$

Using (6.4.2) and (6.4.6), the right-hand side of (6.4.9) reduces to

$$(-1)^{m+1}\sum_{s_m=0}^{x_m-1} S_{\bar{s}_m=0}^{\bar{x}_m-1}\Delta_{s_m} u(s)\Delta_{\bar{s}_m}^{m-1} V(\bar{s}_m, s_m+1;x).$$

Repeating the above arguments successively, we finally obtain

$$(-1)^{2m-1}\sum_{s_m=0}^{x_m-1}\cdots\sum_{s_2=0}^{x_2-1}\Big[\Delta_{s_m}\cdots\Delta_{s_2} u(s)V(s_1, s_2+1,\ldots,s_m+1;x)\Big|_{s_1=0}^{x_1} - \sum_{s_1=0}^{x_1-1}\Delta_{s_m} u(s)V(s+1;x)\Big],$$

which is the same as

$$(-1)^{2m-1}\sum_{s_m=0}^{x_m-1}\cdots\sum_{s_2=0}^{x_2-1}\Delta_{s_m}\cdots\Delta_{s_2} u(\bar{s}_1, x_1) + S_{s=0}^{x-1}\Delta_{s_m} u(s)V(s+1;x)$$

or

$$-u(x) + S_{s=0}^{x-1}\Delta_{s_m} u(s)V(s+1;x).$$

Substituting this in (6.4.8), hence (6.4.7) follows immediately. □

Remark 6.4.1 For all $g(x)$ and $h(x)$, equality in (6.4.5) implies equality in (6.4.7), and hence $V(s;x)$ the solution of problem (6.4.1)–(6.4.2) is the discrete analogue of Riemann's function.

Corollary 6.4.1 (Agarwal [10]) *Let $g(x)$ and $h(x)$ be as in Lemma 6.4.3, and $\phi(x),\ \psi(x)$ be defined on $\mathbb{N}_0^m$ and satisfy for all $x \in \mathbb{N}_0^m$,*

$$\begin{cases} \Delta_x^m \phi(x) \le g(x)\phi(x) + h(x), \\ \Delta_x^m \psi(x) \ge g(x)\psi(x) + h(x), \\ \phi(\bar{x}_i, 0) = \psi(\bar{x}_i, 0), \quad 1 \le i \le m. \end{cases}$$

Then for all $x \in \mathbb{N}_0^m$, we have

$$\phi(x) \le \psi(x).$$

Lemma 6.4.4 (Agarwal [10]) *Let $g(x)$ be as in Lemma 6.4.3, and $V(s;x)$ be the solution of problem (6.4.1)–(6.4.2). Let $W(s;x)$ be hold for all $s \le x-1,\ (s,x) \in \mathbb{N}_0^m \times \mathbb{N}_0^m$ and*

$$\begin{cases} (-1)^m \Delta_s^m W(s;x) \ge g(s)W(s+1;x), & (6.4.10)\\ W(\bar{s}_i, x_i; x) = 1, \quad 1 \le i \le m. & (6.4.11) \end{cases}$$

Then for all $s \le x-1,\ (s,x) \in \mathbb{N}_0^m \times \mathbb{N}_0^m$,

$$V(s;x) \le W(s;x).$$

Proof Let $\phi(s;x)$ be defined and non-negative for all $s \le x-1,\ (s;x) \in \mathbb{N}_0^m \times \mathbb{N}_0^m$ so that

$$(-1)^m \Delta_s^m W(s;x) = g(s)W(s+1;x) + \phi(s;x). \tag{6.4.12}$$

Next we define the iterates as follows

$$\begin{cases} W_0(s;x) = V(s;x), \\ W_{n+1}(s;x) = 1 + S_{\ell=s}^{x-1} W_n(\ell+1;x) + \S_{\ell=s}^{x-1}\phi(\ell;x); \ \ n = 0, 1, \cdots. \end{cases}$$

Obviously, $W_n(s;x) \ge V(s;x)$ for all $n \ge 1$, and as in Lemma 6.4.2, the sequence $\{W_n(s;x)\}$ converges to $W(s;x)$ which is the solution of problem (6.4.10)–(6.4.11). □

Next, we shall assume that the functions appearing in the inequalities are real-valued, non-negative and defined on $\mathbb{N}_0^m$.

Theorem 6.4.1 (Agarwal [10]) *Assume that the following inequality holds for all* $x \in \mathbb{N}_0^m$,

$$u(x) \leq p(x) + q(x) S_{s=0}^{x-1} f(s) u(s). \tag{6.4.13}$$

Thus for all $x \in \mathbb{N}_0^m$,

$$u(x) \leq p(x) + q(x) S_{s=0}^{x-1} f(s) p(s) V(s+1; x), \tag{6.4.14}$$

where $V(s; x)$ *is the solution of*

$$\begin{cases} (-1)^m \Delta_s^m V(s; x) = f(s) V(s+1; x), & s \leq x-1, \\ V(\bar{s}_i, x_i; x) = 1, \quad 1 \leq i \leq m. \end{cases}$$

Proof Define a function $v(x)$ on $\mathbb{N}_0^m$ as follows

$$v(x) = S_{s=0}^{x-1} f(s) u(s).$$

For this function, we have

$$\Delta_x^m v(x) = f(x) u(x), \quad v(\bar{x}_i, 0) = 0, \quad 1 \leq i \leq m. \tag{6.4.15}$$

Since $u(x) \leq p(x) + q(x) v(x)$, and $f(x) \geq 0$, from (6.4.15) it follows

$$\Delta_x^m v(x) \leq f(x) p(x) + f(x) q(x) v(x), \quad v(\bar{x}_i, 0) = 0, \quad 1 \leq i \leq m.$$

Now applying Lemma 6.4.3 to the above inequality, we conclude

$$v(x) \leq S_{s=0}^{x-1} f(s) p(s) V(s+1; x). \tag{6.4.16}$$

Therefore (6.4.14) follows from (6.4.16) and the inequality $u(x) \leq p(x) + q(x) v(x)$. □

Remark 6.4.2 Note that the inequality (6.4.14) is the best possible in the sense that the equality in (6.4.13) implies the equality in (6.4.14).

Theorem 6.4.2 (Agarwal [10]) *Assume the following inequality holds for all* $x \in \mathbb{N}_0^m$,

$$u(x) \leq p(x) + q(x) \sum_{i=1}^{r} E_i(x, u), \tag{6.4.17}$$

where

$$E_i(x, u) = S_{x^1=0}^{x-1} f_{i1}(x^1) S_{x^2=0}^{x^1-1} f_{i2}(x^2) \cdots S_{x^i=0}^{x^{i-1}-1} f_{ii}(x^i) u(x^i). \tag{6.4.18}$$

Then, for all $x \in \mathbb{N}_0^m$,

$$u(x) \le p(x) + q(x)S_{s=0}^{x-1}\left[\sum_{i=1}^{r} \Delta_s^m E_i(s,p)\right] V(s+1;x), \tag{6.4.19}$$

where $V(s;x)$ *is the solution of*

$$\begin{cases} (-1)^m \Delta_s^m V(s;x) = \left[\sum_{i=1}^{r} \Delta_s^m E_i(s,q)\right] V(s+1;x), \quad s \le x-1, \\ V(\bar{s}_i, x_i; x) = 1, \quad 1 \le i \le m. \end{cases}$$

Proof The proof uses the arguments of Theorem 2.1.37 and Theorem 6.4.1. □

In the need of next theorem, we now introduce a definition of Condition (c).

Definition 6.4.1 We say that condition (c) is satisfied if for all $x \in \mathbb{N}_0^m$, (6.4.17) holds, where

$$\begin{cases} f_{ii}(x) = f_i(x), \quad 1 \le i \le r; \\ f_{i+1,i}(x) = f_{i+2,i}(x) \\ \qquad\qquad = \cdots = f_{r,i}(x) = g_i(x), \quad 1 \le i \le r-1. \end{cases}$$

In the next result for all $x \in \mathbb{N}_0^m$, we shall denote

$$\phi_j(x) = \max\Bigg\{0, \sum_{i=1}^{r-j+1} q(x)f_i(x) - g_{r-j+1}(x);$$
$$g_i(x) - g_{r-j+1}(x), \ 1 \le i \le r-\Bigg\}, \ 1 \le j \le r$$

where $g_r(x) = 0$ for all $x \in \mathbb{N}_0^m$.

Theorem 6.4.3 (Agarwal [10]) *Assume the condition (c) holds. Then for all* $x \in \mathbb{N}_0^m$,

$$u(x) \le p(x) + q(x)\psi_j(x), \quad 1 \le j \le r, \tag{6.4.20}$$

where

$$\psi_j(x) = S_{s=0}^{x-1}\left[p(s)\sum_{i=1}^{r-j+1} f_i(s) + g_{r-j+1}(s)\psi_{j-1}(s)\right] V_j(s+1;x), 1 \le j \le r,$$

and $V_j(s;x)$, $1 \le j \le r$, *are the solutions of*

$$\begin{cases} (-1)^m \Delta_s^m V_j(s;x) = \phi_j(s) V_j(s+1;x), & s \le x-1, \\ V_j(\bar{s}_i, x_i; x) = 1, & 1 \le i \le m. \end{cases}$$

Proof The proof is similar to that of Theorems 2.1.39 and 6.4.1. □

Theorem 6.4.4 (Agarwal [10]) *Assume the following inequality holds for all* $x \in \mathbb{N}_0^m$,

$$u(x) \le p_0(x) + \sum_{i=1}^{r} p_i(x) S_{s=0}^{x-1} q_i(s) u(s). \tag{6.4.21}$$

Then for all $x \in \mathbb{N}_0^m$,

$$u(x) \le F_r[p_0(x)], \tag{6.4.22}$$

where

$$\begin{cases} F_i \;= D_i D_{i-1} \cdots D_0, \\ D_0[w] = w, \\ D_j[w] = w + \left[F_{j-1}[p_j]\right] S_{s=0}^{x-1} q_j(s) w(s) V_j(s+1;x), \end{cases}$$

and $V_j(s;x)$, $1 \le j \le r$, *are the solutions of*

$$\begin{cases} (-1)^m \Delta_s^m V_j(s;x) = q_j(s) F_{j-1}[p_j(s)] V_j(s+1;x), & s \le x-1, \\ V_j(\bar{s}_i, x_i; x) = 1, & 1 \le j \le r. \end{cases}$$

Proof The proof is similar to that of Theorems 2.1.40 and 6.4.1. □

6.4.2 The Multi-Dimensional Wendroff Type Inequalities

In this section, we shall introduce some discrete Wendroff type inequalities in multiple independent variables.

Let $W(s;x)$ be any function defined for all $s \le x-1$; $(s;x) \in \mathbb{N}_0^m \times \mathbb{N}_0^m$ and

$$\begin{cases} (-1)^m \Delta_s^m W(s;x) \ge f(s) q(s) W(s+1;x), & s \le x-1, \\ W(\bar{s}_j, x_i; x) = 1, & 1 \le i \le m. \end{cases} \tag{6.4.23}$$

Then from Lemma 6.4.4 it follows that in (6.4.14), $V(s+1;x)$ can be replaced by $W(s+1;x)$. However, finding a suitable $W(s;x)$ in advance which satisfies (6.4.23)

seems to be quite difficult. Therefore, for the function $V(s;x)$, we shall give an upper estimate which is quite adequate in practical applications.

Lemma 6.4.5 (Agarwal [10]) *Let $V(s;x)$ be as in Theorem 6.4.1. Then for all $s \le x-1,\ (s;x) \in \mathbb{N}_0^m \times \mathbb{N}_0^m$,*

$$V(s;x) \le \prod_{\ell_1=s_1}^{x_1-1}\left[1 + S_{\bar{\ell}_1=\bar{s}_1}^{\bar{x}_1-1} f(\ell)q(\ell)\right]. \tag{6.4.24}$$

Proof Since $f(x)q(x) \ge 0$ for all $x \in \mathbb{N}_0^m$, Lemma 6.4.2 implies that $V(s;x) \ge 1$. Therefore, $(-1)^m \Delta_s^m V(s;x) \ge 0$, which, by following the proof of Lemma 6.4.1, gives us that $(-1)^i \Delta_{s_1} \cdots \Delta_{s_i} V(s;x) \ge 0,\ 1 \le i \le m$. Now since

$$\begin{aligned}&(-1)^m \Delta_{s_m}\left[\frac{\Delta_{\bar{s}_m}^{m-1} V(s;x)}{V(\bar{s}_m+1, s_m;x)}\right]\\ &+(-1)^m \Delta_{\bar{s}_m}^{m-1} V(s;x)\left[\frac{1}{V(\bar{s}_m+1, s_m;x)} - \frac{1}{V(s+1;x)}\right] = f(s)q(s),\end{aligned} \tag{6.4.25}$$

we obtain

$$(-1)^m \Delta_{s_m}\left[\frac{\Delta_{\bar{s}_m}^{m-1} V(s;x)}{V(\bar{s}_m+1, s_m;x)}\right] \le f(s)q(s). \tag{6.4.26}$$

In (6.4.26), keeping $\bar{s}_m$ fixed and setting $s_m = \ell_m$ and summing over $\ell_m = s_m$ to $\ell_m = x_m - 1$, we have

$$(-1)^{m+1}\left[\frac{\Delta_{\bar{s}_m}^{m-1} V(s;x)}{V(\bar{s}_m+1, s_m;x)}\right] \le \sum_{\ell_m=s_m}^{x_m-1} f(\bar{s}_m, \ell_m)q(\bar{s}_m, \ell_m).$$

Repeating the above arguments successively with respect to $s_{m-1}, \cdots, s_2$, we finally conclude

$$(-1)^{m+1}\left[\frac{\Delta_{\bar{s}_1} V(s;x)}{V(\bar{s}_1, s_1+1;x)}\right] \le \sum_{\ell_1=s_1}^{x_1-1} f(\bar{\ell}_1, s_1)q(\bar{\ell}_1, s_1),$$

which is the same as

$$V(s;x) \le \left[1 + \sum_{\ell_1=s_1}^{x_1-1} f(\bar{\ell}_1, s_1)q(\bar{\ell}_1, s_1)\right] V(\bar{s}-1, s_1+1;x).$$

Therefore, the above inequality easily gives us (6.4.24). $\square$

Corollary 6.4.2 (Agarwal [10]) *Let $V(s;x)$ be as in Theorem 6.4.1. Then for all $s \le x-1,\ (s;x) \in \mathbb{N}_0^m \times \mathbb{N}_0^m$,*

$$V(s;x) \le \min_{1\le i\le m} \left\{ \prod_{\ell_i=s_i}^{x_i-1} \left[1 + S_{\bar{\ell}_i=s_i}^{\bar{x}_i-1} f(\ell)q(\ell)\right] \right\}.$$

Theorem 6.4.5 (Agarwal [7, 10]) *Assume the following inequality (6.4.18) holds for all $x \in \mathbb{N}_0^m$. Then for all $x \in \mathbb{N}_0^m$,*

$$u(x) \le p(x) + q(x) S_{s=0}^{x-1} f(s)p(s) \min_{1\le i\le m} \left\{ \prod_{\ell_i=s_i+1}^{x_i-1} \left[1 + S_{\bar{\ell}_i=s_i+1}^{\bar{x}_i-1} f(\ell)q(\ell)\right] \right\}. \tag{6.4.27}$$

Remark 6.4.3 In fact, for $m = 1$, (6.4.27) is the same as (2.1.193) with a =0.

Corollary 6.4.3 *Let in Theorem 6.4.1, $p(x)$ be non-decreasing and $q(x) \ge 1$. Then for all $x \in \mathbb{N}_0^m$,*

$$u(x) \le p(x)q(x) \min_{1\le i\le m} \left\{ \prod_{\ell_i=0}^{x_i-1} \left[1 + S_{\bar{\ell}_i=0}^{\bar{x}_i-1} f(\ell)q(\ell)\right] \right\}. \tag{6.4.28}$$

Proof For such $p(x)$ and $q(x)$, inequality (6.4.19) gives us

$$\begin{aligned} u(x) &\le p(x)q(x)\left[1 + S_{s=0}^{x-1} f(s)q(s)V(s+1;x)\right] \\ &= p(x)q(x)\left[1 + S_{s=0}^{x-1} (-1)^m \Delta_s^m V(s;x)\right]. \end{aligned} \tag{6.4.29}$$

Now using $V(\bar{s}_i, x_i; x) = 1, \quad 1 \le i \le m$, we have

$$\begin{aligned} u(x) &\le p(x)q(x)\left[1 + (-1)^{2m-1} \sum_{s_1=0}^{x_1-1} \Delta_{s_1} V(s_1, 0, \ldots, 0; x)\right] \\ &= p(x)q(x)\left[1 + (-1)^{2m-1}\left(V(x_1, 0, \ldots, 0; x) - V(0;x)\right)\right] \\ &= p(x)q(x)V(0;x), \end{aligned} \tag{6.4.30}$$

which thus yields the inequality (6.4.28). □

Theorem 6.4.6 (Agarwal [10]) *Assume the following inequality (6.4.17) holds for all $x \in \mathbb{N}_0^m$. Then, for all $x \in \mathbb{N}_0^m$,*

$$u(x) \le p(x) + q(x) S_{s=0}^{x-1} \left[\sum_{i=0}^{r} \Delta_s^m E_i(s, p) \right] \times \min_{1 \le j \le m} \left\{ \prod_{\ell_j = s_j + 1}^{x_j - 1} \left[1 + S_{\ell_j = \bar{s}_j + 1}^{\bar{x}_j - 1} \sum_{i=1}^{r} \Delta_s^m E_i(\ell, q) \right] \right\}. \tag{6.4.31}$$

Furthermore, if $p(x)$ is non-decreasing and $q(x) \ge 1$, then for all $x \in \mathbb{N}_0^m$,

$$u(x) \le p(x) q(x) \min_{1 \le j \le m} \left\{ \prod_{\ell_j = 0}^{x_j - 1} \left[1 + \sum_{i=1}^{r} \Delta_{\ell_j} E_i(\bar{x}_j, \ell_j, q) \right] \right\}. \tag{6.4.32}$$

Remark 6.4.4 Results involving $V_j(s; x)$, $1 \le j \le r$, in Theorems 6.4.3 and 6.4.4 can be stated analogously.

Theorem 6.4.7 (Agarwal-Thandapani [10, 19]) *Assume the following inequality holds for all $x, X \in \mathbb{N}_0^m$ with $x \le X$,*

$$u(X) \ge u(x) - q(X) S_{\ell = x+1}^{X} f(\ell) u(\ell). \tag{6.4.33}$$

Then for all $x, X \in \mathbb{N}_0^m$ with $x \le X$,

$$u(X) \ge u(x) \left[\min_{1 \le j \le m} \left\{ \prod_{\ell_j = s_j + 1}^{X_j} \left[1 + q(X) S_{\ell_j = \bar{s}_j + 1}^{\bar{X}_j} f(\ell) \right] \right\} \right]^{-1}. \tag{6.4.34}$$

Proof With the transformation $x = X - \alpha$, $\ell = X - \beta$ where $0 \le \alpha, \beta \le X$, $\alpha, \beta \in \mathbb{N}_0^m$, inequality (6.4.33) can be written as

$$u(X) \ge u(X - \alpha) - q(X) S_{\beta = 0}^{\alpha_1} f(X - \beta) u(X - \beta).$$

Therefore, if $u(x - \alpha_1) = \bar{u}(\alpha_1)$, $f(X - \beta_1) = \bar{f}(\beta_1)$ where $0 \le \alpha_1, \beta_1 \le X$, then it follows that

$$\bar{u}(\alpha) \le u(X) + S_{\beta = 0}^{\alpha - 1} q(X) \bar{f}(\beta) \bar{u}(\beta). \tag{6.4.35}$$

Since the inequality (6.4.35) satisfies the hypotheses of Corollary 6.4.3, from (6.4.30) it follows that

$$\bar{u}(\alpha) \le u(X) V(0, \alpha), \tag{6.4.36}$$

where $V(\beta;\alpha)$ is the solution of the equation

$$V(\beta;\alpha) = 1 + S_{\tau=\beta}^{\alpha-1} q(X)\bar{f}(\tau)V(\tau+1;\alpha). \tag{6.4.37}$$

However, from Corollary 6.4.2 it follows

$$V(\beta;\alpha) \le \min_{1\le j\le m} \left\{ \prod_{\tau_j=\beta_j}^{\alpha_j-1} \left[1 + q(X) S_{\bar{\tau}_j=\bar{\beta}_j}^{\bar{\alpha}_j-1} \bar{f}(\tau)\right)\right] \right\}.$$

Using the above estimate in (6.4.36), we then obtain

$$\bar{u}(\alpha) \le \min_{1\le j\le m} u(X) \left\{ \prod_{\tau_j=0}^{\alpha_j-1} \left[1 + q(X) S_{\bar{\tau}_j=0}^{\bar{\alpha}_j-1} \bar{f}(\tau)\right)\right] \right\}$$

which yields

$$\begin{aligned} u(x) &\le u(X) \min_{1\le j\le m} \left\{ \prod_{\tau_j=0}^{X_j-x_j-1} \left[1 + q(X) S_{\bar{\tau}_j=0}^{\bar{X}_j-\bar{x}_j-1} f(X-\tau)\right] \right\} \\ &= u(X) \min_{1\le j\le m} \left\{ \prod_{\ell_j=x_j+1}^{X_j} \left[1 + q(X) S_{\bar{\ell}_j=\bar{x}_j+1}^{\bar{X}_j} f(\ell)\right] \right\}. \end{aligned}$$

□

In the sequel, we shall consider linear systems of m discrete inequalities involving functions of n independent variables, which are due to Agarwal [9]. The origin of these inequalities is so called Gronwall's type discrete inequalities. When $m = 1$, in [7] a discrete analogue of Riemann's function has been obtained and employed to establish some best possible inequalities. This approach is motivated by the continuous inequalities [4, 621] and easily provides Wendroff type explicit estimates. Further, this technique has the advantage that it requires fewer restrictions on the functions that appear in the inequalities than are needed in direct methods, see, e.g., [6, 20, 529, 620, 670, 671]. When $n = 1$, several comparison results for the nonlinear systems were proved in [2]. These results are very useful in establishing asymptotic behaviour, dependence on parameters, etc., for the systems of discrete equations. However, it does not provide explicit upper estimates on the unknown functions. For the general case m, n, nothing much seems to be known. We state a lemma which is needed and is of independent interest. Then, we provided the best possible upper estimates on the known m-vector function $u(x)$ satisfying (6.4.38). This estimates is in terms of solution of an $m \times m$ matrix summation equation (equivalent, a discrete analogue of $m \times m$ matrix Riemann's function), and is in fact a discrete analogue of the result proved in [128]. Third, we shall use the method of

splitting and the method of maxima respectively to obtain explicit upper estimates on $u(x)$ satisfying (6.4.38). These techniques have been used for the continuous inequalities in [58, 163, 292, 294, 639, 683]. Fourth, we use the best possible result and matrix norms to obtain explicit upper estimates on $u(x)$ satisfying (6.4.38). Last, we consider a more general linear inequality and show that the above methods can be used to obtain upper estimates on the function $u(x)$ satisfying (6.4.84).

A point $(x_1, \ldots, x_n)$ in $\mathbb{N}_0^n$ is denoted by x, whereas $\bar{x}_i$ represents $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ and $(\bar{x}_i, t)$ stands for $(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)$. For all $s, x \in \mathbb{N}_0^n, 0 \le s \le x$, i.e., $0 \le s_i < x_i, 1 \le i \le n$ and any $f(x)$ defined on $\mathbb{N}_0^n$, $\sum_{p=s}^{x-1} f(p)$ represents the n-fold sum $\sum_{p_1=s_1}^{x-1} f(p_1, \ldots, p_n)$ and $\triangle_p^n f(p)$ denotes $\triangle_{p_1} \cdots \triangle_{p_n} f(p_1, \ldots, p_n)$ where $\triangle$ is the forward difference operator $\triangle f(t) = f(t+1) - f(t)$. The empty sums and products are taken to be 0 and 1, respectively.

We shall need the following lemma.

Lemma 6.4.6 (Agarwal [9]) *Let the $m \times m$ matrix $A(x)$ be defined and non-negative on $\mathbb{N}_0^n$. Let the m-vector functions $h(x)$ and $u(x)$ be defined on $\mathbb{N}_0^n$. Further, let for all $x \in \mathbb{N}_0^n$ the following inequality hold*

$$\triangle_x^n u(x) \le A(x)u(x) + h(x),$$

where

$$u(\bar{x}_i, 0) = 0, \quad 1 \le i \le n.$$

Then for all $x \in \mathbb{N}_0^n$,

$$u(x) \le \sum_{s=0}^{x-1} V(s+1; x)h(s),$$

where the $m \times m$ matrix $V(s, x), s \le x - 1, (s, x) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ is a solution of

$$\begin{cases} (-1)^n \triangle_s^n V(s; x) = V(s+1; x)A(s), \\ V(\bar{s}_i, x_i; x) = I \quad (m \times m \text{ identity matrix}),\ 1 \le i \le n, \end{cases}$$

or equivalently,

$$V(s; x) = I + \sum_{p=s}^{x-1} V(p+1; x)A(p).$$

Proof We refer to the proof of Lemma 6.4.3 of [7]. □

Theorem 6.4.8 (Agarwal [9]) *Let the $m \times m$ matrices $G(x)$ and $H(x)$ be defined and non-negative on $\mathbb{N}_0^n$. Let the m-vector functions $a(x)$ and $u(x)$ be defined on $\mathbb{N}_0^n$. Further, let for all $x \in \mathbb{N}_0^n$, the following inequality holds,*

$$u(x) \le a(x) + G(x) \sum_{s=0}^{x-1} H(s)u(s). \tag{6.4.38}$$

Then for all $x \in \mathbb{N}_0^n$,

$$u(x) \le a(x) + G(x) \sum_{s=0}^{x-1} V(s+1;x)H(s)a(s), \tag{6.4.39}$$

where $V(s;x)$ satisfies

$$V(s;x) = I + \sum_{p=s}^{x-1} V(p+1;x)H(p)G(p). \tag{6.4.40}$$

Proof Define a m-vector function $\phi(x)$ such that

$$\phi(x) = \sum_{s=0}^{x-1} H(s)u(s), \tag{6.4.41}$$

then

$$\begin{cases} \triangle_x^n \phi(x) = H(x)u(x), \\ \phi(\bar{x}_i, 0) = 0, \quad 1 \le i \le n. \end{cases}$$

Since $u(x) \le a(x) + G(x)\phi(x)$, we obtain

$$\triangle_x^n \phi(x) \le H(x)a(x) + H(x)G(x)\phi(x).$$

Now applying Lemma 6.4.6 to the above inequality gives us

$$\phi(x) \le \sum_{s=0}^{x-1} V(s+1;x)H(s)a(s).$$

Therefore, using (6.4.41) in (6.4.38), (6.4.39) follows. □

Obviously, the equality in (6.4.38) implies the equality in (6.4.39).

Lemma 6.4.7 (Agarwal [7]) *Let $\phi_1(t)$ and $\phi_2(t)$ be defined on $\mathbb{N}_0$, then for all $t \in \mathbb{N}_0$,*

$$\sum_{\ell=0}^{t-1} \phi_1(\ell)\Delta\phi_2(\ell) = \phi_1(\ell)\phi_2(\ell)\Big|_{\ell=0}^{t} - \sum_{\ell=0}^{t-1} \Delta\phi_1(\ell)\phi_2(\ell+1).$$

Lemma 6.4.8 (Agarwal [7]) *Let $g(x)$ be defined on $\mathbb{N}_0^n$, then the function $v(s,x)$, $s \le x-1$, $(s,x) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ is a solution of*

$$(-1)^n \Delta_s^n v(s,x) = g(s)v(s+1,x) \tag{6.4.42}$$

for all $s_i = x_i$, $1 \le \imath \le n$,

$$v(s,x) = 1; \tag{6.4.43}$$

if and only if

$$v(s,x) = 1 + S_{p=s}^{x-1} g(p)v(p+1,x). \tag{6.4.44}$$

Lemma 6.4.9 (Agarwal [7]) *Let $g(x)$ and $h(x)$ be defined and non-negative on $\mathbb{N}_0^n$ and the following inequality holds,*

$$\Delta_x^n u(x) \le g(x)u(x) + h(x) \tag{6.4.45}$$

where

$$u(x_1, \cdots, x_{i-1}, 0, x_{i+1}, \cdots, x_n) = 0, 1 \le i \le n. \tag{6.4.46}$$

Then,

$$u(x) \le S_{s=0}^{x-1} h(s)v(s+1,x) \tag{6.4.47}$$

where $v(s,x)$ is the solution of problem (6.4.42)–(6.4.43).

Proof From (6.4.42) and (6.4.45), we have

$$S_{s=0}^{x-1} v(s+1,x)\Delta_s^n u(s) - S_{s=0}^{x-1} u(s)(-1)^n \Delta_s^n v(s,x) \le S_{s=0}^{x-1} h(s)v(s+1,x). \tag{6.4.48}$$

An application of Lemma 6.4.7 provides

$$\begin{aligned} &S_{s=0}^{x-1} u(s)(-1)^n \Delta_s^n v(s,x) \\ &= (-1)^n \sum_{s_1=0}^{x_1-1} \cdots \sum_{s_{n-1}=0}^{x_{n-1}-1} \Big[u(s)\Delta_{s_1\cdots s_{n-1}}^{n-1} v(s,x)\Big|_{s_n=0}^{x_n} \\ &\quad - \sum_{s_n=0}^{x_n-1} \Delta_{s_n} u(s) \Delta_{s_1\cdots s_{n-1}}^{n-1} v(s_1,\cdots,s_{n-1},s_n+1,x)\Big]. \end{aligned} \tag{6.4.49}$$

Using (6.4.43) and (6.4.46), the right-hand side of (6.4.49) reduces to

$$(-1)^{n+1} \sum_{s_n=0}^{x_n-1} \sum_{s_1=0}^{x_1-1} \cdots \sum_{s_{n-1}=0}^{x_{n-1}-1} \Delta_{s_n} u(s) \Delta_{s_1\cdots s_{n-1}}^{n-1} v(s_1,\cdots,s_{n-1},s_n+1,x).$$

Repeating the above arguments successively, we obtain

$$(-1)^{2n-1} \sum_{s_n=0}^{x_n-1} \cdots \sum_{s_2=0}^{x_2-1} \Big[\Delta_{s_n\cdots s_2}^{n-1} u(s) v(s_1,s_2+1,\cdots,s_n+1,x)\Big|_{s_1=0}^{x_1} - \sum_{s_1=0}^{x_1-1} \Delta_s^n u(s) v(s+1,x)\Big]$$

which is same as

$$(-1)^{2n-1} \sum_{s_n=0}^{x_n-1} \cdots \sum_{s_2=0}^{x_2-1} \Delta_{s_n\cdots s_2}^{n-1} u(x_1,s_2,\cdots,s_n) + (-1)^{2n} S_{s=0}^{x-1} \Delta_s^n u(s) v(s+1,x)$$

or

$$-u(x) + S_{s=0}^{x-1} \Delta_s^n u(s) v(s+1,x).$$

Substituting this in (6.4.48), the result (6.4.47) follows.

Remark 6.4.5 For all $g(x)$ and $h(x)$, the equality in (6.4.45) implies the equality in (6.4.47) and hence $v(s,x)$ the solution of problem (6.4.42)–(6.4.43) is a discrete analogue of Riemann's function.

In what follows, we shall assume that the functions which appear in the inequalities are real-valued, non-negative and defined on $\mathbb{N}_0^n$.

Theorem 6.4.9 (Agarwal [7]) *Let for all $x \in \mathbb{N}_0^n$, the following inequality be satisfied,*

$$\phi(x) \le a(x) + b(x) \sum_{r=1}^{m} E^r(x,\phi) \tag{6.4.50}$$

where

$$E^r(x,\phi) = S_{x^1=0}^{x-1} f_{r1}(x^1) S_{x^2=0}^{x^1-1} f_{r2}(x^2) \cdots S_{x^r=0}^{x^{r-1}-1} f_{rr}(x^r)\phi(x^r).$$

Then,

$$\phi(x) \le a(x) + b(x) S_{s=0}^{x-1}[\sum_{r=1}^{m} \Delta_s^n E^r(s,a)] v(s+1,x) \tag{6.4.51}$$

where $v(s,x)$ is the solution of

$$\begin{cases} (-1)^n \Delta_s^n v(s,x) = [\sum_{r=1}^{m} \Delta_s^n E^r(s,b)] v(s+1,x), \ s \le x-1, \\ v(s,x) = 1; \ s_i = x_i, l \le i \le n. \end{cases}$$

Proof Define a function $u(x)$ such that

$$u(x) = \sum_{r=1}^{m} E^r(x,\phi),$$

then

$$\Delta_x^n u(x) = \sum_{r=1}^{m} \Delta_x^n E^r(x,\phi). \tag{6.4.52}$$

Since $\phi(x) \le a(x) + b(x)u(x)$ and $u(x)$ is non-decreasing in x, from (6.4.52) we get successively

$$\begin{aligned} \Delta_x^n u(x) &\le \sum_{r=1}^{m} \Delta_x^n E^r(x, a+bu) \\ &= \sum_{r=1}^{m} \Delta_x^n E^r(x,a) + \sum_{r=1}^{m} \Delta_x^n E^r(x,bu) \\ &\le \sum_{r=1}^{m} \Delta_x^n E^r(x,a) + [\sum_{r=1}^{m} \Delta_x^n E^r(x,b)] u(x). \end{aligned} \tag{6.4.53}$$

Now an application of Lemma 6.4.9 provides

$$u(x) \le S_{s=0}^{x-1}[\sum_{r=1}^{m} \Delta_x^n E^r(s,b)]v(s+1,x). \tag{6.4.54}$$

Therefore (6.4.51) follows from (6.4.54) and the inequality $\phi(x) \le a(x) + b(x)u(x)$. □

Remark 6.4.6 For $m = 1$, (6.4.51) is best possible, i.e., the equality in (6.4.50) implies the equality in (6.4.51).

Theorem 6.4.10 (Agarwal [7]) *Let for all $x \in \mathbb{N}_0^n$, the following inequality be satisfied*

$$\phi(x) \le p_0(x) + \sum_{i=1}^{m} p_i(x) S_{s=o}^{x-1} q_i(s)\phi(s). \tag{6.4.55}$$

Then,

$$\phi \le F_m[p_0(x)] \tag{6.4.56}$$

where

$$\begin{aligned} F_i \;\; &= D_i D_{i-1} \cdots D_0, \\ D_0 w &= w, \\ D_j w &= w + (F_{j-1}[p_j]) S_{s=0}^{x-1} q_j(s) w(s) v_j(s+1,x) \end{aligned}$$

and $v_j(s,x)$, $1 \le j \le m$ are the solutions of

$$\begin{cases} (-1)^n \Delta_s^n v_j(s,x) = q_j(s) Fj - 1[p_j(s)] v_j(s+1,x), \quad s \le x-1, \\ v_j(s,x) = 1; \quad s_i = x_i, \quad 1 \le i \le n, \quad 1 \le j \le m. \end{cases} \tag{6.4.57}$$

Proof The proof is by finite induction. For $m = 1$, we have from Theorem 6.4.9 that

$$\phi \le p_0(x) + p_1(x) S_{s=0}^{x-1} q_1(s) p_0(s) v_1(s+1,x)$$

where $v_1(s,x)$ is the solution of (6.4.57) when $j = 1$, and hence (6.4.56) is true. Now assume that the result is true for some k, where $1 < k \le m-1$, then for all $k+1$, we have

$$\phi(x) \le [p_0(x) + p_{k+1}(x) S_{s=0}^{x-1} q_{k+1}(s)\phi(s)] + \sum_{i=1}^{k} p_i(x) S_{s=0}^{x-1} q_i(s)\phi(s)$$

which, along with (6.4.56), yields

$$\phi(x) \le F_k[p_0(x) + p_{k+1}(x)S_{s=0}^{x-1}q_{k+1}(s)\phi(s)].$$

Next, using the definition of F_k and the fact that $S_{s=0}^{x-1}q_{k+1}(s)\phi(s)$ is non-decreasing for all $x \in \mathbb{N}_0^n$, the above inequality can be written as

$$\begin{aligned}\phi(x) &\le F_k[p_0(x)] + F_k[p_{k+1}(s)S_{s=0}^{x-1}q_{k+1}(s)\phi(s)]\\ &\le F_k[p_0(x)] + F_k[p_{k+1}(x)]S_{s=0}^{x-1}q_{k+1}(s)\phi(s)\end{aligned}$$

and again an application of Theorem 6.4.9 provides

$$\begin{aligned}\phi(x) &\le F_k[p_0(x)] + F_k[p_{k+1}(x)]S_{s=0}^{x-1}q_{k+1}(s)F_k[p_0(s)]v_{k+1}(s,x)\\ &= F_{k+1}[p_0(x)].\end{aligned}$$

□

Corollary 6.4.4 (Agarwal [7]) *Let $v(s,x)$ be same as in Theorem 6.4.9. Then,*

$$v(s,x) \le \min_{1\le i\le n}\left\{\prod_{p_i=s_i}^{x_i-1}[1 + \sum_{p_1=s_1}^{x_1-1}\cdots\sum_{p_{i-1}=s_{i-1}}^{x_{i-1}-1}\sum_{p_{i+1}=s_{i+1}}^{x_{i+1}-1}\cdots\sum_{p_n=s_n}^{x_n-1}\sum_{r=1}^{m}\Delta_p^n E^r(p,b)]\right\}.$$

In the next result, the methods of splitting is used.

Theorem 6.4.11 (Agarwal [9]) *Let, in addition to hypotheses of Theorem 6.4.8, $u(x) \ge 0$ for all $x \in \mathbb{N}_0^n$. Then for all $x \in \mathbb{N}_0^n$,*

$$u_j(x) \le a_j(x) + g_j^*(x)r_0(x), \tag{6.4.58}$$

where

$$\begin{cases} g_j^*(x) = \max\limits_{1\le k\le m} g_{jk}(x), \quad 1 \le j \le m,\\ r_0(x) = \sum\limits_{s_1=0}^{x_1-1} a^*(\bar{x}_1, s_1)\Pi_{t_1=1+s_1}^{x_1-1}(1 + b(\bar{x}_1, t_1)),\end{cases} \tag{6.4.59}$$

and

$$\begin{cases} a^*(\bar{x}_1, x_1) = \sum\limits_{k=1}^{m}\sum\limits_{l=1}^{m}\sum\limits_{\bar{s}_1=0}^{\bar{x}_1-1} h_{kl}(\bar{s}_1, x_1)a_l(\bar{s}_1, x_1),\\ b(\bar{x}_1, x_1) = \max\limits_{1\le\alpha\le m}\sum\limits_{k=1}^{m}\sum\limits_{l=1}^{m}\sum\limits_{\bar{s}=0}^{\bar{x}_1-1} h_{kl}(\bar{s}, x_1)g_{l\alpha}(\bar{s}_1, x_1).\end{cases} \tag{6.4.60}$$

Proof In component form, inequality (6.4.38) is the same as

$$u_j(x) \le a_j(x) + \sum_{k=1}^{m} g_{jk}(x) r_k(x), \quad 1 \le j \le m, \tag{6.4.61}$$

where

$$r_k(x) = \sum_{l=1}^{m} \sum_{s=0}^{x-1} h_{kl}(s) u_l(s), \quad 1 \le k \le m. \tag{6.4.62}$$

Define

$$r(x) = \sum_{k=1}^{m} r_k(x). \tag{6.4.63}$$

Then it follows from by (6.4.62) and (6.4.61) that

$$\begin{aligned}
\triangle_{x_1} r(x) &= \sum_{k=1}^{m} \triangle_{x_1} r_k(x) \\
&= \sum_{k=1}^{m} \sum_{l=1}^{m} \sum_{\bar{s}_1=0}^{\bar{x}_1-1} h_{kl}(\bar{s}_1, x_1) u_l(\bar{s}_1, x_1) \\
&\le \sum_{k=1}^{m} \sum_{l=1}^{m} \sum_{\bar{s}_1=0}^{\bar{x}_1-1} h_{kl}(\bar{s}_1, x) \left[a_l(\bar{s}_1, x_1) + \sum_{\alpha=1}^{m} g_{l\alpha}(\bar{s}_1, x_1) r_\alpha(\bar{s}_1, x_1) \right] \\
&\le a^*(\bar{x}_1, x_1) + \sum_{\alpha=1}^{m} r_\alpha(x) \sum_{k=1}^{m} \sum_{l=1}^{m} \sum_{\bar{s}_1=0}^{\bar{x}_1-1} h_{kl}(\bar{s}_1, x_1) g_{l\alpha}(\bar{s}_1, x_1),
\end{aligned}$$

since each $r_\sim$ is an increasing function of each of its component variables.

Using (6.4.61) and (6.4.59), we derive

$$\triangle_{x_1} r(x) \le a^*(\bar{x}_1, x_1) + b(\bar{x}_1, x_1) r(x). \tag{6.4.64}$$

Summing the inequality (6.4.64) with respect to x_1 for arbitrary $\bar{x}_1 \ge 0$ (e.g., see, [4]), we conclude

$$r(x) \le r_0(x). \tag{6.4.65}$$

Now from (6.4.61) and (6.4.59), it follows

$$u_j(x) \le a_j(x) + g_j^*(x)\sum_{k=1}^{m} r_k(x)$$
$$= a_j(x) + g_j^*(x)r(x). \tag{6.4.66}$$

Therefore, using (6.4.65) in (6.4.66), we can get the required inequality (6.4.58). □

In the next result, we use the method of maxima. In addition to hypotheses of Theorem 6.4.11, we shall assume that $a(x) \ge 0$ for all $x \in \mathbb{N}_0^n$. Taking maxima in (6.4.61) over $1 \le j \le m$ to obtain

$$u(x) \le a(x) + \sum_{k=1}^{m} g_k(x)r_k(x), \tag{6.4.67}$$

where

$$u(x) = \max_{1\le j\le m} u_j(x),\ a(x) = \max_{1\le j\le m} a_j(x),\ g_k(x) = \max_{1\le j\le m} g_{jk}(x). \tag{6.4.68}$$

Since $h_{kl}(x) \ge 0$, from (6.4.62) it follows

$$r_k(x) \le \sum_{l=1}^{m}\sum_{s=0}^{x-1} h_{kl}(s)u(s) = \sum_{s=0}^{x-1} h_k(s)u(s), \tag{6.4.69}$$

where

$$h_k(x) = \sum_{l=1}^{m} h_{kl}(x). \tag{6.4.70}$$

Inserting (6.4.69) into (6.4.67), we get

$$u(x) \le a(x) + \sum_{k=1}^{m} g_k(x)\sum_{s=0}^{x-1} h_k(s)u(s). \tag{6.4.71}$$

For the inequality (6.4.71), Agarwal [7] have obtained recursive upper bounds for $u(x)$ in Theorem 6.4.10; however, these bounds are in terms of a discrete analogue of Riemann's function which is generally not known. We remark that recursive Wendroff type estimates are possible; however, these are rather complicated. To obtain such an easy estimate, we define

$$h(x) = \max_{1\le k\le m} h_k(x), \qquad g(x) = \sum_{k=1}^{m} g_k(x), \tag{6.4.72}$$

so that (6.4.71) can be written as

$$u(x) \leq a(x) + g(x) \sum_{s=0}^{x-1} h(s)u(s). \tag{6.4.73}$$

In fact, inequality (6.4.73) is exactly the same as (6.4.39) with $m = 1$ and hence from Theorem 6.4.8, it follows

$$u(x) \leq a(x) + g(x) \sum_{s=0}^{x-1} v(s+1; x)h(s)a(s). \tag{6.4.74}$$

In Corollary 6.4.4, an upper estimate on the function $v(s, x)$ can be obtained as

$$v(s; x) \leq \min_{1 \leq j \leq n} \left\{ \Pi_{p_i = s_i}^{x_i - 1} [1 + \sum_{\bar{p}_i = \bar{s}_i}^{\bar{x}_1 - 1} h(p)g(p) \right\}. \tag{6.4.75}$$

Using (6.4.75) in (6.4.74), we obtain an explicit upper estimate on $u(x)$

$$u(x) \leq a(x) + g(x) \sum_{s=0}^{x-1} h(s)a(s) \min_{1 \leq i \leq n} \left\{ \Pi_{p_i = s_i}^{x_i - 1} [1 + \sum_{\bar{p}_i = \bar{s}_i}^{\bar{x}_1 - 1} h(p)g(p)] \right\}. \tag{6.4.76}$$

We summarize the above considerations in the following theorem.

Theorem 6.4.12 (Agarwal [9]) *Let the $m \times m$ matrices $G(x), H(x)$ and the m-vector functions $a(x), u(x)$ be defined and non-negative on $\mathbb{N}_0^n$. Further, let for all $x \in \mathbb{N}_0^n$, the inequality (6.4.38) be hold. Then for all $x \in \mathbb{N}_0^n$, the inequality (6.4.76) holds, where $g(x), h(x), a(x)$, and $u(x)$ are defined in (6.4.72), (6.4.71) and (6.4.68), respectively.*

Theorem 6.4.13 (Agarwal [9]) *Let, in addition to hypotheses of Theorem 6.4.12, $a(x)$ be non-decreasing and $g(x) \geq 1$ for all $x \in \mathbb{N}_0^n$. Then for all $x \in \mathbb{N}_0^n$,*

$$u(x) \leq a(x)g(x) \min_{1 \leq i \leq n} \left\{ \prod_{p_i=0}^{x_i-1} [1 + \sum_{\bar{p}_i=0}^{\bar{x}_i-1} h(p)g(p)] \right\}. \tag{6.4.77}$$

Proof Since $a(x)$ is non-decreasing and $g(x) \geq 1$, (6.4.74) gives us

$$u(x) \leq a(x)g(x) \left[1 + \sum_{s=0}^{x-1} v(s+1; x)h(s)g(s) \right]$$

which, with (6.4.40), gives us

$$u(x) \leq a(x)g(x)v(0; x).$$

Therefore (6.4.77) now follows from (6.4.75). □

Theorem 6.4.14 (Agarwal [9]) *Let the $m \times m$ matrices $G(x)$ and $H(x)$ be defined and non-negative on $\mathbb{N}_0^n$. Let the m-vector functions $a(x)$ and $u(x)$ be defined and $a(x) \leq 0$ on $\mathbb{N}_0^n$. Further, let for all $x \in \mathbb{N}_0^n$, the inequality (6.4.38) be hold. Then, for all $x \in \mathbb{N}_0^n$, the inequality*

$$u_j(x) \leq a_j(x) + \sum_{i=1}^{m} g_{ji}(x) \sum_{s=0}^{x-1} \left(\sum_{k=1}^{m} \sum_{l=1}^{m} h_{kl}(s) a_l(s) \right)$$
$$\times \min_{1 \leq i \leq n} \left\{ \Pi_{p_i=s_i}^{x_i-1} [1 + \sum_{\bar{p}_i=\bar{s}_i}^{\bar{x}_i-1} \|H(p)\| \|G(p)\|] \right\} \tag{6.4.78}$$

holds, where $\|G\|$ is any $m \times m$ matrix norm such that $|g_{ij}| \leq \|G\|$.

Proof Let $\|G\|$ denote any $m \times m$ matrix norm which, in addition to the usual conditions, also satisfies $\|g_{ij}\| \leq \|G\|$ for all $1 \leq i, j \leq m$.

In component form, inequality (6.4.39) is the same as

$$u_j(x) \leq a_j(x) + \sum_{i=1}^{m} \sum_{k=1}^{m} \sum_{l=1}^{m} g_{ji}(x) \sum_{s=0}^{x-1} v_{ik}(s+1; x) h_{kl}(s) a_l(s). \tag{6.4.79}$$

Hence, if all $g_{ji}(x), h_{kl}(x)$, and $a_l(x)$ are defined and non-negative for all $x \in \mathbb{N}_0^n$, then it follows that

$$u_j(x) \leq a_j(x) + \sum_{i=1}^{m} \sum_{k=1}^{m} \sum_{l=1}^{m} g_{ji}(x) \sum_{s=0}^{x-1} \|V(s+1; x)\| h_{kl}(s) a_l(s)$$

or

$$u_j(x) \leq a_j(x) + \sum_{i=1}^{m} g_{ji}(x) \sum_{s=0}^{x-1} \left(\sum_{k=1}^{m} \sum_{l=1}^{m} h_{kl}(s) a_l(s) \right) \|V(s+1; x)\|. \tag{6.4.80}$$

Next, from (6.4.30) we derive

$$\|V(s; x)\| \leq 1 + \sum_{p=s}^{x-1} \|V(p+1; x)\| \|H(p)\| \|G(p)\| \tag{6.4.81}$$

which is a one-dimensional inequality. Hence, Corollary 6.4.2 yields that

$$\|V(s;x)\| \le \min_{1\le i\le n}\left\{\Pi_{p_i=s_i}^{x_i-1}[1+\sum_{\bar{p}_i=\bar{s}_i}^{\bar{x}_1-1}\|H(p)\|\|G(p)\|]\right\}. \tag{6.4.82}$$

Inserting (6.4.82) into (6.4.80), we then obtain (6.4.78). Therefore, the proof is complete □

We also note that if $\|u\|$ is any m-vector norm and $\|G\|$ is the $m\times m$ matrix compatible norm, and the conditions of Theorem 6.4.12 are satisfied, then we conclude

$$\|u(x)\| \le \|a(x)\| + \|G(x)\|\sum_{s=0}^{x-1}\min_{1\le i\le n}\left\{\coprod_{p_i=s_i}^{x_i-1}\left[1+\sum_{\bar{p}_i=\bar{s}_i}^{\bar{x}_i-1}\|H(p)\|\|G(p)\|\right]\right\}\|H(s)\|\|a(s)\|. \tag{6.4.83}$$

The next result is a general linear inequality.

Theorem 6.4.15 (Agarwal [9]) *Let the $m\times m$ matrix $K(x,s)$ be defined and non-negative for all $(x,s)\in\mathbb{N}_0^n\times\mathbb{N}_0^n$. Let the m-vector functions $a(x)$ and $u(x)$ be defined and non-negative on $\mathbb{N}_0^n$. Further, let for all $x\in\mathbb{N}_0^n$, the following inequality holds,*

$$u(x) \le a(x) + \sum_{s=0}^{x-1}K(x,s)u(s). \tag{6.4.84}$$

Then for all $x\in\mathbb{N}_0^n$,

$$u(x) \le \left[1+\sum_{s=0}^{x-1}V(s+1;x)K(x,s)\right]a(x), \tag{6.4.85}$$

where $a(x)=\sup\{a(p):0\le p\le x\}, K(x,s)=\sup\{K(p,s):0\le p\le x\}$ and $V(s;x)$ satisfies

$$V(s;x) = I + \sum_{s=0}^{x-1}V(s+1;x)K(x,p). \tag{6.4.86}$$

Proof For any fixed point X in $\mathbb{N}_0^n$, it follows that for all $0\le x\le X$,

$$u(x) \le a(X) + \sum_{s=0}^{x-1}K(X,s)u(s). \tag{6.4.87}$$

Thus, Theorem 6.4.8 implies that for all $0 \le x \le X$,

$$u(x) \le \left[I + \sum_{s=0}^{x-1} V(s+1; x, X) K(X, s) \right] a(X), \tag{6.4.88}$$

where $V(s; x, X)$ satisfies

$$V(s; x, X) = I + \sum_{p=s}^{x-1} V(p+1; x, X) K(X, p). \tag{6.4.89}$$

In particular, (6.4.88) holds for $x = X$. Replacing X by x in (6.4.88) and (6.4.89) and noting that $V(s; x, x) = V(s; x)$, we can get the desired inequality (6.4.85). $\square$

Remark 6.4.7 The above techniques can be employed to inequality (6.4.87) to obtain explicit upper estimates on the function $u(x)$ satisfying (6.4.84).

6.4.3 Linear Multi-Dimensional Inequalities

In this section, we shall introduce some multi-dimensional linear inequalities.

The multidimensional version of Lemma 6.4.3 can be stated in the following lemma.

Lemma 6.4.10 (Agarwal [8, 10]) *Let the $n \times n$ matrix $A(x)$ be defined and non-negative on $\mathbb{N}_0^m$. Let n vector functions $\mathcal{H}(x)$ and $U(x)$ be defined on $\mathbb{N}_0^m$. Furthermore, let the following inequality hold, for all $x \in \mathbb{N}_0^m$,*

$$\Delta_x^m U(x) \le A(x) U(x) + \mathcal{H}(x)$$

where

$$U(\bar{x}_i, 0) = 0, \quad 1 \le i \le m.$$

Then for all $x \in \mathbb{N}_0^m$,

$$U(x) \le S_{s=0}^{x-1} V(s+1, x) \mathcal{H}(s)$$

where the $n \times n$ matrix $V(s; x)$, $s \le x-1$, $(s; x) \in \mathbb{N}_0^m \times \mathbb{N}_0^m$ is a solution of

$$\begin{cases} (-1)^m \Delta_s^m V(s; x) = V(s+1; x) A(s), \\ V(\bar{s}_i, x_i; x) = 1, \quad 1 \le i \le m, \end{cases} \tag{6.4.90}$$

or equivalently,

$$V(s;x) = 1 + S_{\ell=s}^{x-1} V(\ell+1;x)A(\ell).$$

Theorem 6.4.16 (Agarwal [8, 10]) *Let the $n \times n$ matrices $G(x)$ and $H(x)$ be defined and non-negative on $\mathbb{N}_0^m$ and the n vector functions $P(x)$ and $U(x)$ be defined on $\mathbb{N}_0^m$. Furthermore, let the following inequality hold, for all $x \in \mathbb{N}_0^m$,*

$$U(x) \le P(x) + G(x) S_{s=0}^{x-1} H(s)U(s). \tag{6.4.91}$$

Then for all $x \in \mathbb{N}_0^m$,

$$U(x) \le P(x) + G(x) S_{s=0}^{x-1} V(s+1;x)H(s)P(s), \tag{6.4.92}$$

where $V(s;x)$ satisfies

$$V(s;x) = 1 + S_{s=0}^{x-1} V(\ell+1;x)H(\ell)P(\ell). \tag{6.4.93}$$

Proof The proof is similar to that of Theorem 6.4.1. □

Theorem 6.4.17 (Agarwal [10]) *Let in addition to hypotheses of Theorem 6.4.16, $U(x) \ge 0$ for all $x \in \mathbb{N}_0^m$. Then for all $x \in \mathbb{N}_0^m$,*

$$u_i(x) \le p_i(x) + \max_{1 \le j \le n} g_{ij}(x) q(x) \tag{6.4.94}$$

where

$$q(x) = \sum_{s_1=0}^{x_1-1} \alpha(\bar{x}_1, s_1) \prod_{\ell_1=s_1+1}^{x_1-1} (1 + \beta(\bar{x}_1, \ell_1))$$

and

$$\begin{cases} \alpha(\bar{x}_1, x_1) = \displaystyle\sum_{j,r=1}^{n} S_{\bar{s}_1=0}^{\bar{x}_1-1} h_{jr}(\bar{s}_1, x_1) p_r(\bar{s}_1, x_1), \\ \beta(\bar{x}_1, x_1) = \displaystyle\max_{1 \le \tau \le n} \sum_{j,r=1}^{n} S_{\bar{s}_1=0}^{\bar{x}_1-1} h_{jr}(\bar{s}_1, x_1) g_{r\tau}(\bar{s}_1, x-1). \end{cases}$$

Proof The proof is similar to that of Theorem 2.2.6. □

Theorem 6.4.18 (Agarwal [8, 10]) *Let in addition to hypothesis of Theorem 6.4.17, $P(x) \geq 0$ for all $x \in \mathbb{N}_0^m$. Then for all $x \in \mathbb{N}_0^m$,*

$$u^*(x) \leq p^*(x) + g^*(x) S_{s=0}^{x-1} h^*(s) p(s) \times \min_{1 \leq j \leq m} \left\{ \prod_{\ell_j = s_j + 1}^{x_j - 1} \left[1 + S_{\bar{\ell}_j = \bar{s}_j + 1}^{\bar{x}_j - 1} h^*(\ell) g^*(\ell) \right] \right\} \tag{6.4.95}$$

where

$$\begin{cases} u^*(x) = \max\limits_{1 \leq i \leq n} u_i(x), \quad p^*(x) = \max\limits_{1 \leq i \leq n} p_i(x), \\ g^*(x) = \sum\limits_{j=1}^{n} \left[\max\limits_{1 \leq i \leq n} g_{ij}(x) \right], \; h^*(x) = \max\limits_{1 \leq j \leq n} \left[\sum\limits_{r=1}^{n} h_{jr}(x) \right]. \end{cases}$$

Proof The proof is similar to that of Theorem 2.2.7. □

Theorem 6.4.19 (Agarwal [8, 10]) *Let the hypotheses of Theorem 6.4.18 hold. Then for all $x \in \mathbb{N}_0^m$,*

$$u_j(x) \leq p_j(x) + \sum_{i=1}^{n} g_{ji}(x) S_{s=0}^{x-1} \left[\sum_{r=1}^{n} \sum_{\eta=1}^{n} h_{r\eta}(s) p_\eta(s) \right] \times \min_{1 \leq k \leq m} \left\{ \prod_{\ell_k = s_k + 1}^{x_k - 1} \left[1 + S_{\bar{\ell}_k = \bar{s}_k + 1}^{\bar{x}_k - 1} \|H(\ell)\| \|G(\ell)\| \right] \right\} \tag{6.4.96}$$

where $\|G\|$ is any $n \times n$ matrix norm such that $|g_{ij}| \leq \|G\|$.

Proof In component form, the inequality (6.4.92) is the same as

$$u_j(x) \leq p_j(x) + \sum_{i=1}^{n} \sum_{\tau=1}^{n} \sum_{\eta=1}^{n} g_{ji}(x) S_{s=0}^{x-1} v_{i\tau}(s+1; x) h_{\tau\eta}(s) p_\eta(s).$$

Hence it follows that

$$\begin{aligned} u_j(x) &\leq p_j(x) + \sum_{i=1}^{n} \sum_{\tau=1}^{n} \sum_{\eta=1}^{n} g_{ji}(x) S_{s=0}^{x-1} \|V(s+1; x)\| h_{\tau\eta}(s) p_\eta(s) \\ &= p_j(x) + \sum_{i=1}^{n} S_{s=0}^{x-1} \left[\sum_{\tau=1}^{n} \sum_{\eta=1}^{n} h_{\tau\eta}(s) p_\eta(s) \right] \|V(s+1; x)\|. \end{aligned} \tag{6.4.97}$$

Next from (6.4.93), we can derive

$$\|V(s;x)\| \leq 1 + S_{\ell=s}^{x-1}\|V(\ell+1;x)\|\|H(\ell)\|\|G(\ell)\|$$

which is a one-dimensional inequality. Hence, Corollary 6.4.2 gives us that

$$\|V(s;x)\| \leq \min_{1\leq k\leq m}\left\{\prod_{\ell_k=s_k}^{x_k-1}\left[1 + S_{\bar{\ell}_k=\bar{s}_k}^{\bar{x}_k-1}\|H(\ell)\|\|G(\ell)\|\right]\right\}. \tag{6.4.98}$$

Inserting (6.4.98) into (6.4.97), thus (6.4.96) follows. □

Remark 6.4.8 (Agarwal [10]) Let $\|U\|$ be any vector norm and $\|G\|$ be the matrix compatible norm, and the conditions of Theorem 6.4.18 hold, then

$$\|U(x)\| \leq \|P(x)\| + \|G(x)\|S_{s=0}^{x-1}\|H(s)\|\|P(s)\|$$
$$\times \min_{1\leq j\leq m}\left\{\prod_{\ell_j=s_j+1}^{x_j-1}\left[1 + S_{\bar{\ell}_j=\bar{s}_j+1}^{\bar{x}_j-1}\|H(\ell)\|\|G(\ell)\|\right]\right\}. \tag{6.4.99}$$

Theorem 6.4.20 (Agarwal [8, 10]) *Let the $n\times n$ matrix $K(x,s)$ be defined and non-negative on $\mathbb{N}^m\times\mathbb{N}_0^m$. Let n vector functions $P(x)$ and $U(x)$ be defined and non-negative on $\mathbb{N}_0^m$. Furthermore, let the following inequality hold, for all $x\in\mathbb{N}_0^m$,*

$$U(x) \leq P(x) + S_{s=0}^{x-1}K(x,s)U(s). \tag{6.4.100}$$

Then for all $x\in\mathbb{N}_0^m$,

$$U(x) \leq \left[1 + S_{s=0}^{x-1}V(s+1;x)K^*(x,s)\right]P^*(x) \tag{6.4.101}$$

where $P^*(x) = \sup\{P(\ell) : 0\leq \ell\leq x\}$, $K^*(x,s) = \sup\{K(\ell,s) : 0\leq\ell\leq x\}$, and $V(s;x)$ satisfies

$$V(s;x) = 1 + S_{\ell=s}^{x-1}V(\ell+1;x)K^*(x,\ell). \tag{6.4.102}$$

Proof For any fixed point X in $\mathbb{N}_0^m$, it follows that for all $0\leq x\leq X$,

$$U(x) \leq P^*(X) + S_{s=0}^{x-1}K^*(X,s)U(s).$$

Thus Theorems 6.4.16 implies that for all $0\leq x\leq X$,

$$U(x) \leq \left[1 + S_{s=0}^{x-1}V(s+1;x)K^*(X,s)\right]P^*(X), \tag{6.4.103}$$

where

$$V(s;x) = 1 + S_{s=0}^{x-1} V(\ell+1;x) K^*(X,\ell). \tag{6.4.104}$$

In particular, (6.4.103) and (6.4.104) hold for $x = X$. Thus, replacing X by x in the resulting equation (6.4.103) and (6.4.104), we can get (6.4.101). ☐

Theorem 6.4.21 (Agarwal [10]) *Assume the inequality (6.4.13) holds for all $x \in \mathbb{N}_0^m$, and $p(x)$ and $q(x)$ be non-decreasing for all $x \in \mathbb{N}_0^m$. Then for all $x \in \mathbb{N}_0^m$,*

$$u(x) \le p(x) \min_{1\le j\le m} \left\{ \prod_{\ell_j=0}^{x_j-1} \left[1 + q(x) S_{\bar{\ell}_j=0}^{\bar{x}_j-1} f(\ell)\right] \right\}.$$

Theorem 6.4.22 (Agarwal [10]) *Assume the function $f(k,\ell,u,v)$ is defined for all $k,\ell \in \mathbb{N}$, and $u, v \in \mathbb{R}$, and non-decreasing in u, v. Further, let the functions $\phi(k,\ell)$ and $\psi(k,\ell)$ be defined for all $k,\ell \in \mathbb{N}$, and satisfy the inequalities*

$$\begin{cases} \phi(k+1,\ell) \le f(k,\ell,\phi(k,\ell)), \\ \psi(k+1,\ell) \ge f(k,\ell,\psi(k,\ell),\psi(\ell,k)), \\ \phi(0,\ell) \le \psi(0,\ell). \end{cases}$$

Then for all $k,\ell \in \mathbb{N}$,

$$\phi(k,\ell) \le \psi(k,\ell). \tag{6.4.105}$$

Theorem 6.4.23 (Agarwal [10]) *Assume the following inequality holds for all $x,\ X-1 \in \mathbb{N}_0^m$ with $x \le X-1$,*

$$u(X) \ge u(x) - q(X) S_{\ell=x}^{X-1} f(\ell) u(\ell). \tag{6.4.106}$$

Then for all $x,\ X-1 \in \mathbb{N}_0^m$ with $x \le X-1$,

$$u(X) \ge u(x) \prod_{\ell_1=x_1}^{X_1-1} \left[1 - q(X) S_{\bar{\ell}_1=\bar{x}_1}^{\bar{X}_1-1} f(\ell)\right] \tag{6.4.107}$$

as long as $1 - q(X) S_{\bar{\ell}_1=\bar{x}_1}^{\bar{X}_1-1} f(\ell) > 0$.

Theorem 6.4.24 (Agarwal [10]) *Assume the following inequality holds for all $x,\ X \in \mathbb{N}_0^m$ with $x \le X$,*

$$u(X) \ge u(x) - q(X) W^{-1}\left[S_{\ell=x+1}^{X} f(\ell) W(u(\ell))\right] \tag{6.4.108}$$

where the function W is positive, increasing, convex and sub-multiplicative on $(0, +\infty)$ *and* $\lim_{u\to+\infty} = +\infty$. *Then for all* $c,\ X \in \mathbb{N}_0^m$ *with* $x \le X$,

$$u(X) \ge \alpha(X)W^{-1}\Big[\alpha^{-1}(X)W(u(x))\prod_{\ell_1=x_1+1}^{X_1}\Big[(1+\beta(X)W(q(X)\beta^{-1}(X)))$$

$$\times S_{\bar{\ell}_1=\bar{x}_1+1}^{\bar{X}_1}f(\ell)\Big]^{-1}\Big], \tag{6.4.109}$$

where the functions $\alpha(x)$ *and* $\beta(x)$ *are positive and* $\alpha(x)+\beta(x) = 1$ *for all* $x \in \mathbb{N}_0^m$.

Theorem 6.4.25 (Agarwal [10]) *Assume the following inequality holds for all* $x,\ X \in \mathbb{N}_0^m$ *with* $x \le X$,

$$u(X) \ge u(x) - q(X)S_{\ell=x+1}^{X}f(\ell)W(u(\ell)) \tag{6.4.110}$$

where the function W is continuous, positive and non-decreasing on $[0, +\infty)$. *Then for all* $x,\ X \in \mathbb{N}_0^m$ *with* $x \le X$,

$$u(X) \ge G^{-1}\left[G(u(x)) - q(X)S_{\ell=x+1}^{X}f(\ell)\right] \tag{6.4.111}$$

where

$$G(w) = \int_{W_0}^{W}\frac{dt}{W(t)},\ W > 0$$

and arbitrary $W_0 \ge 0$ *as long as*

$$G(u(x)) - q(X)S_{\ell=x+1}^{X}f(\ell) \in Dom\ (G^{-1}).$$

Theorem 6.4.26 (Agarwal [10]) *Assume the following inequality holds for all* $(k, \ell) \in \mathbb{N}\times\mathbb{N}$,

$$\Delta_k^r\Delta_\ell^r u(k,\ell) \le p(k) + q(\ell) + \sum_{i=0}^{r}\sum_{\tau=0}^{k-1}\sum_{\eta=0}^{\ell-1}h(\tau,\eta)\Delta_\tau^i\Delta_\eta^i u(\tau,\eta), \tag{6.4.112}$$

where $\Delta_k p(k) \ge 0,\ \Delta_\ell q(\ell) \ge 0,\ p(0) = q(0)$. *Then for all* $(k,\ell) \in \mathbb{N}\times\mathbb{N}$,

$$\Delta_k^r\Delta_\ell^r u(k,\ell) \le B_i(k,\ell), \quad 1 \le i \le r+1, \tag{6.4.113}$$

where

$$B_1(k,\ell) = [p(k)+q(\ell)]\prod_{\tau=0}^{k-1}\left[1 + \sum_{\eta=0}^{\ell-1}[h(\tau,\eta)+r]\right]$$

and

$$B_i(k,\ell) = [p(k) + q(\ell)] + \sum_{\tau=0}^{k-1}\sum_{\eta=0}^{\ell-1} h((\tau,\eta) + (r-i+1))B_{i-1}(\tau,\eta), \quad 2 \le i \le r+1.$$

Theorem 6.4.27 (Agarwal [10]) *Assume that $U(x, T(\mathcal{A}))$ is the solution of $\Delta_x^m \mathcal{U}(x) = \mathcal{F}(x, \mathcal{U}(x))$, $\Delta_x^m \mathcal{V}(x) = \mathcal{F}(x, \mathcal{V}(x))$ where $T(\mathcal{A})$ denotes the term $\sum_{i=1}^m (-1)^{i+1} \sum_i \mathcal{A}([\bar{x}_i])$. Furthermore, let $\mathcal{V}(x,0)$ be the solution of the problem $\Delta_x^m \mathcal{V}(x) = F(x, \mathcal{V}(x))$, $\mathcal{V}((i)x) = 0$, where the function $F(x, \mathcal{V})$ for all $0 \le x \le X$, $\mathcal{V} \in \mathbb{R}_+^n$ is defined as $F(x,\mathcal{V}) = \sup_{|u-T(\mathcal{A})| \le \mathcal{V}} |\mathcal{F}(x, U)|$. Then for all x, $0 \le x \le X$,*

$$|U(x, T(\mathcal{A})) - T(\mathcal{A})| \le \mathcal{V}(x, 0). \tag{6.4.114}$$

Theorem 6.4.28 (Agarwal [10]) *Assume that the following conditions hold*

(i) For all $x, 0 \le x \le X$ and $\mathcal{U}, \mathcal{V} \in \mathbb{R}^n$,

$$|\mathcal{F}(x, \mathcal{U}(x)) - \mathcal{F}(x, \mathcal{U}(x))| \le \mathcal{G}(x, |\mathcal{U} - \mathcal{V}|),$$

where the function $\mathcal{G}(x, \mathcal{W})$ is defined for all $x, 0 \le x \le X, x \in \mathbb{R}_+^n$; and for all fixed $x, 0 \le x \le X$, and $1 \le i \le n$, the function $g_i(x, w_1, \cdots, w_n)$ is non-decreasing with respect to $w_1, \cdots, w_n$.

(ii) There exist functions $\mathcal{U}^1(x), \mathcal{U}^2(x), \mathcal{Z}^1(x)$ and $\mathcal{Z}^2(x)$ which are defined for all $x, 0 \le x \le X$ and satisfy the inequalities

$$|\Delta_x^m \mathcal{U}^1(x) - \mathcal{F}(x, \mathcal{U}^1(x))| \le \mathcal{Z}^1(x),$$

and

$$|\Delta_x^m \mathcal{U}^2(x) - \mathcal{F}(x, \mathcal{U}^2(x))| \le \mathcal{Z}^2(x)$$

(iii) $\mathcal{U}(x)$is a solution of the difference equation

$$\Delta_x^m \mathcal{U}(x) = \mathcal{G}(x, \mathcal{U}(x)) + \mathcal{Z}^1(x) + \mathcal{Z}^2(x)$$

which satisfies the inequality

$$|\sum_{i=1}^m (-1)^{i+1} \sum_i (\mathcal{U}^1((i)x) - \mathcal{U}^2((i)x))| \le \sum_{i=1}^m (-1)^{i+1} \sum_{i=1}^m \mathcal{U}((i)x).$$

Then for all $x, 0 \le x \le X$,

$$|\mathcal{U}^1(x) - \mathcal{U}^2(x)| \le \mathcal{U}(x).$$

Theorem 6.4.29 (Agarwal [10]) *Assume that condition (i) of Theorem 6.4.28 holds, and $U(x, T(\mathcal{A}))$ is the same as those in Theorem 6.4.27. Furthermore, assume that $U(x, T(\mathcal{B}))$ is the solution of $\Delta_x^m \mathcal{U}(x) = \mathcal{F}(x, \mathcal{U}(x))$ satisfying $U((i)x) = \mathcal{B}([\bar{x}_i])$. Then for all x, $0 \le x \le X$,*

$$|U(x, T(\mathcal{A})) - U(x, T(\mathcal{B})) \le \mathcal{V}(x), \tag{6.4.115}$$

where $\mathcal{V}(x)$ is a solution of $\Delta_x^m \mathcal{V}(x) = \mathcal{G}(x, \mathcal{V}(x))$ satisfying

$$|T(\mathcal{A}) - T(\mathcal{B})| \le T(\mathcal{V}). \tag{6.4.116}$$

Theorem 6.4.30 (Agarwal [10]) *Assume that for all x, $0 \le x \le X, \mathcal{U} \in \mathbb{R}^n$,*

$$|\mathcal{F}(x, \mathcal{U})| \le \mathcal{G}(x, |\mathcal{U}|) \tag{6.4.117}$$

where the function $\mathcal{G}(x, \mathcal{V})$ is defined for all x, $0 \le x \le X$, $\mathcal{V} \in \mathbb{R}_+^n$; and for all fixed x, $0 \le x \le X$, and $1 \le i \le n$, the function $g_i(x, v_1, \cdots, v_n)$ is non-decreasing with respect to $v_1, \cdots, v_n$. Furthermore, let $\mathcal{U}(x)$ be any solution of $\Delta_x^m \mathcal{U}(x) = \mathcal{F}(x, \mathcal{U}(x))$ and $\mathcal{V}(x)$ be a solution of $\Delta_x^m \mathcal{V}(x) = \mathcal{G}(x, \mathcal{V}(x))$ such that $|T(\mathcal{U})| \le T(\mathcal{V})$. Then

(i) if $\mathcal{V}(x)$ is bounded, so is $\mathcal{U}(x)$; and
(ii) if $\mathcal{V}(x) \to 0$ as $\|x\| = (x_1^2 + \cdots + x_m^2)^{1/2} \to +\infty$, so is $\mathcal{U}(x)$.

Definition 6.4.2 We say that a given function $u(x), x \in \mathbb{N}^m$, satisfies the property (L), if there exist $\rho_1 > 0, \cdots, \rho_m > 0$ such that

$$S_{x=0}^{+\infty} |u(x)| \rho_1^{-x_1} \times \cdots \times \rho_m^{-x_m} < +\infty,$$

where $\rho_1 > 0, \cdots, \rho_m > 0$ depend on the function $u(x)$.

Theorem 6.4.31 (Agarwal [10]) *Assume the following inequality holds for all $x \in \mathbb{N}_0^m$,*

$$u(x) \le p(x) + q(x) S_{s=0}^{x-1} g(s) u(s) + S_{s=0}^{x-1} f(x - s - 1) u(s) \tag{6.4.118}$$

where $p(x)$ and $q(x)$ are non-decreasing and $f(x)$ satisfies the property (L). Then for all $x \in \mathbb{N}_0^m$,

$$u(x) \le p_1(x) \min_{1 \le j \le m} \left\{ \prod_{\ell_j=0}^{x_j-1} \left[1 + q_1(x) S_{\bar{s}_j=0}^{\bar{x}_j-1} g(s) \right] \right\} \tag{6.4.119}$$

where

$$p_1(x) = p(x) \left[1 + S_{s=0}^{x-1} r(s) \right], \qquad q_1(x) = q(x) \left[1 + S_{s=0}^{x-1} r(s) \right]$$

and $r(x)$ *is given by* $r(x) = (2\pi i)^{-m} \int \cdots \int R(z_1, \cdots, z_m) z_1^{x_1 - 1} \times \cdots \times z_m^{x_m - 1} dz_1 \times \cdots \times dz_m$.

We shall note that the application of Riemann's function to study Gronwall type inequalities in several independent variables is known from the 1980s, see, e.g., Thandapani and Agarwal [621], and references therein. The discrete analogue of Riemann's function and its applications to several inequalities discussed are from Agarwal [7]. Lemma 6.4.5 which provides an upper estimate on the Riemann's function and its usefulness to obtain Wendroff's type estimate in Theorem 6.4.5 are also discussed in Agarwal [7]. Using a different approach, Wendroff's type inequalities are also investigated in Agarwal [6], Agarwal and Thandapani [19], Pachpatte and Singare [511], Popenda [529, 531], Singare and Pachpatte [597–599], Thandapani and Agarwal [620], Thandapani [619], Yang [661], Yeh [670, 671]; however, as a consequence of the present approach, Theorem 6.4.6 relaxes some of the conditions needed on the functions appearing in (6.4.31), and the obtained estimate (6.4.32) is sharper.

Theorem 6.4.7 uses the transformation introduced by Beesack [56]. Results are taken from Agarwal and Thandapani [19].

Two independent variable discrete Taylor's formula and the inequalities involving partial differences have appeared in Agarwal and Wilson [20]. Some related results are also available in Thandapani [618].

Multi-dimensional discrete analogue of Riemann's function given in Lemma 6.4.10 and Theorems 6.4.16–6.4.20 are proved in Agarwal [8].

Chapter 7
Linear Multi-Dimensional Discontinuous Integral Inequalities

In this chapter, we shall introduce some multi-dimensional linear discontinuous integral inequalities.

7.1 Linear Multi-Dimensional Discontinuous Volterra Integral Inequalities and Their Generalizations

In this section, we shall introduce some multi-dimensional linear discontinuous Volterra integral inequalities.

7.1.1 Linear Multi-Dimensional Discontinuous Volterra Integral Inequalities

We shall study the problem of obtaining explicit upper bounds of solutions $u(x)$ of linear Volterra integral equations

$$u(x) = f(x) + \int_a^x K(x,t)u(t)dt + \int_a^x g(x,t,u(t))dt, \tag{7.1.1}$$

or the corresponding inequalities

$$u(x) \leq f(x) + \int_a^x K(x,t)u(t)dt + \int_a^x g(x,t,u(t))dt. \tag{7.1.2}$$

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We shall also provide some global existence theorems for Eq. (7.1.1) in both the bounded, measurable case and the L^2-case of f, K, g. Most of results deal with the unperturbed cases where either $g \equiv 0$ or $K \equiv 0$.

Here, $u(x) = (u_1(x), \cdots, u_N(x))^T$ and $x = (x_1, \cdots, x_n)$, so we are dealing with systems of N equations or inequalities in n independent variables. If either $n = 1$ or $N = 1$, (7.1.1) and (7.1.2) have been studied by many authors. For example, if $n = 1,\ N \geq 1$, then we refer to Miller [405], see also Tricomi [625] and Mikhlin [403]. The case $N = 1, n \geq 1$ of (7.1.2) was considered as so-called Gronwall-type inequalities or so by Pachpatte and his coworkers [89, 92, 483, 600], by Yeh [667, 669], Young [677, 678, 680], and others [54, 55, 294, 621].

Some special case for $N > 1$ and $n \geq 1$ are dealt with by Greene [235], Das [163], and Shinde and Pachpatte [589], and by Chandra and Davis [128], Beesack [235], and Conlan and Wang [143] who considered general N, n.

Beesack [59] first considered linear equations, Eq. (7.1.1) with $g \equiv 0$, and then used a lemma proved in [55] to give easy proofs of existence theorems and related matters both for the bounded, measurable case and the Lebesgue square integrable case.

The special case for $K(x, t) = G(x)H(t)$ was investigated with by Chandra and Davis [128]. Beesack [59] obtained the Neumann series solution for the both solution vector $u(x)$, and the general matrix resolvent kernel $\Gamma(x, t)$. A number of explicit bounds are given for $|\Gamma(x, t)|$ and $|u(x, t)|$.

One of the above results involves a generalized exponential function,

$$\exp_n(z) = \sum_{r=0}^{+\infty} z^r/(r!)^n, \tag{7.1.3}$$

where n is the dimension of the space of $x = (x_1, x_2, \cdots, x_n)$.

We now consider a system of N linear equations of Volterra type in the variable $x = (x_1, \cdots, x_n)$, namely,

$$u_i(x) = f_i(x) + \sum_{j=1}^{N} \int_a^x k_{ij}(x, t)u_j(t)dt, \quad 1 \leq i \leq N, \tag{7.1.4}$$

which can be rewritten as the system (7.1.4) in the vector form

$$u(x) = f(x) + \int_a^x K(x, t)f(t)dt, \tag{7.1.5}$$

where K denotes the $N \times N$ matrix (k_{ij}), and u,f are the N-column vectors $(u_1, \cdots, u_N)^T$, $(f_1, \cdots, f_N)^T$. Set

$$J = [a, b] = \{x \in \mathbb{R}^n : a \leq x \leq b\}, \quad T = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}^n : a \leq t \leq x \leq b\},$$

where $a < b$.

In the continuous or the bounded, measurable case considered below, all components of a, b are to be finite (so the cell J is compact), but in the L^2-case, we may have any $a_i = -\infty$ or $b_j = +\infty$.

In the case $n = 1$, system (7.1.4) can be conveniently reduced to a single linear Volterra equation. See, for example, Mikhlin [403], Hoheisel [287], and Tricomi [625]. The last two references actually apply this method of dealing with systems of Volterra equations for the case $n = 1$, and the method consists of an extension of the second of these for $n > 1$. This method was used also in [55] for $N = 1, n \geq 1$, and amounts to showing that, under an appropriate hypothesis, (7.1.5) has a unique solution (of an appropriate class) given by the Neumann series

$$u(x) = \sum_{r=0}^{+\infty} v_r(x), \tag{7.1.6}$$

where

$$v_0(x) = f(x), \quad v_r(x) = \int_a^x K^{(r)}(x,t)f(t)dt, \quad r \geq 1, \tag{7.1.7}$$

and the $K^{(r)}$ are the iterated kernels of K defined recursively by

$$K^{(1)}(x,t) = K(x,t),\ K^{(r)}(x,t) = \int_a^x K(x,s)K^{(r-1)}(s,t)ds, \quad r \geq 2. \tag{7.1.8}$$

As usual, we say that K is continuous, or bounded and measurable on T, or that $K \in L^2(T)$, if and only if each k_{ij} has the corresponding property, and similarly for N-vectors u, f relative to the cell J. If we write

$$|u| = \left(\sum_{i=1}^{N} |u_i|^2\right)^{1/2}, \quad |K| = \left(\sum_{i=1}^{N} |K_{ij}|^2\right)^{1/2}, \tag{7.1.9}$$

then we have the compatibility conditions

$$|Ku| \leq |K||u|, \quad |KM| \leq |K||M|, \tag{7.1.10}$$

where $M = (m_{ij})$. Moreover, if $f \in L^2(J)$ and $K, M \in L^2(T)$, then

$$\int_a^x K(x,t)f(t)dt \in L^2(J), \quad \int_a^x K(x,s)M(s,t)ds \in L^2(T). \tag{7.1.11}$$

This follows from the fact that the corresponding results hold for the corresponding component functions

$$\begin{cases} \left(\int_a^x K(x,t)f(t)dt\right)_i = \sum_{j=1}^{N} \int_a^x k_{ij}(x,t)f_j(t)dt, \\ \left(\int_a^x K(x,s)M(s,t)ds\right)_{i,j} = \sum_{\alpha=1}^{N} \int_a^x k_{i\alpha}(x,t)m_{\alpha j}(s,t)ds. \end{cases}$$

Observe also that, according to the above definitions, if f and K are measurable, then

$$f \in L^2(J) \Leftrightarrow |f| \in L^2(J),$$

and similarly

$$K \in L^2(T) \Leftrightarrow |K| \in L^2(T),$$

precisely as for the case $N = 1$. Finally, for $f \in L^2(J)$ and $K \in L^2(T)$, we define the L^2-norms $\|f\|,\ \|K\|$ by

$$\|f\|^2 = \left(\int_a^b |f(t)|^2 dt\right)^{1/2} \tag{7.1.12}$$

$$\|K\| = \left(\int_a^b \int_a^x |K(x,t)|^2 dtdx\right)^{1/2} = \left(\int_a^b \int_t^b |K(x,t)|^2 dxdt\right)^{1/2}, \tag{7.1.13}$$

and note that

$$\|f\| = \left(\sum_{i=1}^{N} \|f_i\|^2\right)^{1/2}, \quad \|K\| = \left(\sum_{i,j=1}^{N} \|k_{ij}\|^2\right)^{1/2}. \tag{7.1.14}$$

We shall use the following lemma proved in Beesack [55].

Lemma 7.1.1 *Let $f : J \to \mathbb{R}$ be Lebesgue integrable and either non-negative a. e. or non-positive a. e. If $F : I \to \mathbb{R}$ has a non-decreasing derivative on an interval I containing $\int_a^x fdt$ for all $x \in J$, then for all such x,*

$$F(0) + \int_a^x f(t)F'\left(\int_a^t fds\right)dt \le F\left(\int_a^x fdt\right), \tag{7.1.15}$$

$$F(0) + \int_a^x f(t)F'\left(\int_t^x fds\right)dt \le F\left(\int_a^x fdt\right). \tag{7.1.16}$$

We now assume that $f \in L^2(J)$, $K \in L^2(T)$ in (7.1.5) where J need not be bounded, and prove the assertions made concerning (7.1.6), (7.1.7). First, by (7.1.7), (7.1.8) and (7.1.11), an induction on r shows that all $K^{(r)} \in L^2(T)$ and $v \in L^2(J)$.

Moreover, for $r \geq 2$ repeated substitution in (7.1.8) shows that (with $t_0 = x$),

$$\begin{aligned} K^{(r)}(x,t) &= \int_t^x \int_t^{t_1} \cdots \int_t^{t_{r-2}} K(x,t_1)K(t_1,t_2)\cdots K(t_{r-1},t)dt_{r-1}\cdots dt_2 dt_1 \\ &= \int_t^x \int_{t_{r-1}}^x \cdots \int_{t_2}^x K(x,t_1)K(t_1,t_2)\cdots K(t_{r-1},t)dt_1\cdots dt_{r-1}, \end{aligned} \tag{7.1.17}$$

which, by repeated substitution, is equal to

$$\int_t^x \hat{K}^{(r-1)}(x,t_{r-1})K(t_{r-1},t)dt_{r-1},$$

where

$$\hat{K}^{(1)}(x,t) = K(x,t), \quad \hat{K}^{(r)}(x,t) = \int_t^x \hat{K}^{(r-1)}(x,s)K(s,t)ds, \quad (r \geq 2).$$

Hence, precisely as for the case $N = n = 1$, we have $\hat{K}^{(r)} = K^{(r)}$. Thus

$$K^{(r)}(x,t) = \int_t^x K^{(r-1)}(x,s)K(s,t)ds, \quad r \geq 2. \tag{7.1.18}$$

More generally, by (7.1.8), (7.1.18) and induction on j,

$$K^{(r)}(x,t) = \int_t^x K^{(i)}(x,s)K^{(j)}(s,t)ds, \quad r \geq 2, i+j = r, 1 \leq j \leq r-1. \tag{7.1.19}$$

However, that the matrices $K^{(i)}$, $K^{(j)}$ do not commute when $N > 1$. Before discussing L^2-convergence of the series (7.1.6) and related matters, we note that a statement of ordinary convergence of matrix and vector series relative to arbitrary matrix and vector norms may be found in Chap. 4 of Stewart [608].

We now prove the L^2-convergence of the series (7.1.6). It clearly suffices to prove that the scalar series $\sum_{r=0}^{+\infty} |v_r(x)|$ is convergent in $L^2(J)$, from which by comparison:

$$|v_{r,i}(x)| \leq |v_r(x)| \quad \text{for} \quad 1 \leq i \leq N, \quad r \geq 0, \quad x \in J,$$

it follows that the L^2-convergence on J of each of the N component series in (7.1.6). As usual, if we set $S_k = \sum_{r=0}^{k} |v_r|$, then, due to the following inequality,

$$\|S_k - S_p\| \leq \sum_{r=p+1}^{k} \||v_r|\| = \sum_{r=p+1}^{k} \|v_r\|,$$

it suffices to prove that the series $\sum_{r=0}^{k} \|v_r\|$ converges. To prove this, we follow the method used for the case $N = m = 1$ by Tricomi [625] and Hochstadt [286] and for cases $N = 1,\ n \geq 1$ by Beesack [55]. Since $|K| \in L^2(T)$, both functions

$$A^2(x) = \int_a^x |K(x,t)|^2 dt, \quad B^2(t) = \int_t^b |K(x,t)|^2 dx$$

are in $L^2(J)$, with $\int_J A^2 dx = \int_J B^2 dt = \|K\|^2$.

Thus all is now reduced to the scalar $N = 1, n \geq 1$ as in Beesack [55], we know that $r \geq 2$,

$$|K^{(r)}(x,t)|^2 \leq A^2(x)B^2(t)\left(\int_t^x A^2(s)ds\right)^{r-2} \Big/ (r-2)!, \quad a.\,e. \quad on \quad T \tag{7.1.20}$$

(in fact, the proof of (7.1.20) depends on Lemma 7.1.1). Hence from (7.1.7), (7.1.10) and the Cauchy-Schwarz inequality, it follows

$$|v_r(x)|^2 = \left|\int_a^x K^{(r)}(x,t)f(t)dt\right|^2 \leq \int_a^x |K^{(r)}(x,t)|^2 dt \int_a^x |f(t)|^2 dt$$

$$\leq \|f\|^2 \int_a^x A^2(x)B^2(t)\left(\int_t^x A^2(s)ds\right)^{r-2} dt \Big/ (r-2)!,$$

$$|v_r(x)|^2 \leq \|f\|^2 \cdot \|K\|^2 \left(\int_t^x A^2(s)ds\right)^{r-2} \Big/ (r-2)!, \quad a.\,e. \quad \text{on} \quad J. \tag{7.1.21}$$

Indeed, there is a better estimate than (7.1.21). We now use in (7.1.15) in Lemma 7.1.1 with $F(u) = u^{r-1}$ to obtain

$$\|v_r\|^2 = \int_a^b |v_r|^2 dx \leq \|f\|^2 \|K\|^{2r} \Big/ (r-1)!. \tag{7.1.22}$$

By (7.1.7) and (7.1.10), we know that (7.1.22) also holds for $r = 1$, hence for all $r \geq 1$. This proves the convergence of the series $\sum \|v_r\|$ as required, hence the l^2-convergence of the series (7.1.6), and consequently, the a. e. pointwise absolute convergence of the series $\sum v_r$ on J.

Similarly, using (7.1.15) in Lemma 7.1.1 again, we can infer from (7.1.10) that

$$\begin{aligned}\int_T |K^{(r)}(x,t)|^2 dtdx &\le \int_b^a A^2(x) \int_a^x B^2(t) \left(\int_t^x A^2(s)ds\right)^{r-2} dtdx \Big/ (r-2)!\\ &= \int_a^b B^2(t) \int_t^b \left(\int_t^x A^2(s)ds\right)^{r-2} A^2(x)dxdt \Big/ (r-2)!\\ &\le \int_a^b B^2(t) \left(\int_t^b A^2(s)ds\right)^{r-1} dt \Big/ (r-2)!.\end{aligned}$$

Since $\int_J B^2 dt = \int_J A^2 ds = \|K\|^2$, it follows that

$$\|K^{(r)}\| \le \|K\|^r \Big/ \sqrt{(r-1)!}, \quad r \ge 1, \tag{7.1.23}$$

which, as above, shows that the Neumann series for the resolvent kernel,

$$\Gamma(x,t) \equiv \sum_{r=1}^{+\infty} K^{(r)}(x,t) \tag{7.1.24}$$

is also l^2-convergent, as well as a. e. pointwise absolutely convergent on T.

Thus that u, as defined by (7.1.6), is a solution of (7.1.5) follows precisely as for the case $N = n = 1$ by direct substitution into (7.1.5), the term-by-term integration being justified by the monotone convergence of the series $\sum |v_r(x)|$. This integration also gives us the representation

$$u(x) = f(x) + \int_a^x \Gamma(x,s)f(s)ds, \quad x \in J. \tag{7.1.25}$$

Moreover, a similar term-by-term integration gives us, by using (7.1.8) and (7.1.18), the usual resolvent equation

$$\begin{aligned}\Gamma(x,t) &= K(x,t) + \int_t^x K(x,s)\Gamma(s,t)ds\\ &= K(x,t) + \int_t^x \Gamma(x,s)K(s,t)ds, \quad (x,t) \in T.\end{aligned} \tag{7.1.26}$$

Finally the uniqueness, in the class $L^2(J)$, of this solution u follows from the fact that $v \in L^2(J)$ satisfies (7.1.5), then by successively substitutions in (7.1.5), we have by using (7.1.17),

$$v(x) = \sum_{r=0}^{m} v_r(x) + R_{m+1}(x),$$

with v_r, given by (7.1.7) and

$$R_{m+1}(x) = \int_a^x K^{(m+1)}(x, t_{m+1})v(t_{m+1})dt_{m+1}.$$

Hence,

$$\begin{aligned}|R_{m+1}(x)|^2 &\leq \int_a^x |K^{m+1}(x,s)|^2 ds \cdot \int_a^x |v(s)|^2 ds\\ &\leq \|v\|^2\|K\|^2 A^2(x)\left(\int_t^b A^2(s)ds\right)^{m-1} dt \Big/ (m-2)!, \quad a.e. \quad on \quad J.\end{aligned}$$

Thus $R_{m+1}(x) \to 0$ a. e. on J, so $v = u$ in $L^2(J)$ proving the uniqueness.

We summarize results in the following theorem.

Theorem 7.1.1 (Beesack [59]) *Let* $a,\ b \in \overline{\mathbb{R}}^n$ *so* $-\infty \leq a_j < b_j \leq +\infty$ *for* $1 \leq j \leq n$, *let* J *denote the cell* $[a, b],\ T = \{(x,t) : a \leq t \leq x \leq b\}$, *and suppose that* $f \in L^2(J),\ K \in L^2(T)$. *Then the system of integral equations (7.1.4) or (7.1.5) has a unique solution* $u \in L^2(J)$. *This solution is given by the Neumann series case (7.1.6); the series* $\sum_{r=0}^{+\infty} |v_r(x)|$ *is convergent in* $L^2(J)$ *as well as a. e. pointwise convergent on* T. *The solution is also given by the corresponding (7.1.25), where* $\Gamma \in L^2(T)$ *denotes the resolvent kernel defined by the* L^2*-convergent series (7.1.24).*

Remark 7.1.1 By appropriate modifications, the same results can be obtained for systems of Fredholm equations if the kernel K satisfies the usual strong condition that

$$\|K\|^2 = \int_a^b \int_a^b |K(x,t)|^2 dtdx < 1.$$

Remark 7.1.2 The special case $K(x,t) = G(x)H(t)$ was considered rather briefly by Chandra and Davis in [128], with primary attention given to the continuous case. For the L^2-case the necessary assumption was made that the matrices $G(x), H(x)$ commute. For further details concerning the case $K = GH$, see Remark 7.1.4.

We now turn to the case that f and K are continuous or, more generally, essentially bounded measurable on J and T respectively. In this case, we also suppose that the cell J is bounded ($-\infty < a_j < b_j < +\infty$ for $1 \leq j \leq n$), hence compact. Of course, we then have $f \in L^2(J)$ and $K \in L^2(T)$, hence Theorem 7.1.1 and all the preceding estimates still apply, but we now seek better results. First, if

$$M = ess\ \sup\{|K(x,t)| : (x,t) \in T\},$$

then

$$\begin{cases} A^2(x) \le M^2 \displaystyle\int_a^x 1 \cdot dt = M^2 \prod_{i=1}^n (x_i - a_i), \\ B^2(t) \le M^2 \displaystyle\int_t^b 1 \cdot dt = M^2 \prod_{i=1}^n (b_i - t_i). \end{cases}$$

Thus it follows from (7.1.20) that for $r \ge 2$,

$$|K^{(r)}(x,t)| \le M^2 \left\{ \prod_{i=1}^n (x_i - a_i)(b_i - t_i) \right\}^{1/2} \left\{ \frac{\prod_{i=1}^n (x_i - a_i)^2 - \prod_{i=1}^n (t_i - a_i)^2}{2^n} \right\} \Big/ (r-2)!. \tag{7.1.27}$$

Although this is an explicit bound for $K^{(r)}$, it is not useful for determining bounds for either Γ, or for v, or u, but establishes the uniform convergence on T of the Neumann series (7.1.26) for Γ since, from (7.1.27),

$$|K^{(r)}(x,t)| \le M^2 \left\{ \prod_{i=1}^n (b_i - a_i) \right\}^{r-1} \Big/ \sqrt{(r-2)!}, \qquad (x,t) \in T, \quad r \ge 2.$$

Hence, if K is bounded and measurable (or continuous) on T, so also is Γ. By (7.1.25), it then follows that if f is bounded and measurable on J (or continuous on J), so is the solution of (7.1.4). The uniqueness of the solution-within the class of functions—is clear from the uniqueness proved in Theorem 7.1.1 since bounded measurable (or continuous) functions on (compact) J are also in $L^2(J)$. We thus have the following result.

Theorem 7.1.2 (Beesack [59]) *Let $J = [a,b]$ be compact in $\mathbb{R}^n$, and suppose that f, K are bounded and measurable (or continuous) on J and T, respectively. Then the system of integral equations (7.1.4) or (7.1.5) has a unique solution u which is bounded and measurable (or continuous) on J. The solution is given by the Neumann series (7.1.6) which is uniformly absolutely convergence on J, and is also given by the representation (7.1.25), where Γ is bounded and measurable (or continuous) on T. The Neumann series (7.1.24) for Γ is uniformly absolutely convergent on T.*

Note that the only point not already proved is that the uniform absolute convergence on J of the series (7.1.6). However, this follows at once from estimate (7.1.21).

There are, in fact, many alternative bounds to (7.1.27) and (7.1.21) which can be established when f and K are bounded and measurable (or continuous) and J is compact, and we shall obtain three such sets of bounds.

In what follows, the vector and matrix norms need not be the Euclidean norms given by (7.1.9), but can be any compatible pair of norms, i.e., norms

satisfying (7.1.10). For example, two more such pairs are

$$|u| = \max_{1\le i\le N} |u_i|, \quad |K| = \max_{1\le i\le N} \sum_{j=1}^{N} |k_{ij}|$$

and

$$|u| = \sum_{i=1}^{N} |u_i|, \quad |K| = \max_{1\le i\le N} \sum_{j=1}^{N} |k_{ij}|.$$

Moreover, we note that all the bounds given below have a form which is independent of N and so apply equally to the scalar or vector case.

(1) Set

$$A_t(x) = \int_t^x |K(s,t)|ds, \quad B(t) = \sup_{t\le s\le x\le b} |K(x,s)|. \tag{7.1.28}$$

Note that

$$|K^{(2)}(x,t)| \le \int_t^x |K(x,s)||K(s,t)|ds \le B(t)A_t(x). \tag{7.1.29}$$

By induction on $r \ge 3$, we obtain, by using (7.1.8),

$$|K^{(r)}(x,t)| \le B^{r-1}(t) \int_t^x A_t(s) \prod_{i=1}^{n} (x_i - s_i)^{r-3} ds \Big/ [(r-3)!]^n. \tag{7.1.30}$$

We obtain the more useful bound by noting that

$$A_t(x) \le B(t) \int_t^x 1ds = B(t) \prod_{i=1}^{n} (x_i - t_i), \tag{7.1.31}$$

which, together with (7.1.30), yields

$$|K^{(r)}(x,t)| \le B^r(t) \int_t^x \prod_{i=1}^{n} (s_i - t_i)(x_i - s_i)^{r-3} ds \Big/ [(r-3)!]^n,$$

i.e.,

$$|K^{(r)}(x,t)| \le B^r(t) \prod_{i=1}^{n} (x_i - t_i)^{r-1} \Big/ [(r-1!]^n, \quad (x,t) \in T, \tag{7.1.32}$$

this is now valid for all $r \geq 1$, as can be seen by (7.1.28), (7.1.31) and (7.1.29).

It therefore follows from (7.1.24) that

$$|\Gamma(x,t)| \leq B(t)\exp_n\left[B(t)\prod_{i=1}^{n}(x_i - t_i)\right], \quad (x,t) \in T, \tag{7.1.33}$$

where we have introduced the n-exponential function, $\exp_n(z)$, defined for all $z \in \mathbb{C}$,

$$\exp_n(z) = \sum_{r=0}^{+\infty} z^n/(r!)^n, \quad n = 1, 2, \cdots \tag{7.1.34}$$

the function is the special case of the generalized hypergeometric function

$${}_pF_q(\alpha_1, \cdots, \alpha_p; \beta_1, \cdots, \beta_q; z)$$

for $p = 1, q = n$ and $\alpha_1 = 1, \beta_1 = \cdots = \beta_n = 1$. See Slater [601] and also **Remark 7.1.3**.

Similarly, from (7.1.25) and (7.1.33), we obtain for all $x \in J$,

$$|u(x)| \leq |f(x)| + \int_a^x |f(t)|B(t)\exp_n\left[B(t)\prod_{i=1}^{n}(x_i - t_i)\right]dt, \tag{7.1.35}$$

where $B(t)$ is defined by (7.1.28).

(2) Set $A_t(x) = \int_t^x |K(s,t)|ds$ as in (7.1.28) and

$$B_t(x) = \sup_{t \leq s \leq x} |K(x,s)|, \quad C_t(x) = A_t(x)B_t(x). \tag{7.1.36}$$

For $r \geq 3$, we find, as from (7.1.8)–(7.1.17), that

$$|K^{(r)}(x,t)| \leq B_t(x)\int_t^x\int_t^{s_{r-2}}\cdots\int_t^{s_2} B_t(s_{r-2})\cdots B_t(s_2)C_t(s_1)ds_1\cdots ds_{r-2},$$

and

$$|K^{(r)}(x,t)| \leq B_t(x)\int_t^x \bar{A}_r(s_1)C_t(s_1)ds_1,$$

where $\bar{A}_3(s_1) = 1$, and for $r \geq 4$,

$$\bar{A}_r(s_1) = \int_{s_1}^x\int_{s_2}^x\cdots\int_{s_{r-3}}^x B_t(s_{r-2})\cdots B_t(s_2)ds_{r-2}\cdots ds_2.$$

We can now show, by induction on r that for all $r \geq 3$,

$$\bar{A}_r(s_1) \leq \left(\int_{s_1}^{x} B_t(u)du\right)^{r-2} \Big/ (r-3)!, \quad a \leq s_1 \leq x \leq b;$$

the proof needs using in (7.1.16) in Lemma 7.1.1. Hence, for all $r \geq 3$ and for all $(x, t) \in T$,

$$|K^{(r)}(x, t)| \leq B_t(x) \int_t^x \left(\int_{s_1}^{x} B_t(u)du\right)^{r-2} C_t(s_1)ds_1 \Big/ (r-3)!. \tag{7.1.37}$$

For a more useful bound, note that by (7.1.36),

$$C_t(s_1) = B_t(s_1) \int_t^{s_1} |K(s, t)|ds \leq B_t(s_1) \int_t^{s_1} B_t(s)ds.$$

Hence (7.1.37) by using (7.1.16) in Lemma 7.1.1 twice implies

$$\begin{aligned}
|K^{(r)}(x, t)| &\leq B_t(x) \int_t^x \int_t^{s_1} B_t(s)B_t(s_1) \left(\int_{s_1}^{x} B_t(u)du\right)^{r-3} dsds_1 \Big/ (r-3)! \\
&= B_t(x) \int_t^x B_t(s) \left\{\int_s^x B_t(s_1) \left(\int_{s_1}^{x} B_t(u)du\right)^{r-3} ds_1\right\} ds \Big/ (r-3)! \\
&\leq B_t(x) \int_t^x B_t(s) \left(\int_s^x B_t(u)du\right)^{r-2} ds \Big/ (r-2)! \\
&\leq B_t(x) \left(\int_t^x B_t(u)du\right)^{r-1} \Big/ (r-1)!,
\end{aligned}$$

whence, for all $(x, t) \in T$,

$$|K^{(r)}(x, t)| \leq B_t(x) \left(\int_t^x B_t(u)du\right)^{r-1} \Big/ (r-1)!, \tag{7.1.38}$$

and we easily verifies that (7.1.38) holds for all $r \geq 1$.

Now the bounds corresponding to (7.1.33) and (7.1.35) of (7.1.4) are as follows

$$|\Gamma(x, t)| \leq B_t(x) \exp\left(\int_t^x B_t(u)du\right), \quad (x, t) \in T, \tag{7.1.39}$$

and

$$|u(x)| \leq |f(x)| + \int_a^x |f(t)|B_t(x) \exp\left(\int_t^x B_t(u)du\right) dt, \quad x \in J, \tag{7.1.40}$$

where $B_t(u) = \sup_{t\le s\le u} |K(u, s)|$. Note that $B_t(u) \le B(t)$ and $B_t(x) \le B(t)$ as well, in (7.1.40).

(3) Here we use the direct analogue of the method used for the L^2-case, where Hölder's inequality (for $p = q = 2$) was applied to $K^{(2)}, \cdots, K^{(r)}, \cdots$ to obtain bounds. We again use Hölder's inequality (for $p = +\infty,\ q = 1$) to obtain

$$|K^{(2)}(x, t)| \le \int_t^x |K(x, s)||K(s, t)|ds \le D(x)E(t),$$

where

$$D(x) = \sup_{a\le s\le x} |K(x, s)|, \quad E(t) = \int_t^b |K(s, t)|ds. \tag{7.1.41}$$

(Observe that $B_t(x) \le D(x)$, and $A_t(x) \le E(t)$ with $A_t(b) = E(t)$.) Using (7.1.8) and induction on r, we conclude that for all $r \ge 2$, for all $(x, t) \in T$,

$$|K^{(r)}(x, t)| \le D(x)E(t) \left(\int_t^x D(u)du\right)^{r-2} \Big/ (r-2)!, \tag{7.1.42}$$

which is the analogue of the inequality (7.1.20) for the L^2-case, but is somewhat better. We now have for all $(x, t) \in T$,

$$|\Gamma(x, t)| \le |K(x, t)| + D(x)E(t) \exp\left(\int_t^x D(u)du\right). \tag{7.1.43}$$

Thus using (7.1.25) a corresponding bound for $|u(x)|$ can also be obtained. Therefore we note that by using $p = 1,\ q = +\infty$,

$$|K^{(2)}(x, t)| \le D_1(x)E_1(t),$$

with

$$D_1(x) = \int_a^x |K(x, s)|ds, \quad E_1(t) = \sup_{t\le s\le b} |K(s, t)|. \tag{7.1.44}$$

Using (7.1.18) instead of (7.1.8), we can also obtain for all $r \ge 2$, for all $(x, t) \in T$,

$$|K^{(r)}(x, t)| \le D_1(x)E_1(t) \left(\int_t^x E_1(u)du\right)^{r-2} \Big/ (r-2)!, \tag{7.1.45}$$

and

$$|\Gamma(x,t)| \leq |K(x,t)| + D_1(x)E_1(t)\exp\left(\int_t^x E_1(u)du\right). \tag{7.1.46}$$

□

Remark 7.1.3 (Beesack [59]) The bound (7.1.33) for the resolvent kernel involves the n-exponential function $\exp_n(x)$, while the bounds (7.1.39), (7.1.43) and (7.1.46) involve only the special case $n = 1$, the ordinary exponential function. As noted following (7.1.44), $\exp_n(x) =_1 F_n(1;1,\cdots,1;x)$, but it appears that little is known about such generalized hypergeometric functions [601]. It is obvious that, $x \geq 0$ and $n > 1$, they are much smaller than $\exp(x)$, and the series expansion converges are rapidly. Here are some explicit estimates:

$$\begin{cases} \exp_n(x) \leq 1 + x + (x^2/2^n)\exp(x/3^n) \quad \text{if} \quad x \geq 0,\ n \geq 3; \\ \exp_n(x) \leq 1 + x + (x^3/6^n)\exp(x/12^{n/2}) \quad \text{if} \quad x \geq 0,\ n \geq 2. \end{cases}$$

To prove the above second inequality, for example, we write

$$\begin{aligned} \exp_n(x) &= 1 + x + (x^2/2^n) + (x^3/6^n)\left\{1 + \frac{x}{4^n} + \cdots + \frac{x^k}{4.5\ldots(4+3)^n} + \cdots\right\} \\ &\leq 1 + x + \frac{x^2}{2^n} + x^3/6^n\exp(x/A^n), \quad x \geq 0,\ n \geq 2, \end{aligned}$$

provided that A is chosen so that $(A^n)^k \cdot k! \leq [4 \cdot 5 \cdots (k+3)]^n$ holds for all $k \geq 1,\ n \geq 2$.

By induction on k and elementary analysis, we find that $A = 12$ is the best (=largest) A for which this holds.

Now we shall discuss consider the systems of Volterra inequalities of the linear form

$$u_i(x) \leq f_i(x) + \sum_{j=1}^{N}\int_a^x k_{ij}(x,t)u_j(t)dt, \quad 1 \leq i \leq N, \tag{7.1.47}$$

or in vector-matrix form

$$u(x) \leq f(x) + \int_a^x K(x,t)u(t)dt. \tag{7.1.48}$$

Note that special cases of (7.1.47) with $n = 1,\ N = 2$ and degenerate kernels-in fact, separable kernels $K(x,t) = g(x)h(t)$-were considered by Greene [235], Das [163], and Wang [639], and generalizations of these to cases $n \geq 2, N = 2$ were given by Shinde and Pachpatte [589]. The case of general N, n with a separable kernel $K(x,t) = G(x)H(t)$ had been handled somewhat earlier by Chandra and

Davis [128]. For more details of this separable case and references to still earlier results, see Beesack [58]. This case and the general case (7.1.48) under rather restrictive hypotheses were considered by Conlan and Wang [143].

Usually, we hope to obtain upper bounds: $u_i(x) \leq U_i(x, f, k), 1 \leq i \leq N$, valid for any functions u_i satisfying (7.1.47). Also we wish that the functions U_i are as explicit as possible, and in particular, that they do not depend on the solution of a functional equation.

We shall obtain explicit bounds which are due to Beesack [59] for the system (7.1.47) . Recall in all of the works mentioned above except for [128] and [58], it is assumed that all $k_{ij} \geq 0$, $u_i \geq 0$, $f_i \geq 0$; one or both of the last two non-negativity conditions are not required for some of the results of [58, 128]. We begin by extending these results to the general system (7.1.47) or (7.1.48), which are due to Beesack [59].

First, for N-vectors $u = (u_1, u_2, \cdots, u_N)^t$ and $N \times N$ matrices $K = (k_{ij})$, we write $u \geq 0$ (or $0 \leq u$) $\Leftrightarrow u_i \geq 0 \quad (1 \leq i \leq N)$, and

$$K \geq 0 \Leftrightarrow k_{ij} \geq 0, \quad (1 \leq i, j \leq N).$$

Similarly

$$u \leq v \Leftrightarrow v - u \geq 0,$$

then we have the usual order properties

(i) $u \geq 0, v \geq 0 \Rightarrow u + v \geq 0$;
(ii) $u \geq 0, \alpha \in \mathbb{R}, \alpha \geq 0 \Rightarrow \alpha u \geq 0$;
(iii) $u \geq 0, K \geq 0 \Rightarrow Ku \geq 0$;
(iv) $K \geq 0, M \geq 0 \Rightarrow K + M \geq 0, KM \geq 0, \alpha K \geq 0$ if $\alpha \geq 0$.

Theorem 7.1.3 (Beesack [59]) *Let J, T, f, K be either as in Theorem 7.1.1, or as in Theorem 7.1.2 and suppose that $K \geq 0$ on T. If u satisfies the same hypothesis as f on J, and also satisfies (7.1.48) on J, then for all $x \in J$,*

$$u(x) \leq f(x) + \int_a^x \Gamma(x, t) f(t) dt, \tag{7.1.49}$$

where Γ is the resolvent kernel of K defined by (7.1.24).

Proof By (7.1.8), we know that each iterated kernel $K^{(r)} \geq 0$ on T, hence also $\Gamma \geq 0$ on T. Now define the function $\hat{f}$ on J by

$$\hat{f}(x) = f(x) + \int_a^x K(x, t) u(t) dt - u(x),$$

where $\hat{f}(x) \geq 0$ on J by (7.1.48). Also for all $x \in J$,

$$u(x) = (f(x) - \hat{f}(x)) + \int_a^x K(x,t)u(t)dt.$$

Hence $\hat{f} - f$ is a function of the same class as f and u, it follows from the uniqueness in Theorem 7.1.2 that for all $x \in J$,

$$u(x) = f(x) - \hat{f}(x) + \int_a^x \Gamma(x,t)(f(t) - \hat{f}(t))dt.$$

Hence (7.1.49) follows from (7.1.7) by noting $\hat{f} \geq 0,\ \Gamma \geq 0$. □

Note that Theorem 7.1.3 for $N = n = 1$ and f, K continuous appears to have been first proved explicitly by Chu and Metcalf [135] (see, Theorem 1.2.38), whose result was extended to the L^2-case by Beesack [59], where it was shown that when $K(x,t) = G(x)H(t)$, then

$$\Gamma(x,t) = G(x)V(x,t)H(t), \tag{7.1.50}$$

where

$$V(x,t) = I + \int_t^x H(s)G(s)V(s,t)ds. \tag{7.1.51}$$

These results also follow from (7.1.17) and (7.1.24), which further give $K^{(r)}(x,t) = G(x)V_r(x,t)H(t)$, where $V_1(x,t) = I$ and, for all $r \geq 2$,

$$\begin{aligned} V_r(x,t) &= \int_t^x \int_t^{x_1} \cdots \int_t^{t_{r-2}} H(t_1)G(t_1)\cdots H(t_{r-1})G(t_{r-1})dt_{r-1}\cdots dt_1 \\ &= \int_t^x H(t_1)G(t_1)V_{r-1}(t_1,t)dt_1. \end{aligned}$$

By using the second part of (7.1.17), we also obtain

$$V_r(x,t) = \int_t^x V_{r-1}(x,t_{r-1})H(t_{r-1})G(t_{r-1})dt_{r-1} \tag{7.1.52}$$

since by (7.1.24), $\Gamma = \sum_{r=1}^{+\infty} K^{(r)} = G\left(\sum_{r=1}^{+\infty} V_r\right)H$, we set $V = \sum_{r=1}^{+\infty} V_r$.

Then (7.1.50) holds, where

$$V = \sum_{r=1}^{+\infty} V_r = I + \sum_{r=1}^{+\infty} V_{r+1}(x,t) = I + \sum_{r=1}^{+\infty} \int_t^x H(t_1)G(t_1)V_r(t_1,t)dt_1,$$

which gives us (7.1.51).

By using the alternate expression above for V_r, we also obtain

$$V(x,t) = I + \int_t^x V(x,s)H(s)G(s)ds, \tag{7.1.53}$$

as already mentioned in Beesack [58]. □

Corollary 7.1.1 (Chandra-Davis [128]) *If either J is compact and u, f, G, H are bounded and measurable, or continuous, on J, or (inequality for possibly unbounded J), the functions $u, f, G, H \in L^2(J)$, with $G \geq 0, J \geq 0$ on J, and if for all $x \in J$,*

$$u(x) \leq f(x) + G(x) \int_a^x H(t)u(t)dt, \tag{7.1.54}$$

then for all $x \in J$,

$$u(x) \leq f(x) + G(x) \int_a^x V(x,t)H(t)f(t)dt. \tag{7.1.55}$$

Remark 7.1.4 Note that in [128], the L^2-case of this result was stated (as an alternate theorem) under the additional (unnecessary) requirement that the matrices $G(x), \ H(x)$ commute. In the case $N = 1, n \geq 1$ of inequality (7.1.54) (or of the corresponding system of equations), it was shown in [55] that the results of Corollary 7.1.1 hold even under weaker hypothesis: J may be unbounded with u, f measurable on J such that $Hu, \ Hu \in L(J)$ and $HG \in L(G)$ (or just $Hu, \ Hu \in L(J)$ in the case of equations). In [58], it was essentially noted that this same weakening holds for the generate case $N > 1, n = 1$.

The bounds for $u(x)$ given in (7.1.49) or (7.1.55), in order to obtain more elementary bounds from (7.1.49), for example, we follow the method used in [58] for (7.1.55). First, suppose $|v|$ and $|M|$ denote any compatible norms satisfying (7.1.10) for which $|v_i| \leq |v|$ for $1 \leq i \leq N$. If $K \geq 0$ on T and $u = (u_i)$ satisfies (7.1.48), then for all $x \in J$,

$$u_i(x) \leq f_i(x) + \int_a^x |\Gamma(x,t)| \cdot |f(t)|dt, \quad 1 \leq i \leq N. \tag{7.1.56}$$

In fact, by taking components in (7.1.49), it follows from (7.1.56) that

$$u_i(x) \leq f_i(x) + \int_a^x \{\Gamma(x,t)f(t)\}_i dt \leq f_i(x) + \int_a^x |\Gamma(x,t)f(t)|dt.$$

Similarly, suppose that $|M|$ denotes any compatible norm such that $|m_{ij}| \le |M|$ for $1 \le i,\ j \le N, f \ge 0$ on J and $K \ge 0$ on T and (u_i) satisfies (7.1.48), then

$$u_i(x) \le f_i(x) + \int_a^x |\Gamma(x,t)| \cdot \sum_{j=1}^{N} f_j(t)dt, \quad 1 \le i \le N,\ x \in J \tag{7.1.57}$$

and

$$u_i(x) \le f_i(x) + \int_a^x \sum_{j=1}^{N} \Gamma_{ij}(x,t) f_j(t)dt \le f_i(x) + \int_a^x |\Gamma(x,t)| \cdot \sum_{j=1}^{N} f_j(t)dt.$$

Now simpler explicit bounds can be obtained by using (7.1.56) or (7.1.57) with any of the bounds (7.1.33), (7.1.39), (7.1.46), for $|\Gamma(x,t)|$, where $|v|, |M|$ are any compatible norms for which $|v_i| \le |v|$ or $|m_{ij}| \le |M|$ for $1 \le i, j \le N$.

In case (7.1.54), a more elementary bound than (7.1.55) was given in [128], under some commutativity hypothesis which was relaxed by Conlan and Wang [143] and an alternative upper bound to (7.1.49) was given under a corresponding hypothesis. In addition, $K(x,t)$ was also assumed to be non-decreasing in each component x_j of $x = (x_1, \cdots, x_n)$. For brevity, we shall simply say: $K(x,t)$ is non-decreasing in x, etc.

In what follows, we shall obtain and extend all these results. We begin by proving some matrix analogues of Lemma 7.1.1.

Lemma 7.1.2 (Beesack [59]) *Let $F(s) = (f_{ij}(s))$ be integrable and non-negative on the cell $J = [a,b] \subset \mathbb{R}^n$ and let $N \ge 1$. Let*

$$M_a(s) = \int_a^s F(\sigma)d\sigma, \quad M_b(s) = \int_s^b F(\sigma)d\sigma, \quad s \in J. \tag{7.1.58}$$

Then,

$$\begin{cases} r\displaystyle\int_a^b F(s)M_a(s)^{r-1}ds \le M_a(b)^r \ \text{if} \ \ M_a(s)F(s) \ge F(s)M_a(s), \ \ s \in J, & (7.1.59) \\ r\displaystyle\int_a^b M_a(s)^{r-1}F(s)ds \le M_a(b)^r \ \text{if} \ \ F(s)M_a(s) \ge M_a(s)F(s), \ \ s \in J, & (7.1.60) \\ r\displaystyle\int_a^b F(s)M_b(s)^{r-1}ds \le M_b(a)^r \ \text{if} \ \ M_b(s)F(s) \ge F(s)M_b(s), \ \ s \in J, & (7.1.61) \\ r\displaystyle\int_a^b M_b(s)^{r-1}F(s)ds \le M_b(a)^r \ \text{if} \ \ F(s)M_b(s) \ge M_b(s)F(s), \ \ s \in J. & (7.1.62) \end{cases}$$

Proof In fact, inequality (7.1.59) is essentially proved in Conlon and Wang [143], with $F(s) = K(x,s), a = b = x$.

Inequality (7.1.60) follows from (7.1.59) by taking transposes, and noting that

$$A \leq B \Leftrightarrow A^T \leq B^T \quad \text{and} \quad (A^r)^T = (A^T)^r, \quad r = 0, 1, \cdots,$$

while $(\int F ds)^T = \int F^T ds$. Similarly, (7.1.62) is equivalent to (7.1.61) with F replaced by $\hat{F} = F^T$.

Inequality (7.1.61) follows from (7.1.59) by making the change of variables

$$\sigma = -\bar{\sigma}, \quad s = -\bar{s}, \quad a = -\bar{a}, \quad b = -\bar{b}, \quad \bar{F}(\bar{\sigma}) = F(-\bar{\sigma}), (\bar{s} \leq \bar{\sigma} \leq \bar{a}),$$

which thus yields

$$\int_s^b F(\sigma)d\sigma = (-1)^n \int_{\bar{s}}^{\bar{b}} F(-\bar{\sigma})d\sigma = \int_{\bar{b}}^{\bar{s}} \bar{F} \ (\bar{\sigma})d\bar{\sigma}.$$

Since $\bar{F}$ is integrable and non-negative on the cell $[\bar{b}, \bar{a}]$, it follows from (7.1.59) that

$$r \int_b^{\bar{a}} \bar{F}(\bar{s}) \left(\int_b^{\bar{s}} F(\bar{\sigma})d\bar{\sigma} \right)^{r-1} d\bar{s} \leq \left(\int_{\bar{b}}^{\bar{a}} \bar{F}(\bar{\sigma})d\bar{\sigma} \right)^r \tag{7.1.63}$$

since $(\int_{\bar{b}}^{\bar{s}} \bar{F}(\bar{\sigma})d\bar{\sigma})\bar{F}(\bar{s}) \geq \bar{F}(\bar{s})(\int_{\bar{b}}^{\bar{s}} \bar{F}(\bar{\sigma})d\bar{\sigma})$ for $\bar{b} \leq \bar{\sigma} \leq \bar{a}$ holds when

$$(\int_s^b F(\sigma)d\sigma)F(s) \geq F(s)(\int_s^b F(\sigma)d\sigma)$$

for $a \leq s \leq b$. In the same manner, (7.1.63) reduces to the conclusion of (7.1.61), and thus (7.1.61) follows. □

Remark 7.1.5 For the matrices $F(s)$, when $\int_a^s Fd\sigma$ commute, for all $s \in J$, (7.1.59) and (7.1.60) reduce to a single result. Similarly, so do (7.1.61) and (7.1.62) if $F(s)$, $\int_a^s Fd\sigma$ commute for all $s \in J$. When $N = 1$, Lemma 7.1.2 thus reduces to the special case of Lemma 7.1.1 with $F(u) = u^r$.

To give a full generalization of Lemma 7.1.1 to the matrix case, we consider analytic functions of $N \times N$ matrices $X = (x_{ij})$ such as

$$K(X) = \sum_{j=0}^{+\infty} a_j X^j, \quad \text{where} \quad X^0 = I \quad (\text{even if} \quad X = 0). \tag{7.1.64}$$

Here $a_j \in \mathbb{R}$ are such that the ordinary power series $\sum a_j z_j$ is convergent with radius of convergence R satisfying

$$\int_a^b |F(\sigma)|d\sigma < R. \tag{7.1.65}$$

Then we set

$$K'(X) = j\sum_{j=1}^{+\infty} ja_j X^{j-1} \quad \text{for} \quad X = (x_{ij}) \quad \text{with} \quad |X| < R. \tag{7.1.66}$$

In (7.1.65)–(7.1.66), the matrix norm may be any compatible norm.

Lemma 7.1.3 (Beesack [59]) *Let K, K' be given by (7.1.64), (7.1.66) where all $a_j \geq 0$. Let $F(s) = (f_{ij}(s))$ be integrable and non-negative on the cell $[a,b] \subset \mathbb{R}^n$ and suppose R satisfies (7.1.65). Then*

$$\begin{cases} K(0) + \int_a^b F(s)K'\left(\int_a^s F(\sigma)d\sigma\right) ds \leq K\left(\int_a^b F(\sigma)d\sigma\right) \quad \text{if} \quad (7.1.59) \quad \text{holds}, & (7.1.67) \\ K(0) + \int_a^b K'\left(\int_a^s F(\sigma)d\sigma\right) F(s)ds \leq K\left(\int_a^b F(\sigma)d\sigma\right) \quad \text{if} \quad (7.1.60) \quad \text{holds}, & (7.1.68) \\ K(0) + \int_a^b F(s)K'\left(\int_s^b F(\sigma)d\sigma\right) ds \leq K\left(\int_a^b F(\sigma)d\sigma\right) \quad \text{if} \quad (7.1.61) \quad \text{holds}, & (7.1.69) \\ K(0) + \int_a^b K'\left(\int_s^b F(\sigma)d\sigma\right) F(s)ds \leq K\left(\int_a^b F(\sigma)d\sigma\right) \quad \text{if} \quad (7.1.62) \quad \text{holds}. & (7.1.70) \end{cases}$$

Proof The proofs follow from the corresponding parts of Lemma 7.1.2 by using (7.1.64), (7.1.66) and the fact that all $a_j \geq 0$, since term-by-term integration may be carried out when (7.1.65) holds. Under the commutativity assumptions of Remark 7.1.5. The four inequalities above reduce to only two, and when $N = 1$, we obtain a (weaker) version of Lemma 7.1.1.

We now use some of these results to obtain upper bounds on the iterated kernels and on Γ using the matrix order relation "$\leq$".

Lemma 7.1.4 (Beesack [59]) *Let $K = (k_{ij})$ be non-negative, bounded and measurable on T, and set*

$$\hat{K}(x,t) = \sup\{K(s,t) : t \leq s \leq x\}, \quad \bar{K}(x,t) = \sup\{K(x,s) : t \leq s \leq x\}. \tag{7.1.71}$$

Then for all $r \geq 1$,

$$K^{(r)}(x,t) \leq \left(\int_t^x \hat{K}(x,\sigma)d\sigma\right)^{r-1} \hat{K}(x,t) \Big/ (r-1)!, \quad (x,t) \in T, \tag{7.1.72}$$

if either

$$\left(\int_t^s \hat{K}(x,\sigma)d\sigma\right) \hat{K}(x,s) \geq \hat{K}(x,s)\left(\int_t^s \hat{K}(x,\sigma)d\sigma\right), \quad a \leq t \leq s \leq x \leq b, \tag{7.1.73}$$

or

$$\hat{K}(x,s)\left(\int_t^s \hat{K}(x,\sigma)d\sigma\right) \geq \left(\int_t^x \hat{K}(x,\sigma)d\sigma\right)\hat{K}(x,s), \quad a \leq s \leq x \leq b. \tag{7.1.74}$$

Similarly, we have

$$K^{(r)}(x,t) \leq \hat{K}(x,t)\left(\int_t^x \hat{K}(x,\sigma)d\sigma\right)^{r-1} \Big/ (r-1)!, \quad (x,t) \in T, \tag{7.1.75}$$

if either

$$\left(\int_t^s \bar{K}(\sigma,t)d\sigma\right)\bar{K}(s,t) \geq \bar{K}(s,t)\left(\int_t^x \bar{K}(\sigma,t)d\sigma\right), \quad a \leq t \leq s \leq b, \tag{7.1.76}$$

or

$$\bar{K}(s,t)\left(\int_t^x \bar{K}(\sigma,t)d\sigma\right) \geq \left(\int_t^s \bar{K}(\sigma,t)d\sigma\right)\bar{K}(s,t), \quad a \leq s \leq t \leq x \leq b. \tag{7.1.77}$$

Proof Observe that $\hat{K}(x,t)$ is non-decreasing in each component of x, while $\bar{K}(x,t)$ is non-increasing in each component of t. The proofs follow from this observation and (7.1.59), (7.1.60) of Lemma 7.1.2, by induction on r. We omit the details, observing only that to prove (7.1.72) when (7.1.73) holds, or (7.1.75) when (7.1.76) holds, we can use (7.1.8); for the remaining two parts, we may use (7.1.18). □

Corollary 7.1.2 (Beesack [59]) *Under the notation of Lemma 7.1.4, we have for all* $(x,t) \in T$,

$$\Gamma(x,t) \leq \exp\left(\int_t^x \hat{K}(x,\sigma)d\sigma\right)\cdot \hat{K}(x,t), \tag{7.1.78}$$

if either (7.1.73) or (7.1.74) holds, and for all $(x,t) \in T$,

$$\Gamma(x,t) \leq \hat{K}(x,t)\exp\left(\int_t^x \hat{K}(\sigma,t)d\sigma\right), \tag{7.1.79}$$

if either (7.1.76) or (7.1.77) holds.

Proof The conclusions follow readily from (7.1.24). □

Corollary 7.1.3 (Beesack [59]) *If, in addition, f is non-negative, bounded and measurable on $J = [a,b]$, then for any bounded, measurable u satisfying (7.1.48), we have for all $x \in J$,*

$$u(x) \leq f(x) + \int_a^x \exp\left(\int_t^x \hat{K}(x,\sigma)d\sigma\right)\hat{K}(x,t)f(t)dt, \tag{7.1.80}$$

if either (7.1.73) or (7.1.74) holds, and for all $x \in J$,

$$u(x) \leq f(x) + \int_a^x \hat{K}(x,t) \exp\left(\int_t^x \hat{K}(x,\sigma)d\sigma\right) f(t)dt, \tag{7.1.81}$$

if either (7.1.76) or (7.1.77) holds.

Proof This assertion follows at once from Corollary 7.1.2 and (7.1.49). □

Remark 7.1.6 If $K(x,t)$ is non-decreasing in x, then $\hat{K}(x,t) = K(x,t)$, and similarly $\bar{K}(x,t) = K(x,t)$ if K is non-increasing in x was proved by Conlan and Wang [143], under hypothesis (7.1.9), by a somewhat different method. For a comparison of the two methods, we only note here that if $K(x,t)$ is non-decreasing in x, then

$$K^{(r+1)}(x,t) \leq K_r(x,t)K(x,t), \quad r \geq 1,$$

follows easily by induction on r, where K is defined as in [143].

Remark 7.1.7 If $f_i(t) \leq c$ (a constant > 0) for all $t \in J$, $1 \leq i \leq N$, and we write $u_c = (c, c, \cdots, c)^t$, then by applying Lemma 7.1.3 with $K(X) = \exp(X)$, (7.1.80) implies that if (7.1.74) holds,

$$u(x) \leq f(x) + \left\{\exp\left(\int_a^x \hat{K}(x,\sigma)d\sigma\right) - I\right\} u_c.$$

Theorem 7.1.4 (Beesack [59]) *Let* u, f, G, H *be bounded and measurable on* $[a,b]$ *with* f, G, H *non-negative, and suppose u satisfies (7.1.54). Then for all* $a \leq x \leq b$,

$$u(x) \leq f(x) + G(x) \int_a^x \exp\left(\int_t^x H(s)G(s)ds\right) H(t)f(t)dt, \tag{7.1.82}$$

either

$$\left(\int_t^s H(\sigma)G(\sigma)d\sigma\right) H(s)G(s) \geq H(s)G(s) \left(\int_t^s H(\sigma)G(\sigma)d\sigma\right), \quad a \leq t \leq s \leq b, \tag{7.1.83}$$

or

$$H(s)G(s) \left(\int_t^s H(\sigma)G(\sigma)d\sigma\right) \geq \left(\int_t^s H(\sigma)G(\sigma)d\sigma\right) H(s)G(s), \quad a \leq s \leq t \leq b. \tag{7.1.84}$$

Proof Using (7.1.50) and noting (7.1.51), we get

$$\Gamma(x,t) = G(x)V(x,t)H(t), \quad V(x,t) = \sum_{r=1}^{+\infty} V_r(x,t),$$

where

$$V_1(x,t) = I, \quad V_r(x,t) = \int_t^x H(s)G(s)V_{r-1}(s,t)ds, \quad r \geq 2.$$

Therefore

$$V_r(x,t) = \int_t^x V_{r-1}(s,t)H(s)G(s)ds, \quad r \geq 2$$

which was the notation used by Conlan and Wang [143], here $V_r(x,t) = K_{r-1}(x,t)$ for all $r \geq 1$. Hence, as shown by Lemma 5.4.11, if (7.1.83) holds, then for all $r \geq 0, \quad (x,t) \in T$,

$$V_{r+1}(x,t) \leq \left(\int_t^x H(s)G(s)ds \right)^r \Big/ r!, \tag{7.1.85}$$

which was shown in [143] by induction on r, by using the first recursion relation above for V_r together with part (7.1.59) of Lemma 7.1.2, with $F = H \cdot G, a = t$. To obtain (7.1.85) under the alternative hypothesis (7.1.84), we should proceed in the same way, but use the second recursion relation for V_r and (7.1.62) of Lemma 7.1.2 with $F = GH, b = x$.

It now follows that if either (7.1.83) or (7.1.84) holds, then for all $(x,t) \in T$,

$$V(x,t) = \sum_{r=1}^{+\infty} V_r(x,t) \leq \exp\left(\int_t^x HGds \right),$$

which yields (7.1.82) from (7.1.55). □

Remark 7.1.8 The half of this theorem with the hypothesis (7.1.83) and all continuous functions is in [82]. In this continuous case, when $N = 1$ and $G(x) \equiv 1$, the function $V(x,t)$ is the so-called Riemann function appearing in hyperbolic partial differential boundary value problems, see, for example [4, 54, 92, 621, 669, 677].

We conclude giving bounds similar to those given in (I)–(III) below, but using matrix inequalities rather than inequalities involving their norms. Throughout we still assume that $J = [a,b]$ is bounded, and that f, K are bounded, measurable, and non-negative on J, T respectively. We now use (7.1.49) instead of (7.1.25).

(I) Set

$$A(x,t) = \int_t^x K(s,t)ds, \quad B(t) = \sup_{t \leq s \leq x \leq b} K(x,s). \tag{7.1.86}$$

Precisely as for the scalar case, using (7.1.8) and induction on r, we find

$$K^{(r)}(x,t) \le B^{r-1}(t)\int_t^x \prod_{i=1}^n (x_i - s_i)^{r-3} A(s,t)ds \Big/ (r-3)!, \quad r \ge 3, \tag{7.1.87}$$

and further,

$$K^{(r)}(x,t) \le \prod_{i=1}^n (x_i - t_i)^{r-1} B^r(t) \Big/ (r-1)!, \quad r \ge 1. \tag{7.1.88}$$

Thus for all $(x,t) \in T$,

$$\Gamma(x,t) \le B(t)\exp_n\left[\prod_{i=1}^n (x_i - t_i)B(t)\right], \tag{7.1.89}$$

where $\exp_n(X)$ is defined by (7.1.34). If u is any bounded, measurable (N-vector) function on J which is a solution of the inequality (7.1.48), then for all $x \in J$,

$$u(x) \le f(x) + \int_a^x B(t)\exp_n\left[\prod_{i=1}^n (x_i - t_i)B(t)\right] f(t)dt. \tag{7.1.90}$$

The bounds (7.1.89), (7.1.90) do not depend on any pseudo-commutatively hypotheses such as those given below. However, the order of the two matrix factors in (7.1.89), (7.1.90) is essential.

(II) Here we define $A(x,t)$ as in (7.1.86) and set

$$B(x,t) = \sup_{t \le s \le x} K(x,s), \quad C(x,t) = B(x,t)A(x,t),$$

the matrix versions of (7.1.36). In fact, $B(x,t) = \bar{K}(x,t)$ as defined by (7.1.71). An analogue of the most useful result in the scalar case, namely (7.1.38), has already been proved above, as (7.1.75). There are, matrix analogues of (7.1.37), (7.1.39) and (7.1.40), but we omit details here.

(III) Here we define the matrix functions

$$\begin{cases} D(x) = \sup\limits_{a \le s \le x} K(x,s), \quad E(t) = \displaystyle\int_t^b K(s,t)ds, \\ D_1(x) = \displaystyle\int_a^x K(x,s)ds, \quad E_1(t) = \sup\limits_{t \le s \le b} K(s,t). \end{cases} \tag{7.1.91}$$

By using (7.1.8), we obtain for all $r \geq 2$ and all $(x, t) \in T$,

$$K^{(r)}(x,t) \leq D(x) \left(\int_t^x D(u)du \right)^{r-2} E(t) \Big/ (r-2)!, \tag{7.1.92}$$

provided that

$$\left(\int_t^s D(u)du \right) D(s) \geq D(s) \left(\int_t^s D(u)du \right), \quad a \leq t \leq s \leq b. \tag{7.1.93}$$

Similarly by using (7.1.18), we obtain for all $r \geq 2$ and all $(x, t) \in T$,

$$K^{(r)}(x,t) \leq D_1(x) \left(\int_t^x E_1(u)du \right)^{r-2} E_1(t) \Big/ (r-2)!, \tag{7.1.94}$$

provided that

$$E_1(s) \left(\int_s^x E_1(u)du \right) \geq \left(\int_s^x E_1(u)du \right) E_1(s), \quad a \leq s \leq x \leq b. \tag{7.1.95}$$

We note that in (7.1.59) in Lemma 7.1.2 is used in proving (7.1.92) and in (7.1.60) in Lemma 7.1.2 in proving (7.1.94). The inequalities (7.1.92), (7.1.94) are the matrix analogues of (7.1.42), (7.1.45) respectively, but note the order of the factors. Similarly, from (7.1.24), we now obtain the matrix analogues of (7.1.43), (7.1.46) respectively, namely,

$$\begin{cases} \Gamma(x,t) \leq K(x,t) + D(x) \exp\left(\int_t^x D(u)du \right) E(t), \quad (x,t) \in T, \\ \qquad\qquad \text{if} \quad (7.1.93) \quad \text{holds}, & (7.1.96) \\ \Gamma(x,t) \leq K(x,t) + D_1(x) \exp\left(\int_t^x E_1(u)du \right) E_1(t), \quad (x,t) \in T, \\ \qquad\qquad \text{if} \quad (7.1.95) \quad \text{holds}. & (7.1.97) \end{cases}$$

7.1.2 The Volterra Integral Equations and Inequalities

In this section, we shall study the semilinear Volterra integral inequality in the general form

$$x(t) \leq w(t) + \int^t K(t,s)x(s)\,ds + \mathcal{A} \int^t f(x,s,x(s))\,ds \tag{7.1.98}$$

and shall provide the sufficient conditions that Eq. (7.1.98) exists a unique global L^p solution. Note that this result, due to Hu [297], can cover many known results in the literature, for example, it can cover many important results about multi-dimensional Volterra integral inequalities in [59], but the method used here is essentially different from those in [59], i.e., those in Sect. 7.1.1.

Let G be the local compact Hausdorff space with regular Borel measure μ, "$\prec$" be the given order in G, and satisfy

(A_1) $r \prec s \prec t$ $(r, s, t \in G) \Rightarrow r \prec t$;
(A_2) $\{(t, s); t \prec s \prec t\}$ is the $\mu \times \mu$ zero-measurable set in $G \times G$;
(A_3) $G_t \equiv \{s \in G : s \prec t\}$ is a compact set, $G^t \equiv \{s \in G : t \prec s\}$ is a μ measurable set;
(A_4) let $\varphi(t, \cdot)$ be the eigenfunction of G_t, then $\varphi(t, s)$ is $\mu \times \mu$ a measurable function in $G \times G$ (this means that $D = \{(t, s) : s \prec t\}$ is $\mu \times \mu$ measurable in $G \times G$));
(A_5) $\lim_{s\to t} \mu(G_s \Delta G_t) = 0$, where Δ is the symmetric difference;
(A_6) for any $t \in G$, there exist a countable set $G' \subset G$ and some $t' \in G'$ such that $t \prec t'$.

Remark 7.1.9 The notation "$\prec$" is not necessarily the ordinary partial order, even it can not be required to satisfy the property of reflexivity. We can't use the assumption on "connectedness" that plays an important role in [41, 560, 561], so that the present result on G can be also extended to the discrete case.

Remark 7.1.10 It is easy to prove that $G_t \cup \{t\}$ and G^t $(t \in G)$ satisfy $(A_1) - (A_6)$, thus the present results are also valid on $G_t \cup \{t\}$ and G^t $(t \in G)$.

There are special cases about G in the following :

(1) $G = a + \mathbb{R}^n_+$ $(a \in \mathbb{R}^n)$, let $\prec$ be ordinary vector order $\leq$.
(2) $G = \mathbb{R}^n_+$; let $t \prec s \Leftrightarrow t \leq \varphi(s), \varphi = (\varphi_i) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ satisfy $\varphi(t) \leq t$, $\sup_t \varphi_i(t) = +\infty$ $(1 \leq i \leq n)$, and $\leq$ be ordinary vector order.
(3) $G = \mathbb{R}^n$; $t \prec s \Leftrightarrow |t| \leq |s|$, $|\cdot|$ is the Euclidian norm.
(4) $G = Q \times \mathbb{R}^n$, $Q \subset \mathbb{R}^n$ is a compact set; let $(x, y) \prec (u, v) \Leftrightarrow |y| \leq |v|$ $(x, u \in Q, y, v \in \mathbb{R}^n)$.
(5) $G = a + \mathbb{Z}^n_+$ $(a \in \mathbb{Z}^n)$; let $t \prec s \Leftrightarrow t_i < s_i$ $(1 \leq i \leq n, t = (t_i), s = (s_i) \in G)$.

It is easy to verify that (1)–(4) satisfy (A_1)–(A_6) in terms of the Lebesgue measure and (5) satisfies (A_1)–(A_6) in terms of countable measure. Note $G_t = \bar{B}(0, |t|)$ $(t \in \mathbb{R}^n)$ for (3), and $G_t = Q \times \bar{B}(0, |y|)$ $(t = (x, y) \in Q \times \mathbb{R}^n)$ for (4).

Let $(X, |\cdot|)$ be an ordered real Banach space, where $\leq$ is determined by some cone X_+ (see, [172]). For any $K \in L(X)$, let $K \geq 0 \Leftrightarrow KX_+ \subset X_+$. For any $x \in L^1_{loc}(G, X)$, we know that x is integrable on every G_i by using (A_3); let

$$\int^t x(s)\, ds = \int_{G_t} x(s)\, d\mu(s), \quad \int_\tau^t x(s)\, ds = \int_{G_t \cap G^\tau} x(s)\, d\mu(s).$$

By (A_5) and using

$$\Big|\int^t x(s)\,ds - \int^\tau x(s)\,ds\Big| \le \int_{G_t \cap G_\tau} |x(s)|ds,$$

we deduce $t \mapsto \int^t x(s)\,ds$ is continuous.

Lemma 7.1.5 (Hu [297]) *Let* $A \in L^1_{loc}(G, \mathbb{R}_+)$, $k \ge 1$, *then for all* $t \in G$,

$$\int^t A(s)\left[\int^s A(\tau)\,d\tau\right]^{k-1} ds \le \frac{1}{k}\left[\int^t A(s)\,ds\right]^k. \tag{7.1.99}$$

Proof Obviously, (7.1.99) holds for $k = 1$. Suppose $k \ge 2$ and J is the left-hand side of (7.1.99), then for any $i \in \{2, 3, \cdots, k\}$, there holds that

$$\begin{aligned} J &= \int^t \int^{t_1} \cdots \int^{t_k} A(t_1)A(t_2)\cdots A(t_k)\,dt_k \cdots dt_2 dt_1 \\ &= \int^t \int^{t_1} \cdots \int^{t_k} A(t_1)A(t_2)\cdots A(t_k)\,dt_k \cdots dt_{i+1}dt_i dt_{i-1}\cdots dt_2 dt_1 \\ &= \int^t \int^t \cdots \int^t \prod_{j\neq i} \varphi(t_i, t_j)A(t_1)A(t_2)\cdots A(t_k)\,dt_1 dt_2 \cdots dt_k, \end{aligned}$$

where φ satisfies (A_4), whence

$$J = \frac{1}{k}\int^t \int^t \cdots \int^t \sum_{i=1}^k \prod_{j\neq i} \varphi(t_i, t_j)A(t_1)A(t_2)\cdots A(t_k)\,dt_1 dt_2 \cdots dt_k. \tag{7.1.100}$$

On the other hand, we derive from (A_2) that

$$\sum_{i=1}^k \prod_{j\neq i} \varphi(t_i, t_j) \le 1 \quad a.e. \ \text{on} \ G^k$$

which, along with (7.1.100), implies (7.1.99). □

Lemma 7.1.6 (Hu [297]) *Let* $A \in L^1_{loc}(G, \mathbb{R}_+)$, *then for all* $t \in G$,

$$1 + \int^t A(s) \exp\left(\int^t A(\tau)\,d\tau\right) ds \le \exp\left(\int^t A(s)\,ds\right). \tag{7.1.101}$$

Proof Using the Taylor expansion of function $z \mapsto e^z$ to expand the both sides of (7.1.101) into two series, then using (7.1.99) to compare each term of the above two series, we can complete the proof. □

To study the generalized Volterra integral inequalities, we now introduce some knowledge of the corresponding Volterra equations. To this end, we shall give some sufficient conditions that there exists a unique global solution $x \in L^p(G, X)$ to the equation

$$x(t) = w(t) + \int^t K(t,s)x(s)\,ds + \mathcal{A}\int^t f(t,s,x(s))\,ds \tag{7.1.102}$$

with $1 < p < +\infty,\ 1/p + 1/q = 1$, which can be used to study the special type of (7.1.98) in the following:

$$\begin{cases} x(t) = g\left(t, \int^t f(t, sx(s))\,ds\right), & (7.1.103)\\ x(t) = w(t) + \int^t K(t,s)x(s)\,ds + \int^t f(t,s,x(s))\,ds & (7.1.104)\\ x(t) = w(t) + \int^t K(t,s)x(s)\,ds + \int^t\int^s L(t,s)f(s,\tau,x(\tau))\,d\tau ds, & (7.1.105)\\ x(t) = w(t) + \int^t K(t,s)x(s)\,ds, & (7.1.106)\\ x(t) = w(t) + \int^t f(t,s,x(s))\,ds. & (7.1.107) \end{cases}$$

Note that the authors in [59, 560] once studied L^2-solution of Eqs. (7.1.106)–(7.1.107) under special assumptions, but their methods can not be generalized directly to employ to the present case.

For any $u \in L^p(G)$, we set $\|u\|_{p,t} = \|u|_{G_t}\|_p$. For $\mathcal{A},\ f,\ g,\ K,\ L$, we assume:

(H_1) $\mathcal{A} : L^p(G,X) \to L^p(G,X)$, let $x, y \in L^p(G,X)$ satisfy one of the following conditions:

(H_1') there exists a constant $\beta > 0$, independent of x, y such that $|\mathcal{A}x(t) - \mathcal{A}y(t)| \le \beta|x(t) - y(t)|$ (here and hereinafter "$\le$" holds in the sense of "$\mu - a.e.$").

(H_1'') there exists a function $u \in L^p(G, \mathbb{R}_+)$, independent of x, y such that $|\mathcal{A}x(t) - \mathcal{A}y(t)| \le u(t)\|x - y\|_{p,t}$.

(H_2) $f(t,s,x) : D \times X \to X$ is measurable with respect to $(t,s) \in D$; and there exists a measurable function $\psi : D \to \mathbb{R}_+$ such that $|f(t,s,x) - f(t,s,y)| \le \psi(t,s)|x - y|$, $((t,s) \in D, x, y \in X)$ and $v(t) \equiv \|\psi(t,\cdot)|G_t\|_q \in L^p(G)$; $m(t) \equiv \int^t |f(t,s,0)|\,ds \in L^p(G)$.

(H_3) $K : D \to L(X)$ is measurable; $k(t) \equiv \|K(t,\cdot)|G_t\|_q \in L^p(G)$.

(H_4) $g(t,x) : G \times X \to X$ is measurable with respect to t; $g(t,0) \in L^p$: there exists a constant $\beta > 0$ such that $|g(t,x) - g(t,y)| \le \beta|x - y|$ $(t \in G, x, y \in X)$.

(H_5) $L : D \to L(X)$ is measurable; $l(t) \equiv \|L(t,\cdot)|G_t\|_q \in L^p(G)$.

Theorem 7.1.5 (Hu [297]) *Suppose that $w \in L^p(G, X)$, and (H_1)–(H_3) hold, then there exists a unique global L^p-solution in G to Eq. (7.1.102).*

Proof

(1) For any $x \in L^p(G, X)$, let

$$Fx(t) = \int^t f(t, s, x(s))\, ds, \quad t \in G. \tag{7.1.108}$$

We deduce from (H_2) that f satisfy Caratheodory condition, thus $s \mapsto f(t, s, x(s))$ is measurable. Applying (H_2) and the Hölder inequality, we have

$$\begin{aligned} \int^t |f(t, s, x(s))|\, ds &\le \int^t [|f(t, s, 0)| + \psi(t, s)|x(s)|]\, ds \\ &\le m(t) + v(t)\|x\|_{L^p} < +\infty, \quad \mu - a.e., \end{aligned}$$

which implies that $Fx(t)$ is well-defined for almost all $t \in G$, and

$$|Fx(t)| \le m(t) + v(t)\|x\|_{L^p}. \tag{7.1.109}$$

We derive from (A_4) that the function $(t, s) \mapsto \varphi(t, s)f(t, s, x(s))$ is measurable. For any $\tau \in G$, we have

$$\begin{aligned} \int^t \int^t |\varphi(t, s)f(t, s, x(s))| dtds &\le \int^t [m(t) + v(t)\|x\|_{L^p}]\, dt \\ &\le (\mu G_t)^{1/q} \left(\|m\|_p + \|v\|_p\|x\|_{L^p}\right) < +\infty, \end{aligned}$$

which, by Fubini Theorem, yields that

$$Fx(t) = \int^t \varphi(t, s)f(t, s, x(s))\, ds, \qquad t \in G_t$$

is measurable on G_t. On the other hand, we deduce from (H_6) that $Fx(t)$ is measurable on G; and it follows from (7.1.109) that $Fx \in L^p$. Thus, (7.1.108) can define the operator $F : L^p(G, X) \to (G, X)$.

(2) Applying condition (H_3) and (1), and using

$$Ux(t) = \int^t K(t, s)x(s)\, ds, \quad x \in L^p(G, X), \tag{7.1.110}$$

we can define a linear operator $U : L^p(G,X) \to L^p(G,X)$. Thus, by the following relations

$$Tx = w + Ux + \mathcal{A}Fx, \quad x \in L^p(G,X), \tag{7.1.111}$$

we can define the operator $T : L^p(G,X) \to L^p(G,X)$.

(3) Let $h(t) = v(t) + m(t)$ (if (H_1') is satisfied, we set $h(t) = v(t) + m(t) + u(t)$); choose $a > 1$ and set

$$\begin{cases} w(t) = \exp\left(-\dfrac{a}{p}\|h\|_{p,t}^p\right), \quad t \in G, & (7.1.112) \\ \|x\|_{p,w} = \sup\limits_{t\in G} w(t)\|x\|_{p,t}, \quad x \in L^p(G,X). & (7.1.113) \end{cases}$$

It is clear that $\|\cdot\|_{p,w}$ is a norm verifying $\|x\|_{p,w} \le \|x\|_p$; next, we deduce from (A_6) that $\sup_t \|x\|_{p,t} = \|x\|_p$; thus,

$$\|x\|_{p,w} \ge \inf_{t\in G} w(t) \sup_{t\in G} \|x\|_{p,t} \ge \exp\left(-\frac{1}{p}\|h\|_p^p\right)\|x\|_p.$$

This means that $\|\cdot\|_{p,w}$ is equivalent to $\|\cdot\|_p$ on $L^p(G,X)$.

(4) For every $x, y \in L^p(G,X)$, we derive from (H_2),

$$|Fx(t) - Fy(t)| \le \int^t \psi(t,s)|x(s) - y(s)|\,ds \le v(t)\|x-y\|_{p,t} \le h(t)\|x-y\|_{p,t};$$

$$\begin{aligned} \|Fx - Fy\|_{p,t} &\le \left\{\int^t [h(s)\|x-y\|_{p,s}]^p\,ds\right\}^{1/p} \\ &\le \left\{\int^t [h_1(s)]^p\,ds\right\}^{1/p} \|x-y\|_{p,w} \end{aligned} \tag{7.1.114}$$

where $h_1 = h/w$.

Applying Lemma 7.1.6 to $A(t) = a[h(t)]^p$, we obtain

$$\begin{aligned} \int^t [h_1(s)]^p\,ds &= \frac{1}{a}\int^t A(s)\exp\left(\int^s A(\tau)\,d\tau\right)ds \\ &\le \frac{1}{2}\exp\left(\int^t A(s)\,ds\right) = \frac{1}{a}\exp\left(a\|h\|_{p,t}^p\right) \\ &= \frac{1}{a[w(t)]^p}, \end{aligned} \tag{7.1.115}$$

i.e.,

$$w(t)\|h_1\|_{p,t} \le a^{-1/p} = r_a$$

which, together with (7.1.114)–(7.1.115), yields

$$\|Fx - Fy\|_{p,w} \le r_a \|x - y\|_{p,w}. \tag{7.1.116}$$

(5) Using condition (H_3) and (4), we derive for all $x \in L^p(G, X)$,

$$\|Ux\|_{p,w} \le r_a \|x\|_{p,w}. \tag{7.1.117}$$

For any $x, y \in L^p(G, X)$, and if (H_1') holds, then

$$\|\mathcal{A}x - \mathcal{A}y\|_{p,w} \le \beta \|x - y\|_{p,w}. \tag{7.1.118}$$

If (H_1'') holds, then $u \le h$, similarly to the proof of (7.1.116) in (4), we can obtain

$$\|\mathcal{A}x - \mathcal{A}y\|_{p,w} \le r_a \|x - y\|_{p,w}. \tag{7.1.119}$$

Thus combination of (7.1.116)–(7.1.119) implies

$$\|Tx - Ty\|_{p,w} \le (1 + \beta + r_a) r_a \|x - y\|_{p,w}. \tag{7.1.120}$$

Now taking $a > 1$ so large that $(1 + \beta + r_a) r_a < 1$, then T is a contractive mapping with respect to $\|\cdot\|_{p,w}$. Using the Banach Fixed Point Theorem, there exists a unique $x \in L^p(G, X)$ such that $x = Tx$, that is, x is a unique global L^p-solution of (7.1.102). The proof is thus complete.

□

Corollary 7.1.4 (Hu [297]) *If (H_2) and (H_4) hold, then there exists a unique L^p-solution on G for problem (7.1.103).*

Proof Let $\mathcal{A}x(t) = g(t, x(t))$, we infer from (H_4) that $\mathcal{A} : L^p(G, X) \to L^p(G, X)$ and $\mathcal{A}$ satisfies (H_1'). On the other hand, (7.1.103) is equivalent to $x = \mathcal{A}Fx$. Thus we obtain the corollary by virtue of Theorem 7.1.5. □

Corollary 7.1.5 (Hu [297]) *If $w \in L^p$, and (H_2), (H_3) and (H_5) hold, then there exists a unique L^p-solution on G to Eq. (7.1.105).*

Proof Let $\mathcal{A}x(t) = \int^t L(t, s)x(s)\, ds$, we can deduce from (H_5) that $\mathcal{A} : L^p(G, X) \to L^p(G, X)$ and $\mathcal{A}$ satisfies (H_1''). On the other hand, (7.1.105) is equivalent to $x = w + Ux + \mathcal{A}Fx$. Thus we obtain the corollary by virtue of Theorem 7.1.5. □

Taking $\mathcal{A} = identity \quad operator \equiv id$, we get from Theorem 7.1.5 the following corollary.

Corollary 7.1.6 (Hu [297]) *If $w \in L^p$, and (H_2)–(H_3) hold, then there exists a unique L^p-solution on G to Eq. (7.1.104).*

Taking $K = 0$ or $f = 0$, we derive the next corollary from Corollary 7.1.6.

Corollary 7.1.7 (Hu [297]) *If $w \in L^p$, and (H_2) (or (H_3)) holds, then there exists a unique L^p-solution on G for (7.1.107) (or (7.1.106)).*

Remark 7.1.11 If taking $G = a + \mathbb{R}^n_+ \quad (a \in \mathbb{R}^n), X = \mathbb{R}^m, p = 2$, we can conclude Theorem 7 in [59] by virtue of Corollary 7.1.7. But the method in [59] is only designed for L^2 framework, it is difficult to generalize to this case. Thus the results here have essentially improved those in [59], and the method used here is also stimulated by Kosińnski [328].

If (H_3) holds, then $\|U\|_w \leq r_a < 1$ by virtue of (7.1.117), where $\|\cdot\|$ is the relative operator norm with respect to $\|\cdot\|_{p,w}$, U is defined by (7.1.110). Then the unique L^p solution of (7.1.106) can be expressed by Neumann series

$$x = (id - U)^{-1}w = \sum_{j=0}^{+\infty} U^j w. \tag{7.1.121}$$

A straightforward calculation gives us

$$U^j w(t) = \int^t K_j(t,s)w(s)\,ds, \quad j \geq 1 \tag{7.1.122}$$

where $K_1 = K$; when $j \geq 2$ (let $t_0 = t$),

$$K_j(t,s) = \int_s^t \int_s^{t_1} \cdots \int_s^{t_{j-1}} K(t,t_1)K(t_1,t_2)\cdots K(t_{j-1},s)\,dt_{j-1}\cdots dt_1. \tag{7.1.123}$$

Remark 7.1.12 If $w \in L^p$, $K \in L^p$, we can improve (7.1.121), by virtue of more delicate argumentation, as

$$x(t) = w(t) + \int^t \Gamma(t,s)w(s)\,ds \tag{7.1.124}$$

where, for all $(t,s) \in D$,

$$\Gamma(t,s) = \sum_{j=1}^{+\infty} K_j(t,s), \tag{7.1.125}$$

where the right-hand side series of (7.1.125) is absolutely convergent almost everywhere and L^p convergent on D.

Having discussed the Volterra equation, we now turn to discuss the corresponding Volterra integral inequality (7.1.98). These results can be applied to the inequalities relative to Eqs. (7.1.103)–(7.1.107).

The natural order "$\leq$" can be determined by a cone in $L^p(G,X)$, and it is obvious that $\mathcal{A} : L^p(G,X) \to L^p(G,X)$ is an accretive operator.

Theorem 7.1.6 (Hu [297]) *Under the assumptions of Theorem 7.1.5, and suppose $\mathcal{A}$ is an accretive operator, $f(t,s,y)$ is monotone increasing with respect to y; $x \in L^p(G,X)$ satisfies inequality (7.1.98), and $\bar{x}$ is the unique L^p-solution of Eq. (7.1.102). Then $x \leq \bar{x}$.*

Proof Let $y = w + Ux + \mathcal{A}Fx - x$, where U and F are defined by (7.1.110) and (7.1.108) respectively. Then, $y \in L^p$. We deduce from (7.1.98) that $y \geq 0$ and x satisfies

$$x = (w - y) + Ux + \mathcal{A}Fx \equiv T_1 x. \tag{7.1.126}$$

By virtue of the assumptions that $U,\ \mathcal{A},\ F$ are accretive operators in Theorem 7.1.6, we deduce that T is accretive, where T is defined (7.1.111). By $y \geq 0$, we get $T_1 z \leq Tz$ (for all $z \in L^p(G,X)$). Furthermore, we obtain $T_1^n z \leq T^n z \quad (n \geq 1)$ by induction. Using the Banach Fixed Point Theorem and the proof of Theorem 7.1.5, we have

$$x = \lim_n T_1^n 0, \quad \bar{x} = \lim_n T^n 0 \tag{7.1.127}$$

where 0 is the zero element in $L^p(G,X)$. Noting that L^p convergence must imply that a subsequence converges almost everywhere, we get $T_1^n 0 \leq T^n 0$. This, along with (7.1.127), implies $x \leq \bar{x}$. □

Corollary 7.1.8 (Hu [297]) *Under the assumptions of Corollary 7.1.4, suppose that $f(t,s,y)$ and $g(t,y)$ are monotone increasing with respect to y; $x \in L^p(G,X)$ satisfies*

$$x(t) \leq g\left(t, \int^t f(t,s,x(s))\,ds\right), \tag{7.1.128}$$

and $\bar{x}$ is the L^p-solution of Eq. (7.1.103), then

$$x \leq \bar{x}. \tag{7.1.129}$$

Corollary 7.1.9 (Hu [297]) *Under the assumptions of Corollary 7.1.5, suppose that $K(t,s) \geq 0,\ L(t,s) \geq 0;\ \ f(t,s,y)$ is monotone increasing with respect to y; $x \in L^p(G,X)$ satisfies*

$$x(t) \leq w(t) + \int^t K(t,s)x(s)\,ds + \int^t \int^s L(t,s)f(s,\tau,x(\tau))\,d\tau ds, \tag{7.1.130}$$

and $\bar{x}$ is the L^p-solution of Eq. (7.1.105), then

$$x \leq \bar{x}. \tag{7.1.131}$$

Corollary 7.1.10 (Hu [297]) *Suppose $w \in L^p$, and $f(t,s,y)$ is monotone increasing with respect to y and satisfies (H_2); $x \in L^p(G,X)$ satisfies*

$$x(t) \leq w(t) + \int^t f(t,s,x(s))\,ds \tag{7.1.132}$$

and $\bar{x}$ is the L^p-solution of Eq. (7.1.107), then

$$x \leq \bar{x}. \tag{7.1.133}$$

If K satisfies (H_3) and $K(t,s) \geq 0$; $x \in L^p(G,X)$ satisfies

$$x(t) \leq w(t) + \int^t K(t,s)x(s)\,ds \tag{7.1.134}$$

and $\bar{x}$ is the L^p-solution of Eq. (7.1.106), then

$$x \leq \bar{x}. \tag{7.1.135}$$

Remark 7.1.13 If $w \in L^p$, $K \in L^p$, $x \in L^p(G,X)$ satisfies (7.1.134), then combining Corollary 7.1.10 with Remark 7.1.11 gives us

$$x(t) \leq w(t) + \int^t \Gamma(t,s)w(s)\,ds. \tag{7.1.136}$$

This is a general Gronwall inequality, and implies essentially Theorem 3 in [59].

If $w = 0, f(t,s,0) = 0$, then (7.1.132)–(7.1.133) admit zero solutions, which can be deduced from Corollaries 7.1.9–7.1.10.

Corollary 7.1.11 (Hu [297]) *Suppose $f(t,s,0) \equiv 0$, and K, L, f satisfy the assumptions of Corollary 7.1.9; $x \in L^p(G,X)$ satisfies*

$$x(t) \leq w(t) + \int^t K(t,s)x(s)\,ds + \int^t \int^s L(t,s)f(s,\tau,x(\tau))\,d\tau ds, \tag{7.1.137}$$

then

$$x \leq 0. \tag{7.1.138}$$

If $f(t,s,y)$ satisfies (H_2) and is monotone increasing with respect to y; and $x \in L^p(G,X)$ satisfies

$$x(t) \leq \int^t f(t,s,x(s))\,ds, \tag{7.1.139}$$

then

$$x \leq 0. \tag{7.1.140}$$

If K satisfies (H_3) and $K(t,s) \geq 0$; $x, y \in L^p(G, X)$ satisfy

$$x(t) - \int^t K(t,s)x(s)\,ds \leq y(t) - \int^t K(t,s)y(s)\,ds, \tag{7.1.141}$$

then

$$x \leq y. \tag{7.1.142}$$

7.2 Linear Multi-Dimensional Discontinuous Integral Inequalities in Banach Spaces

In this section, we shall introduce some multi-dimensional linear discontinuous integral inequalities in Banach spaces which are due to Chandra and Feishman [130], and Ronkov and Bainov [560].

7.2.1 A Generalization of the Gronwall-Bellman Inequalities in Partially Ordered Banach Spaces

In this section, we shall introduce the results due to Chandra and Feishman [130].

Let B denote a real Banach space, let $K \subset B$ be a cone (see, [320]) of "positive" elements and let L is a linear operator mapping a subset of B into B. A partial ordering may be introduced in B in the following way: for all

$$x, y \in B, \quad x \leq y \quad \text{if and only if} \quad (y - x) \in K.$$

In spaces of common interest like $C[0,T]$, $L^p[0,T]$, etc., a natural choice for K is the cone of non-negative functions. In these cases, the partial ordering assumes a simple meaning. For instance, if $u, v \in C[0,T]$, then $u \leq v$ means that $u(t) \leq v(t)$ for all $t \in [0,T]$.

Consider the operator equation

$$u = Nu + p \tag{7.2.1}$$

where p is a fixed element in B and N maps B into B. Throughout this section, we shall assume that for all $u \in B$, the following holds

$$Nu + p \leq Mu + q \tag{7.2.2}$$

where q is a fixed element in B and M maps B into B. Then clearly any solution of (7.2.1) will satisfy the operator inequality

$$u \leq Mu + q. \tag{7.2.3}$$

Clearly, inequality (7.2.3) may be regarded as an abstract analogue of the classical Gronwall-Bellman inequality (1.1.3) in Theorem 1.1.2.

We make the following assumptions once and for all:

(A1) L has a bounded inverse L^{-1} which is defined on B and leaves the cone K invariant.

(A2) M is Lipschitz continuous with Lipschitz constant β satisfying

$$\beta\|L^{-1}\| < 1.$$

We shall obtain bounds on the solution of inequality (7.2.3) in terms of the solutions of the corresponding equation

$$v = Mv + q. \tag{7.2.4}$$

Lemma 7.2.1 (Chandra-Feishman [129]) *Let* $(A_1), (A_2)$ *and (7.2.2) hold. Let either N or M (or both) be monotonic on B, that is, if for all* $u, v \in B$,

$$u \geq v,$$

then

$$Nu \geq Nv. \tag{7.2.5}$$

If $z \in B$ *is a solution of* $Lu = Nu + p$, *then*

$$z \leq \phi.$$

Proof

Case 1. Let M be monotonic. For $n = 1, 2, \cdots$, set

$$\phi_n = L^{-1}(M\phi_{n-1} + q), \quad \phi_0 = z.$$

Then using (7.2.2)

$$\phi_0 = z = L^{-1}(Nz + p) \leq L^{-1}(Mz + q) = L^{-1}(M\phi_0 + q) = \phi_1.$$

We have used here the fact that

$$\{(Mz + q) - (Nz + p)\} \in K,$$

therefore,

$$L^{-1}\{(Mz + q) - (Nz + p)\}$$

is in K. Now from (7.2.5)

$$\phi_2 - \phi_1 = L^{-1}(M\phi_1 - M\phi_0) \in K.$$

Thus by induction

$$z \leq \phi_1 \leq \phi_2 \leq \cdots \leq \phi_n \leq \cdots,$$

that is, the sequence $\{(\phi_n - z)\}$ is in K. Since K is closed $lim(\phi_n - z) \in K$. But ϕ_n converges to the unique solution of $Lu = Mu + q$. Hence $(\phi - z) \in K$.

Case 2. Let N be monotonic. Again $z \leq \phi_1$. Now from (7.2.2) and (7.2.5),

$$\phi_2 - z = L^{-1}[(M\phi_1 + q) - (N\phi_1 + p) + (N\phi_1 + p) - (Nz + p)] \in K.$$

If we assume

$$(\phi_n - z) \in K,$$

it follows similarly that

$$(\phi_{n+1} - z) \in K.$$

Proceeding to the limit, we have $(\phi - z) \in K$.

□

The following hypothesis is common to Theorems 7.2.1 and 7.2.2 : For all $u, v \in B$, let M satisfy

$$\| Mu - Mv \| \leq \omega(\| u - v \|) \tag{7.2.6}$$

where $\omega(r)$ is non-negative and continuous for all $r \geq 0$.

Theorem 7.2.1 (Chandra-Feishman [130]) *Let $\omega(r) < r$ for all $r > 0$ and let M or N or both be monotonic. Then the unique solution ϕ of Eq. (7.2.4) is an upper bound on all solutions of inequality (7.2.3).*

Proof From the hypothesis on ω, it is clear that M is a nonlinear contraction on B. Hence in view of [110], the proof of Lemma 7.2.1 may be modified to complete the proof of the theorem. □

Theorem 7.2.2 (Chandra-Feishman [130]) *Let M be completely continuous and monotonic on B and let $\omega(r)$ be a non-decreasing function of r with the following property:*

(A) there exists $r^* > 0$ such that for all $r \geq r^*$,

$$\omega(r)+ \| q \| + \| M\theta \| \leq r.$$

If ϕ is the maximal solution of Eq. (7.2.4), then ϕ is an upper bound on all solutions of inequality (7.2.3)

Proof Let u be any solution of Eq. (7.2.1). For $n = 1, 2, \ldots,$ set

$$v_n = Mv_{n-1} + q, \quad v_0 = u, \tag{7.2.7}$$

then

$$v_1 = Mv_0 + q \geq Mu + q \geq u = v_0$$

where we have used inequality (7.2.3). From the monotonicity of M, an induction on n shows that $\{v_n\}$ is non-decreasing sequence in B. If now we suppose that for some n, $\| v_n \| \leq R$, $(R > 0)$, then (7.2.7) in conjunction with (7.2.6) and the monotonicity of the scalar function ω gives us

$$\begin{aligned} \| v_{n+1} \| &\leq \omega(\| v_n \|)+ \| M\theta \| + \| q \| \\ &\leq \omega(R)+ \| M\theta \| + \| q \| . \end{aligned}$$

Therefore, by using the above property (A), we conclude that $\| v_{n+1} \| \leq R$ whenever $R \geq r^*$. Recall that M is a completely continuous operator; hence for the compactness of the sequence $\{v_n\}$, we need only select $R \geq \max(\| u \|, r^*)$. Thus, the monotonic and compact sequence $\{v_n\}$ converges to some $v \in B$, and the continuity of M implies that v is, in fact, a solution of Eq. (7.2.4). But from the maximality, $v \leq \phi$. Thus $u \leq v \leq \phi$, which finishes the proof. □

Let the mapping $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and satisfy for any $x, y \in \mathbb{R}^n$,

$$x \leq y \Longrightarrow f(t, x) \leq f(t, y).$$

Here the relation "$\leq$" between any two points $x = (x_1, \ldots x_n), y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ means that

$$x \leq y \quad \text{if and only if} \quad x_i \leq y_i \qquad i = 1, 2, \ldots, n.$$

Let $B = C^n[0, T]$ whose components are non-negative on $[0, T]$. Define an operator M mapping $C^n[0, T]$ into itself as follows for all $x \in C^n[0, T]$,

$$Mx = \int_0^t f(s, x(s))ds,$$

where f is the above function which is assumed to be continuous and non-decreasing in its second argument. An application of Arzel's Theorem guarantees that M is completely continuous operator, and the monotonicity of M is a consequence of the monotonicity of f. Now for any $u, v \in B$ such that $\| u \| \leq R, \| v \| \leq R$, where $0 < R < +\infty$, we observe that

$$\| Mu - Mv \| \leq 2TF$$

where

$$F = \max_{0 \leq t \leq T, \|x\| \leq R} | f(t, x) | .$$

Thus by choosing for $\omega(r)$ the identically constant function $2TF$, we can satisfy the remaining hypotheses of Theorem 7.2.2.

We should point out, however, that an application of either Theorem 7.2.1 or Theorem 7.2.2, in general, poses the formidable task of choosing properly the scalar function $\omega(r)$. This, in turn, will depend on a judicious selection of "majorant" operator M for a given problem so that the inequality (7.2.2) holds.

Note that Theorems 7.2.1 and 7.2.2 still hold in any Banach spaces, as we have made no assumptions on the spaces involved. We shall mention now some related results in which the structure of the cone K plays a crucial role. In fact, there have been many generalizations of Gronwall inequality in this direction. For instance, using a lattice fixed point theorem, Hanson and Waltman [255] obtained such results for functional inequalities, which, in particular, include a generalization of Gronwall inequality due to Viswanatham [633]. In a similar context, a typical result for operator inequalities in partially ordered linear spaces is contained in Pelczar [522].

Theorem 7.2.3 (Pelczar [522]) *Let M be a monotonic operator mapping B into B. Define the subset $Q \subset B$ as follows:*

$$Q = \{z | \;\; z \in B, z \leq Mz + q\}, \quad q \in B.$$

If Q is non-empty and sup Q *(say $\tilde{v}$) exists, then $\tilde{v}$ is the maximal solution of*

$$v = Mv + q.$$

The assumption that $\sup Q$ exists is, however, a restriction on the cone K which may exclude many spaces of practical interest. For instance, it may not be true in $C[0, \tau]$, the space of continuous functions, which is partially ordered by the cone of non-negative functions in $C[0, \tau]$.

Now we begin to study matrix inequalities.

Let $\mathcal{A}$ denote the linear space of real $n \times n$ symmetric matrices. In $\mathcal{A}$, we can introduce a partial ordering in more than one way. For instance, using cones of non-negative matrices and non-negative definite matrices, respectively, two different types of orderings can be introduced in $\mathcal{A}$. Since a non-negative definite matrix is a natural generalization of non-negative number, we adopt here the second kind of ordering. Then with this ordering in $\mathcal{A}$, we have

$$X, Y \in \mathcal{A}, \quad X \leq Y \quad \text{iff} \quad (X - Y) \in \mathcal{K}.$$

A function $P : \mathcal{A} \to \mathcal{A}$ is called monotonic [47] if $X, Y \in \mathcal{A}$ and $X \leq Y$ imply $P(X) \leq P(Y)$ (that is, $P(Y) - P(X)$ is a non-negatively definite matrix).

Theorem 7.2.4 (Chandra-Feishman [130]) *Let H be a real symmetric matrix. Let G be a monotone and Lipschitz continuous function from $\mathcal{A}$ into $\mathcal{A}$:*

$$\| G(X) - G(Y) \| \leq \rho \| X - Y \| . \tag{7.2.8}$$

Then the inequality

$$X(t) \leq H(t) + \int_0^t G(X(s))ds$$

implies

$$X(t) \leq Y(t)$$

on their common interval of existence, where $Y(t)$ is the unique solution of the corresponding equality.

Proof For $n = 1, 2, \cdots$, set

$$Y_n(t) = H(t) + \int_0^t G(Y_{n-1}(s))ds$$

where $Y_0(t) = X(t) \in \mathcal{A}$. Then $\{Y_n\}, n = 1, 2, \cdots$, are all in $\mathcal{A}$. Next, using the monotonicity of G, we easily verify that

$$X(t) \leq Y_1(t) \leq \cdots \leq Y_n(t).$$

Since G is Lipschitz continuous, $\{Y_n(t)\}$ converges to the unique solution $Y(t)$ of the corresponding equality. This completes the proof. □

The following results used by Bellman [73] may be regarded as corollaries of the above theorem.

Corollary 7.2.1 (The Bellman Inequality [73]) *The matrix inequality*

$$\frac{dX}{dt} \leq F(t) + G(X), \quad X(0) = C,$$

implies

$$X(t) \leq Y(t)$$

where F and C are real and symmetric, G has the properties above, and $Y(t)$ is the unique solution of the initial value problem

$$\frac{dY}{dt} \leq F(t) + G(Y), \quad Y(0) = C.$$

Proof Integrating the inequality gives us

$$X(t) \leq C + \int_0^t F(s)ds + \int_0^t G(X(s))ds.$$

If we set

$$H(t) = C + \int_0^t F(s)ds \in \mathcal{A},$$

then the result follows immediately from Theorem 7.2.4. □

Corollary 7.2.2 (Bellman [73]) *The inequality*

$$\frac{dX}{dt} \leq F(t) + RX + XR^T + \sum_i^m Q_i X Q_i^T, \quad X(0) = C$$

implies

$$X(t) \leq Y(t)$$

where F and C are real and symmetric, R and Q_i $(i = 1, 2, \cdots, m)$ are real constant $n \times n$ matrices, and $Y(t)$ is the unique solution of the corresponding initial value problem.

Proof Following a familiar procedure (multiplying from the left and right by e^{Rt} and $e^{R^T t}$, respectively, etc.), we have

$$X(t) \leq e^{Rt}Ce^{R^T t} + \int_0^t e^{R(t-s)}\left[F(s) + \sum_{i=1}^{m} Q_i X Q_i^T\right] e^{R^T(t-s)} ds$$
$$= H(t) + \int_0^t G(t, s, X(s)) ds$$

where

$$H(t) = e^{Rt}Ce^{R^T t} + \int_0^t e^{R(t-s)}F(s)e^{R^T(t-s)} ds$$

and

$$G(t, s, x) = e^{R(t-s)}\sum_{i=1}^{m} Q_i X Q_i^T e^{R^T(t-s)}.$$

Clearly, $H(t) \in \mathcal{A}$. Also G is Lipschitz continuous in its last argument and monotonic, because if $X \leq Y$, for each i $(i = 1, \cdots, m)$, and for any Q_i,

$$Q_i Y Q_i^T - Q_i X Q_i^T = Q_i(Y - X)Q_i^T \in \mathcal{K}.$$

Again,

$$e^{R(t-s)}\sum_{i} Q_i(Y - X)Q_i^T e^{R^T(t-s)} \in \mathcal{K}.$$

Having established the desired properties for H and G, we may now apply Theorem 7.2.4 to obtain our conclusion. □

7.2.2 Integral Inequalities of Volterra Type for Functions Defined in Partially Ordered Spaces

In this section, we shall introduce the results due to Ronkov and Bainov [560].

To this end, we shall study the Volterra type integral operators action on numeric functions defined in partially ordered topological spaces with a measure. Integral equations and inequalities for such operators have been considered. Note that, Bainov, Myshkis and Zahariev [41] is the first contribution to consider linear integral equations and inequalities of Volterra type for functions defined in metric spaces.

We shall consider complex functions defined in the partially ordered set T, $(T,)$, employing the following notations:

$T_x := \{y : y \in T \textit{ and } y < x\}$ for the segment of the element x and

$$\chi(x, y) = \begin{cases} 1, & \text{if} \quad y \leq x, \\ 0, & \text{otherwise,} \end{cases}$$

for its characteristic function.

Concerning the set T, we shall assume that the following conditions (C1)–(C4) are fulfilled:

(C1). T is a partially ordered connected topological space with positive measure μ.

(C2). For every $x \in T$, the function $\chi(x, \cdot)$ is μ-measurable.

(C3). If x_α is a generalized sequence of elements of T, convergent to x, then

$$||\chi(x_\alpha, \cdot) - \chi(x, \cdot)||_2 \ \textit{tends to} \ 0.$$

(C4). There exists an element $x_0 \in T$ such that $||\chi(x_0, \cdot)||_2 = 0$.

Definition 7.2.1 The operator $\phi : L^2(T) \to L^2(T)$ will be termed as characteristic provided for all $x \in T, \psi \in L^2(T)$ and $\chi(x, \cdot)\psi(\cdot) = \chi(x, \cdot)\phi\psi(\cdot)$ holds.

Remark 7.2.1 If ϕ is a linear characteristic operator, then for any $x \in T$ and every function $\psi \in L^2(T)$, the equality $\chi(x, \cdot)\phi\psi(x) = \chi(x, \cdot)\phi\psi_x(x)$ holds, where $\psi_x(\cdot) := \chi(x, \cdot)\psi(\cdot)$.

Indeed, since

$$\chi(z, \cdot)(\psi(\cdot) - \psi_z(\cdot)) = \psi_z(\cdot) - \psi_z(\cdot) = 0,$$

then

$$\chi(z, \cdot)\phi(\psi(\cdot) - \psi_z(\cdot)) = 0$$

whence, taking into account the fact that the operator ϕ is linear, it follows that

$$\chi(z, \cdot)\phi\psi(\cdot) - \chi(z, \cdot)\phi\psi_z(\cdot) = 0.$$

Definition 7.2.2 By V denote the integral operator defined in $L^2(T)$ in the following way:

$$\begin{aligned} Vf(x) :&= \int_T \chi(x, y)K(x, y)\phi f(y)d\mu(y) \\ &= \int_{T_x} \chi(x, y)K(x, y)\phi f(y)d\mu(y) \end{aligned} \tag{7.2.9}$$

where

(1) The kernel $K(x, y) \in L^2(T \times T)$.
(2) ϕ is a bounded linear characteristic operator.

Remark 7.2.2 V is a compact operator acting from $L^2(T)$ onto $L^2(T)$ since it is composition of the operator ϕ with a Hilbert-Schmidt operator, the latter being as is well-known (see, e.g, [254, Chap. 15, problem 135]) a compact operator from $L^2(T)$ onto $L^2(T)$.

To prove Theorem 7.2.1, we need the following lemma.

Lemma 7.2.2 (Ronkov-Bainov [560]) *The integral equation*

$$\varphi = \lambda V\varphi \tag{7.2.10}$$

(here λ is an arbitrary complex number) only possesses the trivial solution .

Proof Let the function $\varphi \in L^2(T)$ be a solution of the integral equation (7.2.10). By T_0 denote the following subset of T:

$$T_0 := \{x : x \in T \text{ and } \chi(x, \cdot)\varphi(\cdot) = 0\} .$$

We shall show that $T_0 = T$ whence it follows that $\varphi = 0$. Indeed,

$$\begin{aligned} ||\varphi||_2^2 &= \int_T |\varphi|^2 d\mu(x) \\ &= |\lambda|^2 \int_T | \int_T K(x, y)\chi(x, y)\phi\mu(y) \ |^2 \ d\mu(x) = 0 \end{aligned}$$

because, in view of Remark 7.2.1, $\chi(x, \cdot)\phi\varphi(\cdot) = \chi(x, \cdot)\phi\varphi_x(\cdot)$, where $\varphi_x(\cdot) = \chi(x, \cdot)\phi = 0$.

Since T is a connected topological space, then in order to establish that $T_0 = T$, it is sufficient to show that T_0 is not empty, closed and open at the same time.

In view of condition (C4), an element $x_0 \in T$ exists such that $||\chi(x_0, \cdot)||_2 = 0$ whence $||\varphi\chi(x_0, \cdot)||_2 = 0$, i.e., $x_0 \in T$ and therefore T_0 is not empty.

It will be shown that T_0 is a closed set. Indeed, let $\{x_\alpha\}$ be a generalized sequence of elements of T_0, convergent to x. But then

$$||\varphi(\cdot)\chi(x, \cdot)||_2 = ||\varphi(\cdot)\chi(x, \cdot) - \varphi(\cdot)\chi(x_\alpha, \cdot)||_2 \int_{T_x \Delta T_{x_\alpha}} ||\varphi(y)||^2 d\mu(y)$$

and since in view of (C3), $||\chi(x, \cdot) - \chi(x_\alpha, \cdot)||_2$ tends to 0, then $||\varphi(\cdot)\chi(x, \cdot)||_2 = 0$. Hence, $x \in T_0$, which implies that T_0 is closed.

Now we shall show that T_0 is open, too. Let $z_0 \in T_0$. According to the condition (C3), for every $\delta > 0$ a neighborhood $U(z_0, \delta)$ of z_0 exists such that if $z \in U(z_0, \delta)$, then $\mu(T_z \backslash T_{z_0}) < \delta$. We shall show that if $\delta > 0$ is sufficiently small, then

$U(z_0, \delta) \subset T_0$. Let $z \in T$ and consider the function $\varphi_z(y) := \chi(z, y)\varphi(y)$. For the square of the norm of the function φ_z, the following holds:

$$\begin{aligned}||\varphi_z||_2^2 &= \int_T \chi^2(z, x)|\varphi(x)|^2 d\mu(x)\\ &\le |\lambda|^2 \int_T \chi(z, x)\left\{\int_T \chi(x, y)|K(x, y)||\phi\varphi(y)|d\mu(y)\right\}^2 d\mu(x)\\ &= |\lambda|^2 \int_T \left\{\int_T \chi(z, x)\chi(x, y)|K(x, y)\phi\varphi(y)|d\mu(y)\right\}^2 d\mu(x)\\ &\le |\lambda|^2 \int_T \left\{\int_{T_z} |K(x, y)\phi\varphi(y)|d\mu(y)\right\}^2 d\mu(x).\end{aligned}$$

Since $z_0 \in T_0$, then $||\varphi\chi(z_0, \cdot)||_2 = 0$, and therefore, $||\chi(z_0, \cdot)\phi\varphi(\cdot)||_2 = 0$, and

$$\int_{T_z} |K(x, y)\phi\varphi(y)|d\mu(y) = \int_{T_z/T_{z_0}} |K(x, y)\phi\varphi(y)|d\mu(y).$$

On the other hand,

$$\begin{aligned}&\int_{T_z\setminus T_{z_0}} |K(x, y)\phi\varphi(y)|d\mu(y)\\ &\quad\le \left\{\int_{T_z\setminus T_{z_0}} |K(x, y)|^2 d\mu(y)\right\}^{1/2} \left\{\int_{T_z} |\phi\varphi(y)|^2 d\mu(y)\right\}^{1/2}.\end{aligned}$$

However, noting Remark 7.2.1,

$$\begin{aligned}\int_{T_z} |\phi\varphi(y)|^2 d\mu(y) &= \int_T |\chi(z, y)\phi\varphi(y)|^2 d\mu(y) = \int_T |\chi(z, y)\phi\varphi_z(y)|^2 d\mu(y)\\ &\le ||\phi\varphi_z||_2^2 \le ||\phi||_2^2||\varphi_z||_2^2.\end{aligned}$$

Therefore,

$$||\varphi_z||_2^2 \le |\lambda|^2 \int_T \left[\int_{T_z\setminus T_{z_0}} |K(x, y)|^2 d\mu(y)\right] d\mu(x) \cdot ||\phi||_2^2||\varphi_z||_2^2.$$

By assumption $|K(x, y)|^2 \in L(T\times T)$ and since the set of the functions of the form $f(x, y) = \Sigma_{s=1}^n X_s(x)Y_s(y)$ where $X_s(\cdot)$ and $Y_s(\cdot)$ are proportional of characteristic functions of measurable sets with finite measure, is dense in $L(T \times T)$, then, without loss of generality, we can consider that $|K(x, y)|^2 = \sum_{s=1}^n X_s(x)Y_s(y)$.

Thus

$$\int_{T_z}\left[\int_{T_z\backslash T_{z_0}}|K(x,y)|^2 d\mu(y)\right]d\mu(x) = \int_{T_z}\left[\int_{T_z\backslash T_{z0}}\sum_{s=1}^{n}X_s(x)Y_s(y)d\mu(y)\right]d\mu(x)$$
$$\leq \left(\int_{T_z}|\sum_{s=1}^{n}X_s(x)|d\mu(x)\right)\cdot C_1\cdot\mu(T_z\backslash T_{z_0})$$
$$\leq C\cdot\mu(T_z\backslash T_{z_0})$$

where

$$C_1 = \max_{s=1,\cdots,n\ y\in T}|Y_s(x)|;\quad C = C_1\int_T|\sum_{s=1}^{n}X_s(x)|d\mu(x).$$

Let δ be a positive number so small that $\delta\cdot C|\delta|^2||\phi||_2^2 < 1$. Then if $z\in U(z_0,\delta)$, then $\mu(T_z\backslash T_{z_0}) < \delta$ and hence $||\varphi_z||_2^2 \leq \delta\cdot c\cdot|\lambda|^2||\phi||_2^2||\varphi_z||_2^2$ which implies that $||\varphi_z||_2 = 0$. Therefore, $U(z_0,\delta)\subset T_0$, i.e, T_0 is an open set. This completes the proof of the lemma. □

Theorem 7.2.5 (Ronkov-Bainov [560]) *Let the function g be from $L^2(T)$ and for the space T the conditions (C1)–(C4) are fulfilled. Then the integral equation*

$$\varphi = g + V\varphi \tag{7.2.11}$$

possesses a unique solution $\varphi\in L^2(T)$, and

$$\varphi = \Sigma_{n=0}^{+\infty}V_g^n.$$

Proof Since V is a compact operator, then Lemma 7.2.2 implies that the spectral radius of the operator V equals 0, i.e., $\lim_{n\to+\infty}||V^n||^{1/n} = 0$.

However, in this case the Cauchy criterion implies that the series $\sum_{n=0}^{+\infty}V^n$ is convergent with respect to norm, and hence it is convergent. Its sum, as is seen by an immediate verification, is an operator inverse to the operator $E - V$. Therefore, the integral equation (7.2.10), which can be written, otherwise, as

$$(E - V)\varphi = g$$

possesses a unique solution

$$\varphi = \sum_{n=0}^{+\infty}V^n g$$

which completes the proof. □

Definition 7.2.3 If f and g are two real functions from $L^2(T)$, then $f \leq g \Leftrightarrow f(x) \leq g(x)$ for almost every $x \in T$.

Definition 7.2.4 The operator $W \quad (W : \quad L^2(T) \to L^2(T))$ is called monotone if $Wf \leq Wg$ for $f \leq g$.

Definition 7.2.5 The operator $W \quad (W : \quad L^2(T) \to L^2(T))$ will be said to be characteristically monotone if, for every $x \in T$ and for any two real functions $f, g \in L^2(T)$, for which $(f(\cdot) - g(\cdot))\chi(x, \cdot) \leq 0$, the inequality $Wf(x) \leq Wg(x)$ holds.

Obviously, if W is a characteristically monotone operator, then W is monotone.

Theorem 7.2.6 (Ronkov-Bainov [560]) *Let the following assumptions hold:*

1. *For the space T, conditions (C1)–(C4) hold.*
2. *The kernel $K(x, y)$ of the integral operator V is non-negative, while the operator ϕ is monotone.*
3. *For some $x \in T$ and for two real functions $f, g \in L^2(T)$, the inequality*

$$(f(\cdot) - g(\cdot) - Vf(\cdot))\chi(x, \cdot) \leq 0, \quad \Big((f(\cdot) - g(\cdot) - Vf(\cdot))\chi(x, \cdot) \geq 0\Big) \tag{7.2.12}$$

   holds.

Then

$$(f(\cdot) - \varphi(\cdot))\chi(x, \cdot) \leq 0, \quad \Big((f(\cdot) - \varphi(\cdot))\chi(x, \cdot) \geq 0\Big)$$

where φ is the solution of the integral equation (7.2.10).

Proof We shall first show that the operator V is characteristically monotone. Indeed, if $h \in L^2(T), z \in T$ and $h_z(\cdot) = \chi(z, \cdot)h(\cdot) \leq 0$, then in view of Remark 7.2.1, $\chi(z, \cdot)\phi h(\cdot) = \chi(z, \cdot)\phi h_z(\cdot)$ and since $h_z(\cdot) \leq 0$, and ϕ is monotone, then $\chi(z, \cdot)\phi h(\cdot) \leq 0$. On the other hand, $Vh(z) = \int_T K(z, y)\chi(z, y)\phi h(y)d\mu(y)$ and since $K(x, y) \geq 0$, then $Vh(z) \leq 0$.

Next we shall show that inequality (7.2.12) implies that

$$\left(Vf(\cdot) - Vg(\cdot) - V^2 f(\cdot)\right)\chi(x, \cdot) \leq 0.$$

Indeed, if $z < x$, then (7.2.12) yields that $(f(\cdot) - g(\cdot) - Vf(\cdot))\,\chi(z, \cdot) \leq 0$. But since V is characteristically monotone, $Vf(z) - Vg(z) - V^2 f(z) \leq 0$. Therefore,

$$\left(Vf(\cdot) - Vg(\cdot) - V^2 f(\cdot)\right)\chi(x, \cdot) \leq 0.$$

Analogously, by induction, it follows that for every $n = 0, 1, 2, \cdots$, the inequality

$$\left(V^n f(\cdot) - V^n g(\cdot) - V^{n+1} f(\cdot)\right)\chi(x, \cdot) \leq 0$$

holds. Summing by $n = 0, 1, 2, \cdots$, and taking into account that $V^n f \to 0$ as $n \to +\infty$, and that $\varphi = \Sigma_{n=0}^{+\infty} V^n g$ is a solution of the integral equation (7.2.10), thus it follows that $(f(\cdot) - \varphi(\cdot))\, \chi(x, \cdot) \le 0$. The inverse inequality can be proved similarly. □

Theorem 7.2.7 (Ronkov-Bainov [560]) *If, under the conditions of Theorem 7.2.6, the inequality* $f(\cdot) - g(\cdot) - Vf(\cdot) \le 0$ *holds (or* $f(\cdot) - g(\cdot) - Vf(\cdot) \ge 0$*), then*

$$f \le \varphi = \Sigma_{n=0}^{+\infty} V^n g \quad (f \ge \varphi,\ \textit{respectively}).$$

Proof The inequality $f(\cdot) - g(\cdot) - Vf(\cdot) \le 0$ implies that $\big(Vf(\cdot) - Vg(\cdot) - V^2 f(\cdot)\big)$ $\chi(x, \cdot) \le 0$ for every $x \in T$. However then, in view of Theorem 7.2.6, $(f(\cdot) - \varphi(\cdot))\, \chi(x, \cdot) \le 0$ for any $x \in T$, and since the operator V is characteristically monotone, then this implies that $Vf \le V\varphi$. Hence

$$f \le g + Vf \le g + V\varphi = \varphi.$$

The inverse inequality can be proved analogously. □

Corollary 7.2.3 (Ronkov-Bainov [560]) *If in the conditions of Theorem 7.2.6 (Theorem 7.2.7)* $g = 0$, *then it follows that the inequality* $f - Vf \le 0 \quad (\chi(x, \cdot)(f - Vf) \le 0$, *respectively) implies the inequality* $f \le 0 \quad (\chi(x, \cdot) f \le 0$, *respectively). Then, since* V *is a linear operator, then the inequality* $f - Vf \le h - Vh \quad$ *(*h *is a real function from* $L^2(T)$*) implies that* $f \le h$.

Now consider in detail the case when the operator φ is identity operator. Then

$$Vf(x) = \int_{T_s} K(x, y) f(y) d\mu(y)$$

and, in view of Theorem 7.2.5, the integral equation

$$\varphi(x) = g(x) + \int_{T_s} K(x, y)\varphi(y) d\mu(y) \tag{7.2.13}$$

(here g denotes an arbitrary real function from $L^2(T)$) possesses a unique solution $\varphi \in L^2(T)$. We next shall obtain some explicit estimates for this solution which, in view of Corollary 7.2.3, will hold for the solutions f of the corresponding integral inequality

$$f \le g + Vf. \tag{7.2.14}$$

Next we shall employ Theorem 7.2.5 to prove that the solution φ of (7.2.13) is actually the sum of the Neumann series $\sum_{n=0}^{+\infty} V^n g$. The idea is to compare the terms of this series with the ones of an exponential series, whence the demanded estimate follows at once.

Theorem 7.2.8 (Ronkov-Bainov [560]) *Assume that*

(1) For the space T conditions (C1)–(C4) hold, and, for the ordering in T, beside being transitive, it is assumed that it satisfies the requirement if $x < y$ and $y < x$, then $x = y$.
(2) The diagonal $D := \{(x, x) : x \in T\}$ of the space $(T^2, \Sigma^2, \mu_2) := (T, \Sigma, \mu) \times (T, \Sigma, \mu)$ is a μ_2-null set.
(3) The kernel $K(x, y)$ of the integral operator V is a non-negative function from $L^2(T^2)$, which, for a fixed y from T, is a non-decreasing function of x.
(4) $g(x)$ is a non-decreasing, non-negative function from $L^2(T)$.

Then, for the solution φ of the integral equation (7.2.13), the estimate

$$\varphi(x) \leq g(x) \exp\left(\int_{T_x} K(x, y) d\mu(y)\right) \tag{7.2.15}$$

holds.

Proof By Theorem 7.2.5, the integral equation (7.2.13) possesses a unique solution $\varphi = \sum_{n=0}^{+\infty} V^n g$. On the other hand,

$$\exp\left(\int_{T_x} K(x, y) d\mu(y)\right) = \exp\left(V(1)(x)\right) = \sum_{n=0}^{+\infty} \frac{(V(1)(x))^n}{n!}.$$

Hence, in order to obtain the estimate (7.2.15), it is sufficient to prove that the inequality

$$n! V^n g \leq g \cdot (V(1))^n \tag{7.2.16}$$

holds for every natural number $n = 0, 1, \cdots$. (For $n = 0$, we obviously have an equality.)

$$\begin{aligned}
V^n g(x) &= \int_{T_x} K(x, y_1) d\mu(y_1) \int_{T_{y_1}} K(y_1, y_2) \cdots \int_{T_{y_{n-1}}} K(y_{n-1}, y_n) g(y_n) d\mu(y_n) \\
&= \int_{T^n} \chi(x, y_1) \chi(y_1, y_2) \cdots \chi(y_{n-1}, y_n) \cdots K(x, y_1) K(y_1, y_2) \\
&\quad \cdots K(y_{n-1}, y_n) g(y_n) d\mu_n(y) \\
&= \int_{T^n} \chi(x, y_{\alpha_1}) \chi(y_{\alpha_1}, y_{\alpha_2}) \cdots \chi(y_{\alpha_{n-1}}, y_{\alpha_n}) \cdots K(x, y_{\alpha_1}) K(y_{\alpha_1, \alpha_2}) \\
&\quad \cdots K(y_{\alpha_{n-1}}, y_{\alpha_n}) g(y_{\alpha_n}) d\mu_n(y),
\end{aligned}$$

where α is an arbitrary element from the aggregate Π of all permutations of $\{1, 2, \cdots, n\}$. Hence the monotonicity of g and K implies that

$$V^n g(x) \le g(x) \int_{T^n} \chi(x, y_{\alpha_1})\chi(y_{\alpha_1}, y_{\alpha_2})\chi(y_{\alpha_{n-1}}, y_{\alpha_n}) \cdots K(x, y_2) \ \cdots K(x, y_n) d\mu_n(y).$$

Since the number of all permutations of $\{1, 2, \cdots, n\}$ is $n!$, then

$$\begin{aligned} n!V^n g(x) \ &\le \ g(x) \int_T (\Sigma_{\alpha \in \Pi} \chi(x, y_{\alpha_1})\chi(y_{\alpha_1}, y_{\alpha_2}) \cdots \chi(y_{\alpha_{n-1}}, y_{\alpha_n})) \cdots K(x, y_2) \\ &\quad \cdots K(x, y_n) d\mu_n(y). \end{aligned} \tag{7.2.17}$$

Now we shall show that the inequality

$$\Sigma_{\alpha \in \Pi} \chi(x, y_{\alpha_1})\chi(y_{\alpha_1}, y_{\alpha_2}) \cdots \chi(y_{\alpha_{n-1}}, y_{\alpha_n}) \le \chi(x, y_1)\chi(x, y_2) \cdots \chi(x, y_n) \tag{7.2.18}$$

holds almost everywhere in T^n.

Indeed, let $x, \ y_1, \ \cdots, \ y_n$ be elements of T. In view of condition (2), without loss of generality, we may consider that taken two-by-two they are different. However, if the left-hand side of (7.2.18) is different from zero, then for some permutation $\alpha \in \Pi$,

$$y_{\alpha_n} < y_{\alpha_{n-1}} < \cdots < y_{\alpha_1} < x \tag{7.2.19}$$

will be fulfilled. Since $x, \ y_1, \ \cdots, \ y_n$ are different from one another, then there will not be another similar permutation and hence

$$\sum_{\alpha \in \Pi} \chi(x, y_{\alpha_1})\chi(y_{\alpha_1}, y_{\alpha_2}) \cdots \chi(y_{\alpha_{n-1}}, y_{\alpha_n}) = 1.$$

Moreover, (7.2.19) yields that in this case the right-hand side of (7.2.18) also assumes the value one.

This in fact proves that inequality (7.2.18) holds almost everywhere in T^n. Then it follows from inequalities (7.2.17) and (7.2.18) that

$$\begin{aligned} n!V^n g(x) &\le g(x) \int_{T^n} \chi(x, y_1)\chi(x, y_2)K(x, y_2) \cdots \chi(x, y_n)K(x, y_n) d\mu_n(y) \\ &= g(x) \left(\int_{T_x} K(x, z) d\mu(z) \right)^n = g(x)(V(1)(x))^n, \end{aligned}$$

i.e., inequality (7.2.16) is fulfilled for every natural number n. This completes the proof of Theorem 7.2.8. □

Corollary 7.2.4 *If, under the conditions of Theorem 7.2.8, for some function* $f \in L^2(T)$ *inequality (7.2.14) holds, then the estimate*

$$f(x) \leq g(x) \exp\left(\int_{T_s} K(x,y)d\mu(y)\right)$$

holds for it since, in view of Theorem 7.2.3, $f \leq \varphi$.

Note that Theorem 7.2.8 was proved under the assumption that the measure of the diagonal of the space $T \times T$ is zero. A sufficient condition for this assumption to be satisfied is supplied by the following lemma.

Lemma 7.2.3 (Ronkov and Bainov [560]) *Let* $T = (T, \Sigma, \mu)$ *be a space of non-negative measure. If for any positive* ϵ, *a sequence* $\{U_{\epsilon,n}\}$ *exists, consisting of sets that are measurable with respect to* μ *and such that*

$$T = \cup_{n=1}^{\infty} U_{\epsilon,n}, \quad \textit{and} \quad \mu(U_{\epsilon,n}) < \epsilon$$

for every $n = 1, 2, \cdots$, *then the diagonal* $D = \{(x,x) : x \in T\}$ *of the space* $T \times T = (T, \Sigma, \mu) \times (T, \Sigma, \mu)$ *is of zero measure.*

Proof Since T can be represented as a denumerable sum of sets having a finite measure, then T is a space with σ-finite measure. First consider the case when $\mu(T) < +\infty$. Let ϵ be an arbitrary positive number and $T = \cup_{n=1}^{\infty} U_{\epsilon,n}$ and $U_{\epsilon,n} \in \Sigma$ and $\mu(U_{\epsilon,n}) < \epsilon$ for every natural number n. Without loss of generality, we can consider that $U_{\epsilon,n} \cap U_{\epsilon,m} = \emptyset$ when $n \neq m$ because otherwise, we could have set

$$W_{\epsilon,1} := U_{\epsilon,1};\ W_{\epsilon,2} := U_{\epsilon,2} \backslash U_{\epsilon,1};\ W_{\epsilon,3} := U_{\epsilon,2} \backslash (U_{\epsilon,1} \cup U_{\epsilon,2}).$$

Consider the set $E_\epsilon := \cup_{n=1}^{\infty}(U_{\epsilon,n} \times U_{\epsilon,n})$. Obviously, E_ϵ is a measurable set with respect to the measure of the product $\mu_2 = \mu \times \mu$ and $D \subset E_\epsilon$. By E_y denote the following subset of $T : E_y := \{x : (x,y) \in E_\epsilon\}$. As is known (see, e.g., [200], III.II.7), $\mu_2(E_\epsilon) = \int_T \mu(E_y)d\mu(y) < \epsilon\mu(T)$ since $\mu(E_y) < \epsilon$ for every $y \in T$ because $U_{\epsilon,n} \cap U_{\epsilon,m} = \emptyset$ when $n \neq m$ and $\mu(U_{\epsilon,n}) < \epsilon$ for any n, whence, taking into account that $\mu(T) < +\infty$ and ϵ is arbitrary, it follows that $\mu(D) = 0$.

Now consider the case when the measure of T is not finite. Since $D = \cup_{n=1}^{\infty} D_{\epsilon,n}$ where $D_{\epsilon,n} := \{(x,x) : x \in U_{\epsilon,n}\}$, and besides $\mu(U_{\epsilon,n}) < \epsilon$ for any n, then $\mu(D_{\epsilon,n}) = 0$, thus it follows that $\mu(D) = 0$. This completes the proof. □

In Theorem 7.2.8, the function g from the integral equation (7.2.13) was assumed to be non-negative and non-decreasing.

The next theorem provides an estimate for the solution of (7.2.13) without these assumptions.

Theorem 7.2.9 (Ronkov-Bainov [560]) *Let conditions (1), (2), and (3) from Theorem 7.2.8 hold. Then, if g is an arbitrary function from* $L^2(T)$, *then for the solution*

φ of the integral equation (7.2.13), the estimate

$$|\varphi(x)| \leq |g(x)| + \int_{T_x} |g(y)|K(x,y) \exp\left(\int_{T_x\setminus T_y} ||K(x,z)||d\mu(z)\right) d\mu(y) \qquad (7.2.20)$$

holds, this implies immediately the weaker but simpler estimate:

$$|\varphi(x)| \leq |g(x)| + \int_{T_x} |g(y)|K(x,y)d\mu(y) \exp\left(\int_{T_x} ||K(x,y)||d\mu(y)\right). \qquad (7.2.21)$$

Proof For the solution φ of the integral equation (7.2.13), we have $|\varphi(x)| = |g(x) + \int_{T_x} g(y)K(x,y)d\mu(y)|$, which implies

$$|\varphi(x)| \leq |g(x)| + \int_{T_x} |\varphi(y)|K(x,y)d\mu(y). \qquad (7.2.22)$$

Since $K(x,y) \geq 0$, then

$$\int_{T_x} |\varphi(y)|K(x,y)d\mu(y) \leq \int_{T_x} |g(y)|K(x,y)d\mu(y) + \int_{T_x} K(x,y)\left(\int_{T_y} |\varphi(z)|K(x,z)d\mu(z)\right) d\mu(y). \qquad (7.2.23)$$

Obviously, $g_1(x) = \int_{T_x} |g(y)|K(x,y)d\mu(y)$ is a non-negative and non-decreasing function. Therefore the estimate (7.2.21) follows from Corollary 7.2.3 and by inequalities (7.2.23) and (7.2.22).

We know that estimate (7.2.20) is obtained by means of calculations analogous to those carried out in the proof of Theorem 7.2.8. So, if by φ_1 we denote the solution of the integral equation

$$\varphi_1 = g_1 + V\varphi_1,$$

then it follows from Theorem 7.2.5 implies that

$$\varphi_1 = \sum_{n=0}^{+\infty} V^n g_1.$$

Thus inequality (7.2.23) and Theorem 7.2.6 yield that

$$\int_{T_x} |\varphi(y)|K(x,y)d\mu(y) \leq \varphi_1(x) = \sum_{n=0}^{+\infty} V^n g_1(x). \qquad (7.2.24)$$

However,

$$
\begin{aligned}
V^n g_1(x) &= \int_{T_x} K(x, y_1) d\mu(y_1) \int_{T_{y_1}} K(y_1, y_2) d\mu(y_2) \\
&\quad \cdots \int_{T_{y_{n-1}}} K(y_{n-1}, y_n) d\mu(y_n) \int_{T_{y_n}} K(y_n, y_0) |g(y_0)| d\mu(y_0) \\
&= \int_{T^{n+1}} K(x, y_1) K(y_1, y_2) \cdots K(y_n, y_0) |g(y_0)| \\
&\quad \times \chi(x, y_1) \chi(y_1, y_2) \cdots \chi(y_{n-1}, y_n) \chi(y_{n-1}, y_n) \chi(y_n, y_0) d\mu_{n+1}(y) \\
&= \int_{T^{n+1}} K(x, y_{\alpha_1}) K(y_{\alpha_1}, y_{\alpha_2}) \cdots K(y_{\alpha_{n-1}}, y_{\alpha_n}) K(y_{\alpha_n}, y_0) |g(y_0)| \\
&\quad \times \chi(x, y_{\alpha_1}) \chi(y_{\alpha_1}, y_{\alpha_2}) \cdots \chi(y_{\alpha_{n-1}}, y_{\alpha_n}) \chi(y_{\alpha_n}, y_0) d\mu_{n+1}(y),
\end{aligned}
$$

where α denotes any element of the aggregate Π of all permutation of $\{1, 2, \cdots, n\}$. Since $K(x, y)$ for a fixed y is a non-decreasing function of x, then

$$
\begin{aligned}
n! V^n g_1(x) \le \int_{T^{n+1}} & K(x, y_0) K(x, y_1) \cdots K(x, y_n) |g(y_0)| \\
& \times \left(\Sigma_{\alpha \in \Pi} \chi(x, y_{\alpha_1}) \chi(y_{\alpha_1}, y_{\alpha_2}) \cdots \chi(y_{\alpha_{n-1}}, y_{\alpha_n}) \chi(y_{\alpha_n}, y_0)\right) d\mu_{n+1}(y).
\end{aligned}
$$

On the other hand, since almost everywhere in T^n, the inequality

$$
\begin{aligned}
&\sum_{\alpha \in \Pi} \chi(x, y_{\alpha_1}) \chi(y_{\alpha_1}, y_{\alpha_2}) \cdots \chi(y_{\alpha_{n-1}}, y_{\alpha_n}) \chi(y_{\alpha_n}, y_0) \\
&\le \chi(x, y_0) \left(\chi(x, y_{\alpha_1}) - \chi(y_0, y_{\alpha_1})\right) \left(\chi(x, y_{\alpha_2}) - \chi(y_0, y_{\alpha_2})\right) \\
&\quad \cdots \left(\chi(x, y_{\alpha_n}) - \chi(y_0, y_{\alpha_n})\right) \qquad (7.2.25)
\end{aligned}
$$

holds, which is proved quite analogously to inequality (7.2.18) of Theorem 7.2.8, then

$$
\begin{aligned}
n! V^n g_1(x) &\le \int_{T_x} |g(y_0)| K(x, y_0) \Big[\int_{T^n} K(x, y_1) K(x, y_2) \cdots K(x, y_n) \\
&\quad \times \left(\chi(x, y_1) - \chi(y_0, y_1)\right) \left(\chi(x_1, y_2) - \chi(y_0, y_2)\right) \\
&\quad \cdots \left(\chi(x, y_n) - \chi(y_0, y_n)\right) d\mu_n(y) \Big] d\mu(y_0) \\
&= \int_{T_x} |g(y_0)| K(x, y_0) \left(\int_{T_x \setminus T_{y_0}} K(x, z) d\mu(z) \right)^n d\mu(y_0)
\end{aligned}
$$

whence

$$\varphi_1(x) = \sum_{n=0}^{+\infty} V^n g_1(x) \le \int_{T_x} |g(y)| K(x,y) \cdot \exp\left(\int_{T_x \setminus T_y} K(x,z) d\mu(z) \right)^n d\mu(y).$$

Hence the above expression and inequalities (7.2.24) and (7.2.22) imply the estimate (7.2.20). This completes the proof. □

Corollary 7.2.5 (Ronkov-Bainov [560]) *If, under the conditions of Theorem 7.2.9, for some real function from $L^2(T)$, inequality (7.2.14) holds, then the following estimate holds*

$$f(x) \le |g(x)| + \int_{T_x} |g(y)| K(x,y) \exp\left(\int_{T_x \setminus T_y} K(x,z) d\mu(z) \right) d\mu(y). \tag{7.2.26}$$

Remark 7.2.3 (Ronkov-Bainov [560]) If the ordering in T is linear, then there will be an equality in (7.2.25). Then, if by T we denote the real interval $[a, B)$ (here B may be $+\infty$ as well) having the usual ordering and topology, and if μ denotes the Lebesgue measure, then conditions (1) and (2) of Theorem 7.2.9 are obviously fulfilled. Let $K(x,y) = K(y)$ be a non-negative function from $L^2(T)$. In this case, for every solution f of the integral inequality (7.2.14), it is possible to give the more precise estimate

$$f(x) \le g(x) + \int_a^x g(y) K(y) \exp\left(\int_x^y K(z) d\mu(z) \right) d\mu(y) \tag{7.2.27}$$

which follows from the fact that in this case (7.2.25) is an equality hence

$$\begin{aligned} &\int_a^x g(y) K(y) \exp\left(\int_{T_x \setminus T_y} K(z) d\mu(z) \right)^n d\mu(y) \\ &= \sum_{n=0}^{+\infty} V^n \left(\int_a^x g(y) K(y) d\mu(y) \right) = \varphi(x), \end{aligned}$$

which, in view of Theorem 7.2.1, is a solution of the equation

$$\varphi(x) = \int_a^x g(y) K(y) d\mu(y) + \int_a^x \varphi(y) K(y) d\mu(y).$$

However, inequality (7.2.14), since $K(y) \ge 0$, implies that

$$\int_a^x f(y) K(y) d\mu(y) \le \int_a^x g(y) K(y) d\mu(y) + \int_a^x K(y) \left(\int_a^y f(z) K(z) d\mu(z) \right) d\mu(y)$$

whence, in view of Theorem 7.2.7,

$$\int_a^x f(y)K(y)d\mu(y) \le \varphi(x).$$

The above expression and inequality (7.2.14) imply the estimate (7.2.26), which in fact is the well-known Gronwall-Bellman inequality (see, e.g., [338]).

7.3 Linear Multi-Dimensional Discontinuous Integral Inequalities in Measure Spaces

This section introduces some multi-dimensional linear discontinuous integral inequalities in measure spaces. These results are due to Horváth [290], and Györi and Horváth [245].

7.3.1 Gronwall-Bellman Type Integral Inequalities in Measure Spaces

In this section, we introduce theorems from Horváth [290].

Let $(X, \mathcal{A}, \mu)$ be a measure space. $\mathcal{A}$ always denotes a σ-algebra in X. The μ-integrable functions over $A \in \mathcal{A}$ are considered to be almost measurable on A (the function f is said to be almost measurable on A if there exists a measurable subset H of A such that $\mu(A \backslash H) = 0$ and f is measurable on H). We shall discuss explicit bounds of the solutions of integral inequality of the form

$$\phi(x) \le f(x) + g(x) \int_{S(x)} \phi \, d\mu, \quad x \in X \tag{7.3.1}$$

where ϕ, f and g are real-valued functions on X and $S(x) \in \mathcal{A}$ for every $x \in X$. Such a problem is due to Gronwall [239] and Bellman [61], therefore the functional inequality (7.3.1) is usually called the Gronwall-Bellman type integral inequality. Special cases of the problem often appear and play a fundamental role in the study of differential and integral equations.

Recall that the above problem has been investigated by many authors (see, e.g., [42, 338] and the references therein). However, these results mainly concerns the next case: X is an interval in $\mathbb{R}^n$; ϕ, f and g are continuous; the sets $S(x)$ $(x \in X)$ are bounded intervals with the same left-hand end-points and with right-hand end-points x; and integral in (7.3.1) is interpreted in the Riemann senses. We also note that even X is subset of $\mathbb{R}^n$, there exist relatively few results when the sets

$S(x)$ $(x \in X)$ are not intervals [592], and the integral in (7.3.1) is not a Riemann integral [166, 204, 244, 305, 547, 580]; but these cases are important in applications.

In this section, we show that (7.3.1) admits explicit bounds in measure spaces under fairly general conditions. We obtain not only integral inequalities but the unique solvability of special integral equations. Choosing the measure spaces properly, discrete inequalities can also be obtained from these results.

Suppose we are given n measure spaces $(X_i, \mathcal{A}_i, \mu_i)$, $i = 1, \cdots, n$ $(n \geq 2)$. Let $X = X_1 \times \cdots \times X_n$, $\mathcal{R} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$, let the set function μ be defined on $\mathcal{R}$ by $\mu(A_1 \times \cdots \times A_n) = \prod_{i=1}^n \mu_i(A_i)$. Then $\mathcal{R}$ is a semiring in X and μ is a measure on $\mathcal{R}$. The outer measure on the power set of X induced by μ is denoted by μ, too. Let $\mathcal{A}$ be the μ-measurable (in the Caratheodory sense) subsets of X. We call the measure space $(X, \mathcal{A}, \mu)$ the product of $(X_i, \mathcal{A}_i, \mu_i)$, $i = 1, \ldots, n$.

If $(X, \mathcal{A}, \mu)$ is a measure space, then $(X^n, \mathcal{A}^n, \mu^n)$ denotes the n-fold product of $(X, \mathcal{A}, \mu)$ $(n = 2, 3, \cdots)$.

We say that the function $S : X \to \mathcal{A}$ satisfies the condition (C) if the following properties hold

(C_1) $x \notin S(x)$ for every $x \in X$,
(C_2) if $y \in S(x)$, then $S(y) \subset S(x)$ $(x \in X)$,
(C_3) $\{(x_1, x_2) \in X^2 | x_1 \in X,\ x_2 \in S(x_1)\} \in \mathcal{A}^2$.

We now give some important types of functions satisfying the condition (C).

Example 7.3.1 Let $I_1, \cdots, I_p$ $(p = 1, 2, \ldots)$ be intervals in $\mathbb{R}$ (not necessary bounded). Then $I = I_1 \times \cdots \times I_p$ is an interval in $\mathbb{R}^p$. Consider the measure space $(I, \mathcal{B}^p, \mu)$ where $\mathcal{B}^p$ is the σ-algebra of Borel sets of I and μ is a Lebesgue-Stieltjes measure on $\mathcal{B}^p$.

(a_1) Define $S_1 : I \to \mathcal{B}^p$ by $S_1(x_1, \cdots, x_p) = I_1(x_1) \times \cdots \times I_p(x_p) \backslash \{(x_1, \cdots, x_p)\}$, where $I_i(x_i)$ is (not exclusively) $I_i \bigcap (-\infty, x_i)$ or $I_i \bigcap (-\infty, x_i]$ $(i = 1, \cdots, p)$.
(a_2) Define $S_2 : I \to \mathcal{B}^p$ by $S_2(x_1, \cdots, x_p) = I_1(x_1) \times \cdots \times I_p(x_p) \backslash \{(x_1, \cdots, x_p)\}$, where $I_i(x_i)$ is $I_i \bigcap (x_i, +\infty)$ or $I_i \bigcap [x_i, +\infty)$ $(i = 1, \cdots, p)$.

Then S_1 and S_2 satisfy the condition (C).

Example 7.3.2 Let $X = \{x_1, \cdots, x_n\}$ $(n = 1, 2, \cdots)$ be a set and $\mathcal{A}$ its power set. Consider the measure space $(X, \mathcal{A}, \mu)$ where μ is a measure on $\mathcal{A}$.

(b_1) Define $S_3 : X \to \mathcal{A}$ by $S_3(x_1) = \emptyset$ and $S_3(x_i) = \{x_1, \cdots, x_{i-1}\}$ $(i = 2, \cdots, n)$.
(b_2) Define $S_4 : X \to \mathcal{A}$ by $S_4(x_n) = \emptyset$ and $S_4(x_i) = \{x_{i+1}, \cdots, x_n\}$ $(i = 1, 2, \cdots, n-1)$.

Then S_3 and S_4 satisfy the condition (C).

Lemma 7.3.1 (Horváth [290]) *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $A \in \mathcal{A}$ and let $S : X \to \mathcal{A}$ satisfy the condition (C). Then*

$$H_n(A) = \{(x_1, \cdots, x_n) \in X^n | x_1 \in A, x_k \in S(x_{k-1}), k = 2, \cdots, n\}$$

is an element of $\mathcal{A}^n$ $(n = 2, 3, \cdots)$.

Proof Since $H_2(A) = H_2(X) \bigcap (A \times X)$ and $H_{n+1}(A) = (H_n(A) \times X) \bigcap (X^{n-1} \times H_2(X))$, the result follows from (C_3) by an easy induction argument. $\square$

The next inequality provides the key to the proof of the main results.

Theorem 7.3.1 (Horváth [290]) *Let* $(X, \mathcal{A}, \mu)$ *be a measure space, let* $S : X \to \mathcal{A}$ *satisfy the condition (C), and let* $f_i, i = 1, \cdots, n+1, \quad (n = 1, 2, \cdots),$ *be non-negative,* μ*-integrable functions on X. Then*

$$\int_X \Big(\int_{S(x_1)} \Big(\cdots \Big(\int_{S(x_n)} f_1(x_1) \cdots f_n(x_n) f_{n+1}(x_{n+1})\, d\mu(x_{n+1}) \Big) \cdots \Big) d\mu(x_2) \Big) d\mu(x_1)$$
$$\leq \frac{1}{(n+1)!} \prod_{i=1}^{n+1} \int_X f_i \, d\mu.$$

Proof Let us denote the set of the permutations of the integers $1, 2, \cdots, n$ by P_n. Suppose $\pi \in P_{n+1}$, it is obvious that

$$\begin{aligned}
&\int_X \Big(\int_{S(x_1)} \Big(\cdots \Big(\int_{S(x_n)} f_1(x_1) \cdots f_n(x_n) f_{n+1}(x_{n+1})\, d\mu(x_{n+1}) \Big) \cdots \Big) d\mu(x_2) \Big) d\mu(x_1) \\
&= \int_X \Big(\int_{S(x_{\pi(1)})} \Big(\cdots \Big(\int_{S(x_{\pi(n)})} f_1(x_{\pi(1)}) \cdots f_{n+1}(x_{\pi(n+1)})\, d\mu(x_{\pi(n+1)}) \Big) \times \cdots \Big) \\
&\quad \times d\mu(x_{\pi(2)}) \Big) \mu(x_{\pi(1)}).
\end{aligned} \tag{7.3.2}$$

Now Lemma 7.3.1 allows us to use Fubini's theorem. This thus implies that

$$(7.3.2) = \int_{H_\pi} f_1(x_{\pi(1)}) \cdots f_{n+1}(x_{\pi(n+1)})\, d\mu^{n+1} \tag{7.3.3}$$

where

$$H_\pi = \{(x_1, \cdots, x_{n+1} \in X^{n+1} | x_{\pi(1)} \in X, x_{\pi(1)} \in S(x_{\pi(k-1)}), k = 2, \cdots, n+1\}.$$

If $\tau \in P_{n+1}$ and $\tau \neq \pi$, then there exists $1 \leq k \leq n$ such that k is the smallest natural number for which $\tau(k) \neq \pi(k)$, and there exists $k < l, m \leq n+1$ such that $\tau(k) \neq \pi(k)$ and $\tau(l-1) = \pi(m)$. Suppose that $(x_1, \cdots, x_{n+1}) \in H_\pi \bigcap H_\tau$. Then $x_{\pi(k)} \in S(x_{\tau(l-1)})$ and $x_{\tau(l-1)} \in S(x_{\pi(m-1)})$, and hence, by the condition (C_2), $x_{\pi(k)} \in S(x_{\pi(m-1)})$. Since $S(x_{\pi(m-1)}) \subset S(x_{\pi(m-2)}) \subset \cdots \subset S(x_{\pi(k)})$, it follows that $x_{\pi(k)} \in S(x_{\pi(k)})$, and this contradicts the condition (C_1). We have thus shown that $H_\pi \bigcap H_\tau = \emptyset$, if π and τ are different elements of P_{n+1}. Parts (7.3.2)–(7.3.3) and

the last statement imply that

$$(n+1)!\int_X\Big(\int_{S(x_1)}\Big(\cdots\Big(\int_{S(x_n)} f_1(x_1)\cdots f_{n+1}(x_{n+1})\,d\mu(x_{n+1})\Big)\cdots\Big)d\mu(x_2)\Big)d\mu(x_1)$$

$$=\sum_{\pi\in P_{n+1}}\int_{H_\pi} f_1(x_{\pi(1)})\cdots f_{n+1}(x_{\pi(n+1)})d\mu^{n+1}$$

$$\le\int_{X^{n+1}} f_1(x_1)\cdots f_{n+1}(x_{n+1})\,d\mu^{n+1}=\prod_{i=1}^{n+1}\int_X f_i\,d\mu$$

which is the required inequality. The proof is complete. □

Let $(X,\mathcal{A},\mu)$ be a measure space, let Y be a set, and h be a mapping of X into Y. Then $\mathcal{R}=\{B\subset Y|h^{-1}(B)\in\mathcal{A}\}$ is a σ-algebra and $\nu(B)=\mu(h^{-1}(B))$ defines a measure ν on $\mathcal{R}$. The measure space $(Y,\mathcal{A},\nu)$ is called the image of $(X,\mathcal{A},\mu)$ under the mapping h, and ν is denoted by $\mu(h^{-1})$.

We shall need the following result whose proof is a straightforward computation, so we omit it.

Lemma 7.3.2 (Horváth [290]) *Let $(X_i,\mathcal{A}_i,\mu_i)$, $(i=1,2)$ be measure spaces, let Y_1 and Y_2 be sets, h_i be a mapping of X_i into Y_i $(i=1,2)$, and let the measure spaces $(Y_i,\mathcal{R}_i,\mu(h_i^{-1}))$ be the image of $(X_i,\mathcal{A}_i,\mu_i)$ under the mapping h_i $(i=1,2)$. The product of $(X_i,\mathcal{A}_i,\mu_i)$, $(i=1,2)$ is denoted by $(X,\mathcal{A},\mu)$ and the product of $(Y_i,\mathcal{R}_i,\mu(h_i^{-1}))$, $(i=1,2)$ is denoted by $(Y,\mathcal{R},\nu)$. If h_1 and h_2 are one-to-one mappings and $h=(h_1,h_2):X\to Y$, then $\nu=\mu(h^{-1})$.*

We now introduce a general Gronwall-Bellman type integral inequality.

Theorem 7.3.2 (Horváth [290]) *Let $(X,\mathcal{A},\mu)$ be a measure space, $S:X\to\mathcal{A}$ satisfy the condition (C). Suppose f and g are real-valued functions on X, and μ-integrable over $S(x)$ for every $x\in X$. Then*

(a) the integral equation

$$y(x)=f(x)+g(x)\int_{S(x)} y\,d\mu \tag{7.3.4}$$

has one and only one solution on X (this means that $s:X\to\mathbb{R}$ is μ-integrable over $S(x)$ and $y=s$ satisfies (7.3.4) for every $x\in X$),

(b) s can be written in the form $\sum_{n=0}^{+\infty}s_n$, where $s_0=f$ and $s_{n+1}=g(x)\int_{S(x)}s_n\,d\mu$ $(x\in X,n\in\mathbb{N})$,

(c) *for non-negative f and g, we have $x \in X$,*

$$0 \le s(x) \le f(x) + g(x) \int_{S(x)} f\, d\mu$$
$$\times \begin{cases} 1, & \text{if } \int_{S(x)} g\, d\mu = 0 \\ \frac{1}{\int_{S(x)} g\, d\mu}\Big(\exp\Big(\int_{S(x)} g\, d\mu\Big) - 1\Big), & \text{if } \int_{S(x)} g\, d\mu \neq 0. \end{cases} \tag{7.3.5}$$

(d) *Suppose g is non-negative. If ϕ is a real-valued function on X such that ϕ is μ-integrable over $S(x)$ for every $x \in X$, and*

$$\phi(x) \le f(x) + g(x) \int_{S(x)} \phi\, d\mu, \quad \mu - a.\ e. \quad \text{on} \quad X, \tag{7.3.6}$$

then $\phi(x) \le s(x)$ whenever (7.3.6) holds at $x \in X$ (so that $\phi \le s$, $\quad \mu - a.\ e.$ on X).

Proof First let f and g be non-negative. Let H be the set of points $x \in X$ such that

$$\phi(x) \le f(x) + g(x) \int_{S(x)} \phi\, d\mu.$$

Since g is non-negative, it follows from (7.3.6) that for $x \in H$,

$$\phi(x) \le f(x) + g(x) \int_{S(x)} f\, d\mu + g(x) \int_{S(x)} \Big(g(x_1) \int_{S(x_1)} \phi\, d\mu\Big)\, d\mu(x_1)$$

which, by induction, implies that

$$\begin{aligned} \phi(x) \le f(x) + g(x)\Big(&\int_{S(x)} f\, d\mu + \sum_{k=1}^{n} \int_{S(x)} \Big(\int_{S(x_k)} \Big(\cdots\Big(\int_{S(x_1)} f(x_0)g(x_1)\cdots \\ &\times g(x_k)\, d\mu(x_0)\Big)\, d\mu(x_1)\Big)\cdots\Big)\, d\mu(x_k)\Big) \\ &+ g(x)\int_{S(x)}\Big(\int_{S(x_{n+1})}\Big(\cdots\Big(\int_{S(x_2)}\Big(\int_{S(x_1)} \phi(x_0)g(x_1)\cdots \\ &\times g(x_{n+1})\, d\mu(x_0)\Big)\, d\mu(x_1)\Big)\cdots\Big)\, d\mu(x_n)\Big)\, d\mu(x_{n+1}) \\ = &\sum_{k=0}^{n+1} s_k(x) + R_{n+1}(x), \quad x \in H, \ \ n = 1, 2. \end{aligned} \tag{7.3.7}$$

By Theorem 7.3.1, we obtain

$$
\begin{aligned}
\sum_{k=0}^{n+1} s_k(x) &\le f(x) + g(x)\left(\int_{S(x)} f\,d\mu + \sum_{k=1}^{n} \frac{1}{(k+1)!}\int_{S(x)} f\,d\mu \left(\int_{S(x)} g\,d\mu\right)^k\right)\\
&= f(x) + g(x)\int_{S(x)} f\,d\mu \sum_{k=0}^{n} \frac{1}{(k+1)!}\left(\int_{S(x)} g\,d\mu\right)^k\\
&\le f(x) + g(x)\int_{S(x)} f\,d\mu\\
&\quad\times\begin{cases} 1, & \text{if } \int_{S(x)} g\,d\mu = 0,\\ \frac{1}{\int_{S(x)} g\,d\mu}\left(\exp\left(\int_{S(x)} g\,d\mu\right)-1\right), & \text{if } \int_{S(x)} g\,d\mu \ne 0.\end{cases}
\end{aligned} \tag{7.3.8}
$$

To estimate R_{n+1}, we now apply Theorem 7.3.1 again

$$
R_{n+1}(x) \le g(x)\frac{1}{(n+2)!}\int_{S(x)} \phi\,d\mu\left(\int_{S(x)} g\,d\mu\right)^{n+1}, \qquad x \in X,\quad n \in \mathbb{N}. \tag{7.3.9}
$$

Let $s = \sum_{n=0}^{+\infty} s_n$. Relation (7.3.8) implies that s is a non-negative real-valued function on X such that s is μ-integrable over $S(x)$ for every $x \in X$, and satisfies (C). It can be verified by direct substitution that s is a solution of (7.3.4). Thus $\phi(x) \le s(x), (x \in H)$ is an immediate consequence of (7.3.7), (7.3.9), and $\lim_{n\to+\infty}(z^n/n!) = 0$ $(z \in \mathbb{C})$.

Since $|f|$ and $|g|$ are non-negative real-valued functions on X with the same properties as f and g, it now follows easily that $s = \sum_{n=0}^{+\infty} s_n$ is a solution of (7.3.4) in the general case, too.

Suppose that $t : X \to \mathcal{R}$ is another solution of (7.3.4). Then for all$x \in X$,

$$
|s(x) - t(x)| \le |g(x)| \int_{S(x)} |s - t|\,d\mu.
$$

By what we have already proved, it follows from this that $|s(x) - t(x)| \le 0$ $(x \in X)$, so that $s = t$. We have thus shown (a), (b) and (c).

It remains only to prove (d). Let H be the set of points $x \in X$ such that

$$
\phi(x) \le f(x) + g(x)\int_{S(x)} \phi\,d\mu.
$$

Let $B = \{x \in H | \phi(x) > s(x)\}$, and let $\Psi : X \to \mathcal{R}$,

$$
\Psi(x) = \begin{cases} 0, & x \notin B\\ \phi(x) - s(x), & x \in B.\end{cases}
$$

Then Ψ is a non-negative real-valued function on X such that Ψ is μ-integrable over $S(x)$ for every $x \in X$. Thus it follows from (7.3.4) and (7.3.6) that for each $x \in B$,

$$\Psi(x) = \phi(x) - s(x) \leq g(x) \int_{S(x)} (\phi - s)\, d\mu \leq g(x) \int_{S(x)} \Psi\, d\mu,$$

which yields, for all $x \in X$,

$$0 \leq \Psi(x) \leq g(x) \int_{S(x)} \Psi\, d\mu. \tag{7.3.10}$$

We have shown that (7.3.10) implies $\Psi(x) = 0 \quad (x \in X)$, and therefore $B = \emptyset$. The proof is complete. □

Remark 7.3.1 Let $(X, \mathcal{A}, \mu)$ be a measure space, and $S : X \to \mathcal{R}$ satisfy the condition (C). Suppose f, g and h are real-valued functions on X such that h is non-negative, and hf, hg are μ-integrable over $S(x)$ for every $x \in X$. Then the integral equation

$$y(x) = f(x) + g(x) \int_{S(x)} hy\, d\mu$$

is equivalent to

$$y(x) = f(x) + g(x) \int_{S(x)} y\, d\mu_h,$$

where the measure μ_h is defined on $\mathcal{A}$ by $\mu_h(A) = \int_A h\, d\mu$. This shows that we can apply Theorem 7.3.2 in the considered situation, too.

The following result is a typical example of Gronwall-Bellman type integral inequalities involving Stieltjes integral.

Theorem 7.3.3 (Jones [305]) *Let $a, b \in \mathbb{R}$, $a < b$, and ϕ, f, g, h be real-valued functions on [a, b], with g, h non-negative. Let $\alpha : [a, b] \to \mathbb{R}$ be increasing left-continuous function. Suppose that ϕ, f, g, h are right-continuous at the discontinuity points of α. If for all $x \in [a, b]$,*

$$\phi(x) \leq f(x) + g(x) \int_{[a,x)} h\phi\, d\alpha,$$

then for all $x \in [a, b]$,

$$\phi(x) \leq f(x) + g(x) \int_{[a,x)} f(s)h(s) \exp\left(\int_{[s,x)} gh\, d\alpha \right) d\alpha(s). \tag{7.3.11}$$

If $X = [a, b]$ and $S(x) = [a, x) \quad (x \in X)$ (see Example 7.3.1 (a)), then the conditions of Theorem 7.3.2 are closely similar to those of Theorem 7.3.3, but the estimates (7.3.5) and (7.3.11) are rather different. This follows from essential difference between the proofs. In the proof of Theorem 7.3.3 (and most of the similar theorems), it is substantial that the integration is performed over intervals, and differentiation is employed. We illustrate by examples that the character of the estimates (7.3.5) and (7.3.11) is not the same.

Let $X = [0, +\infty)$, $\mathcal{A}$ be σ-algebra of Borel sets in X, μ be the Lebesgue measure of $\mathcal{A}$, $S(x) = [0, x)$ for every $x \in X$. Let u and v denote that functions on the right-hand side of (7.3.11) and (7.3.5), respectively.

(a) If $f, g : X \to [0, +\infty)$ are continuous, $f = g$ and $h(x) = 1$ $(x \in X)$, then $u = v$.
(b) If $f(x) = e^x$ and $g(x) = h(x) = 1$ $(x \in X)$, then $u(x) = e^x(1 + x)$ $(x \in X)$ and $v(x) = e^x + (1/x)(e^x - 1)^2$ $(x \in (0, +\infty))$, so that $u(x) < v(x)$ for every $x \in (0, +\infty)$.
(c) If $f(x) = e^{-x}$ and $g(x) = h(x) = 1$ $(x \in X)$, then $u(x) = ch(x)$ $(x \in X)$ and $v(x) = e^{-x} + (2/x)(ch(x) - 1)$ $(x \in (0, +\infty))$, so that $v(x) < u(x)$ for every $x \in (0, +\infty)$.

The next result is an extension of Theorem 7.3.2.

Theorem 7.3.4 (Horváth [290]) *Let $(X, \mathcal{A}, \mu)$ be a measure space, let $S : X \to \mathcal{A}$ satisfy the condition (C). Suppose $h : X \to X$ is a one to one mapping such that $h(S(x)) \subset S(x)$ for every $x \in X$. Suppose f and g are real-valued functions on X such that $f \circ h$, $g \circ h$ are μ-integrable over $S(x)$ for every $x \in X$. Then*

(a) The integral equation

$$y(x) = f(x) + g(x) \int_{S(x)} y \circ h \, d\mu \tag{7.3.12}$$

has one and only one solution s on X (this means that s is real-valued, $s \circ h$ is μ-integrable over $S(x)$, and $y = s$ satisfies (7.3.12) for every $x \in X$).

(b) s can be written in the form $\sum_{n=0}^{+\infty} s_n$, where $s_0 = f$ and $s_{n+1}(x) = g(x) \int_{S(x)} s_n \circ h \, d\mu$ $(x \in X, \quad n \in \mathbb{N})$.

(c) For non-negative f and g, we have

$$0 \leq s(x) \leq f(x) + g(x) \int_{S(x)} f \circ h \, d\mu$$
$$\times \begin{cases} 1, & \text{if } \int_{S(x)} g \circ h \, d\mu = 0, \\ \frac{1}{\int_{S(x)} g \circ h \, d\mu} \left(\exp\left(\int_{S(x)} g \circ h \, d\mu \right) - 1 \right), & \text{if } \int_{S(x)} g \circ h \, d\mu \neq 0. \end{cases}$$

(d) *Suppose g is non-negative. If ϕ is a real-valued function on X such that $\phi \circ h$ is μ-integrable over $S(x)$ for every $x \in X$, and*

$$\phi(x) \leq f(x) + g(x) \int_{S(x)} \phi \circ h \, d\mu, \quad \mu(h^{-1}) - a.\ e. \quad \text{on} \quad X, \tag{7.3.13}$$

then $\phi(x) \leq s(x)$ whenever (7.3.13) holds at $x \in X$ (so that $\phi \leq s, \quad \mu(h^{-1}) - a.\ e. \quad$ on $\quad X$).

Proof Let $(X, \mathcal{B}, \mu(h^{-1}))$ be the image of $(X, \mathcal{A}, \mu)$ under the mapping h. Since h is one-to-one, $h(S(x)) \in \mathcal{B}$ for every $x \in X$. It follows from $(C_1), (C_2)$, and the injectivity of f that $x \notin h(S(x))$ and if $y \in h(S(x))$, then $h(S(y)) \subset h(S(x))$ $(x \in X)$. By (C_3) and Lemma 7.3.2, $\{(x_1, x_2) \in X \times X | x_1 \in X_1, x_2 \in h(S(x))\} \in \mathcal{B}^2$. We have thus shown that the function $x \mapsto h(S(x))$ from X into $\mathcal{B}$ satisfies the condition (C). Since the integral equation (7.3.12) is equivalent to

$$y(x) = f(x) + g(x) \int_{h(S(x))} y d\mu(h^{-1})$$

and the inequality (7.3.13) is equivalent to

$$\phi(x) \leq f(x) + g(x) \int_{S(x)} y \, d\mu(h^{-1}), \quad \mu(h^{-1}) - a.\ e. \ \text{on} \ X,$$

then the theorem is an immediate consequence of Theorem 7.3.2. The proof is complete. □

7.3.2 Gronwall-Bellman Type Integral Inequalities for Abstract Lebesgue Integrals

In this section, we introduce the results on Gronwall-Bellman type integral inequalities for abstract Lebesgue integrals which are due to Györi and Horváth [245].

It is well-known that the Stieltjes type inequalities are employed to study functional differential equations, generalized differential equations, impulse differential equations, and Volterra-Stieltjes integral equations (see, [204, 238, 278]). However, there seems to be relatively few results dealing with integral inequalities for abstract Lebesgue (especially Lebesgue-Stieltjes) integrals, even in the one variables case [290]. One reason could be that the proofs of these inequalities are usually based on methods involving abstract Lebesgue integrals.

Next, we shall discuss linear integral inequalities of the Gronwall-Bellman type for scalar functions of several variables involving abstract Lebesgue integrals. The interesting features of these results are that some delay and advance effects are also

included in the inequalities, which could be important in applications to differential equations with aftereffects.

For any two elements of $\mathbb{R}^p, a = (a_1, \cdots, a_p)$ and $b = (b_1, \cdots, b_p)$, we write $a \le b \quad (a < b)$ if $a_i \le b_i \quad (a_i < b_i)$ for all $i = 1, \cdots, p$. All integral in $\mathbb{R}^p$ is the Cartesian product of p (not necessary bounded) intervals of real numbers. For points $a, b \in \mathbb{R}^p$ with $a \le b$, the intervals $(a, b), [a, b), (a, b], [a, b]$ are defined in the usual way.

If I is an interval in $\mathbb{R}^p$ and $x \in I$, then the intervals $\{u \in I | u < x\}$ and $\{u \in I | x < u\}$ are denoted by $R(x)$ and $L(x)$. If it is necessary, we write $R_I(x)$ and $L_I(x)$ instead of $R(x)$ and $L(x)$.

Let f be a function from a subset H of $\mathbb{R}^p$ to $\mathbb{R}^q$. We say that f is increasing if $f(x) \le f(y)$ whenever $x, y \in H, x \le y$. Function f is strictly increasing if $f(x) < f(y)$ whenever $x, y \in H, x \le y$, and $x \ne y$.

We assume that any σ-algebra in an interval of $\mathbb{R}^p$ contain the Borel sets in the interval.

In the proofs of the main results, we use the following statement, which is a special case of Theorem 2.1 in [290], i.e., Theorem 7.3.1.

Theorem 7.3.5 (Horváth [290]) *Let I be an interval in $\mathbb{R}^p$, (I, A, μ) be a measure space, and b be an increasing and continuous function from I to $\mathbb{R}^p$ such that $b(x) \le x$ for every $x \in I$. If f is a non-negative μ-integrable function on I, then for all $x \in X,\ n \in \mathbb{N}$*

$$\int_{R(b(x))} \left(\int_{R(b(x_1))} \left(\cdots \left(\int_{R(b(x_n))} f(x_1) \cdots f(x_{n+1}) d\mu(x_{n+1}) \right) \cdots \right) d\mu(x_2) \right) d\mu(x_1)$$
$$\le \frac{1}{(n+1)!} \left(\int_{R(b(x))} f d\mu \right)^{n+1}. \tag{7.3.14}$$

Proof Let $X = I$ and $S : I \to A, S(x) = R(b(x_1))$ in Theorem 2.1 in [290], i.e., in Theorem 7.3.1. □

Let I be an interval in $\mathbb{R}^p$, and let b an increasing function from I to $\mathbb{R}^p$ such that $b(x) \le x$ for every $x \in I$.

It is easy to see that the set $J = \cup\{[b(x), y] | x, y \in I\}$ is an interval which contains I.

The following result is a many-sided generalization of the usual Gronwall-Bellman type integral inequalities (functions of several variables, general intervals, abstract Lebesgue integral).

Theorem 7.3.6 (Györi-Horváth [245]) *Let I be an interval in $\mathbb{R}^p$, b be an increasing and continuous function from I to $\mathbb{R}^p$ such that $b(x) \le x$ for every $x \in I$, $J = \cup\{[b(x), y] | x, y \in I\}$, and let (J, A, μ) be a measure space. Suppose f is a real-valued function on I and ϕ_0 is a real-valued function on $J \backslash I$ such that the function*

$\tilde{f} : J \to \mathbb{R}$,

$$\tilde{f} = \begin{cases} \phi_0(x), & x \in J \backslash I, \\ f(x), & x \in I \end{cases}$$

is μ-integrable over $R_J(b(x))$ for every $x \in I$. Suppose g is a real-valued function on I, and it is μ-integrable over $R_I(b(x))$ for every $x \in I$. Then we have the following.

(a) The integral equation

$$y(x) = f(x) + g(x) \int_{R_J(b(x))} y d\mu, \quad (x \in I) \tag{7.3.15}$$

has one and only one solution s on J such that the restriction of s to $J \backslash I$ is ϕ_0 ($s : J \to \mathbb{R}$ is μ-integrable over $R_J(b(x))$ for every $x \in I$, and $y = s$ satisfies (7.3.15) for all $x \in I$).

(b) Introducing the functions

$$\tilde{g} = \begin{cases} 0, & x \in J \backslash I, \\ g(x), & x \in I, \end{cases}$$

and

$$S(x) = \begin{cases} 0, & x \in J \backslash I, \\ R_J(b(x)), & x \in I \end{cases}$$

s can be written in the form $\sum_{n=0}^{+\infty} s_n$, where $s_0 = \tilde{f}$ and $s_{n+1}(x) = \tilde{g}(x) \int_{S(x)} s_n d\mu, \quad x \in J, \quad n \in \mathbb{N}$.

(c) For non-negative f, g and ϕ_0, we have for all $x \in I$,

$$0 \leq s(x) \leq f(x) + g(x) \int_{R_I(b(x))} \left(f(u) \exp\left(\int_{(u,b(x))} g d\mu \right) \right) d\mu(u)$$
$$+ g(x) \int_{R_{J(b(x))} \backslash R_I(b(x))} \phi_0 d\mu \exp\left(\int_{R_I(b(x))} g d\mu \right). \tag{7.3.16}$$

(d) Suppose g is non-negative. If ϕ is a real-valued function on J such that ϕ_0 is the restriction of ϕ to $J \backslash I$, ϕ is μ-integrable over $R_J(b(x))$ for every $x \in I$, and

$$\phi(x) \leq f(x) + g(x) \int_{R_J(b(x))} \phi d\mu, \quad \mu - a.\ e.\ on\ I, \tag{7.3.17}$$

then $\phi(x) \leq s(x)$ whenever (7.3.17) holds at $x \in I$ (so that $\phi \leq s, \quad \mu - a.\ e.\ on\ I$).

Proof We can apply Theorem 7.3.4 (see also Theorem 3.1 in [290]) with $J, A, \mu, S, \tilde{f}, \tilde{g}, \phi$, and hence assertions $(a), (b), (d)$ follow readily. Thus it remains to prove (c).

By (b), for every $x \in I$ and $n \in \mathbb{N}$,

$$\begin{aligned}\sum_{k=0}^{n+1} s_k(x) &= \tilde{f}(x) + \tilde{g}(x)\Big(\int_{S(x)} \tilde{f} d\mu \\ &\quad + \sum_{k=1}^{n} \int_{S(x)} \left(\int_{S(x_k)} \left(\cdots\left(\int_{S(x_1)} \tilde{f}(x_0)\tilde{g}(x_1)\cdots\tilde{g}(x_k)d\mu(x_0)\right)\cdots\right)\right. \\ &\quad \left.\times d\mu(x_{k-1})\right) d\mu(x_k)\Big) \\ &= f(x) + g(x)\Big(\int_{R_I(b(x))} f d\mu + \int_{R_J(b(x))\setminus R_I(b(x))} \phi_0 d\mu \\ &\quad + \sum_{k=1}^{n} \int_{R_I(b(x))} \left(\int_{R_I(b(x_k))} \left(\cdots\left(\int_{R_I(b(x_1))} f(x_0)g(x_1)\cdots g(x_k)d\mu(x_0)\right)\right.\right. \\ &\quad \left.\left.\times\cdots\right) d\mu(x_{k-1})\right) \\ &\quad d\mu(x_k) + \sum_{k=1}^{n} \int_{R_I(b(x))} \Big(\int_{R_I(b(x_k))} \Big(\cdots\Big(\int_{R_I(b(x_k))\setminus R_I(b(x_1))} \\ &\quad \phi_0(x_0)g(x_1)\cdots g(x_k)d\mu(x_0)\Big)\cdots\Big)d\mu(x_{k-1})\Big)d\mu(x_k)\Big).\end{aligned}$$

To estimate $\sum_{k=0}^{n+1} s_k$ on I, first we use that $b(x) \le x$ for every $x \in I$, then apply Fubini's theorem, and finally use Theorem 7.3.5,

$$\begin{aligned}\sum_{k=0}^{n+1} s_k(x) &\le f(x) + g(x)\Big[\int_{R_I(b(x))} f d\mu + \int_{R_J(b(x))\setminus R_I(b(x))} \phi_0 d\mu \\ &\quad + \sum_{k=1}^{n} \int_{R_I(b(x))} \Big(\int_{R_I(b(x_k))} \Big(\cdots\Big(\int_{R_I(b(x_1))\setminus R_I(b(x_1))} f(x_0)g(x_1)\cdots g(x_k)d\mu(x_0)\Big) \\ &\quad \cdots\Big)d\mu(x_{k-1})\Big)d\mu(x_k) + \Big(\int_{R_J(b(x))\setminus R_I(b(x))} \phi_0 d\mu\Big)\Big(\int_{R_I(b(x))} g d\mu \\ &\quad + \sum_{k=2}^{n} \int_{R_I(b(x))} \left(\int_{R_I(b(x_k))} \left(\cdots\left(\int_{R_I(b(x_2))} g(x_1)\cdots g(x_k)d\mu(x_1)\right)\cdots\right)\right. \\ &\quad \left.\times d\mu(x_{k-1})\right) d\mu(x_k)\Big)\Big]\end{aligned}$$

$$
\begin{aligned}
&= f(x) + g(x)\Big\{ \int_{R_I(b(x))} f d\mu + \int_{R_I(b(x))} \Big(f(x_0) \int_{(x_0,b(x))} g(x_1) d\mu(x_1) \Big) d\mu(x_0) \\
&\quad + \sum_{k=2}^{n} \int_{R_I(b(x))} \Big(f(x_0) \int_{(x_0,b(x))} \Big(\int_{(x_0,x_k)} \Big(\cdots \Big(\int_{(x_0,x_2)} g(x_1) \cdots g(x_k) d\mu(x_1) \Big) \cdots \Big) \\
&\quad \times d\mu(x_{k-1}) \Big) d\mu(x_k) \Big) d\mu(x_0) \Big\} \\
&\quad + g(x) \Big(\int_{R_J(b(x)) \setminus R_I(b(x))} \phi_0 d\mu \Big) \Big\{ 1 + \int_{R_I(b(x))} g d\mu \\
&\quad + \sum_{k=2}^{n} \int_{R_I(b(x))} \Big(\int_{R_I(b(x_k))} \Big(\cdots \Big(\int_{R_I(b(x_2))} g(x_1) \cdots g(x_k) d\mu(x_1) \Big) \cdots \Big) d\mu(x_{k-1}) \Big) \\
&\quad \times d\mu(x_k) \Big\} \\
&\le f(x) + g(x) \sum_{k=0}^{n} \int_{R_I(b(x))} \Big(f(x_0) \frac{1}{k!} \Big(\int_{(x_0,b(x))} g d\mu \Big)^k \Big) d\mu(x_0) \\
&\quad + g(x) \Big(\int_{R_J(b(x)) \setminus R_I(b(x))} \phi_0 d\mu \Big) \sum_{k=0}^{n} \frac{1}{k!} \Big(\int_{R_I(b(x))} g d\mu \Big)^k, \quad x \in I, n = 2, 3, \cdots.
\end{aligned}
$$

Therefore, we can see that (c) follows from (b) and the previous estimate. The proof is complete. □

Remark 7.3.2 The estimate in (c) of Theorem 7.3.6 is a straightforward generalization of the estimates in the classical Gronwall-Bellman type integral inequalities. It is worth nothing that the integral inequality (7.3.17) is discussed in a very general case, and hence, the obtained result depends on the technique which is applied. A similar result is included in Theorem 7.3.4 (see also Theorem 3.1 in [290]), but the explicit estimates has another form.

Let I be an interval in $\mathbb{R}^p$, b be an increasing and continuous function from I to $\mathbb{R}^p$ such that $b(x) \le x$ for every $x \in I$, $J = \cup\{[b(x), y] | x, y \in I\}$, and let (J, A, μ) be a measure space. Suppose f is a real-valued function on I, ϕ_0 is a real-valued function on $J \setminus I$, and h is a real-valued function on J such that h is non-negative and the function $\tilde{f} : J \to \mathbb{R}$,

$$
\tilde{f} = \begin{cases} h(x)\phi_0(x), & x \in J \setminus I, \\ h(x)f(x), & x \in I \end{cases}
$$

is μ-integrable over $R_J(b(x))$ for every $x \in I$. Suppose g is a real-valued function on I, and hg is μ-integrable over $R_I(b(x))$ for every $x \in I$. Then the integral equation

$$
y(x) = f(x) + g(x) \int_{R_J(b(x))} h y d\mu, \quad x \in I,
$$

is equivalent to

$$y(x) = f(x) + g(x) \int_{R_J(b(x))} y d\mu_h, \quad x \in I,$$

where the measure μ_h is defined on A by $\mu_h(A) = \int_A h d\mu$ ((3) can be transformed similarly), so the previous theorem can be applied in this case too.

The next result follows from Theorem 7.3.6 easily, but it is interesting enough to be stated separately.

Theorem 7.3.7 (Györi-Horváth [245]) *Let I be an interval in $\mathbb{R}^p$, b be an increasing and continuous function from I to $\mathbb{R}^p$ such that $x \le b(x)$ for every $x \in I$, $J = \cup\{[y, b(x)] | x, y \in I\}$ (J is an interval which contains I), and (J, A, μ) be a measure space. Suppose f is a real-valued function on I and ϕ_0 is a real-valued function on $J \backslash I$ such that the function $\tilde{f} : J \to \mathbb{R}$,*

$$\tilde{f} = \begin{cases} \phi_0(x), & x \in J\backslash I, \\ f(x), & x \in I, \end{cases}$$

is μ-integrable over $L_J(b(x))$ for every $x \in I$. Suppose g is a real-valued function on I, and it is μ-integrable over $L_I(b(x))$ for every $x \in I$. Then we have the following conclusions.

(a) The integral equation

$$y(x) = f(x) + g(x) \int_{L_J(b(x))} y d\mu, \quad x \in I, \tag{7.3.18}$$

has one and only one solution s on J such that the restriction of s to $J\backslash I$ is ϕ_0 ($s : J \to \mathbb{R}$, is μ-integrable over $L_J(b(x))$ for every $x \in I$, and $y = s$ satisfies (7.3.18) for all $x \in I$).

(b) Introducing the functions

$$\tilde{g} : J \to \mathbb{R}, \;\; \tilde{g} = \begin{cases} 0, & x \in J\backslash I, \\ g(x), & x \in I, \end{cases}$$

and

$$s : J \to A, \;\; s(x) = \begin{cases} 0, & x \in J\backslash I, \\ L_J(b(x)), & x \in I, \end{cases}$$

s can be written in the form $\sum_{n=0}^{+\infty} s_n$, where $s_0 = \tilde{f}$ and $s_{n+1}(x) = \tilde{g}(x) \int_{S(x)} s_n d\mu, \;\; x \in J, \;\; n \in \mathbb{N}$.

(c) For non-negative f, g and ϕ_0, we have for all $x \in I$,

$$0 \le s(x) \le f(x) + g(x) \int_{L_I(b(x))} \left(f(u) \exp\left(\int_{(u,b(x))} g d\mu \right) \right) d\mu(u)$$
$$+g(x) \int_{L_{J(b(x))} \setminus L_I(b(x))} \phi_0 d\mu \exp\left(\int_{L_I(b(x))} g d\mu \right). \tag{7.3.19}$$

(d) Suppose g is non-negative. If ϕ is a real-valued function on J such that ϕ_0 is the restriction of ϕ to $J \setminus I$, ϕ is μ-integrable over $L_J(b(x))$ for every $x \in I$, and

$$\phi(x) \le f(x) + g(x) \int_{L_J(b(x))} \phi d\mu, \quad \mu - \ a.\,e\,.\ on \ \ I, \tag{7.3.20}$$

then $\phi(x) \le s(x)$ whenever (7.3.20) holds at $x \in I$ (so that $\phi \le s$, $\quad \mu -$ a. e. on I).

If, in addition to the hypothesis of Theorem 7.3.6, the range of b is a subset I, then we have the following corollary.

Corollary 7.3.1 (Györi-Horváth [245]) *Let I be an interval in $\mathbb{R}^p$, (J, A, μ) be a measure space. Suppose b is an increasing and continuous function from I to I such that $b(x) \le x$ for every $x \in I$. Suppose f and g are real-valued functions on I, and μ-integrable over $R(b(x))$ for every $x \in I$. Then we have the following conclusions.*

(a) The integral equation

$$y(x) = f(x) + g(x) \int_{R(b(x))} y d\mu, \quad x \in I, \tag{7.3.21}$$

has one and only one solution s on I ($s : I \to \mathbb{R}$, is μ-integrable over $R(b(x))$ for every $x \in I$, and $y = s$ satisfies (7.3.21) for all $x \in I$).

(b) s can be written in the form $\sum_{n=0}^{+\infty} s_n$, where $s_0 = f$ and $s_{n+1}(x) = g(x) \int_{R(b(x))} s_n d\mu, \quad x \in I, \quad n \in \mathbb{N}$.

(c) For non-negative f and g, we have

$$0 \le s(x) \le f(x) + g(x) \int_{R(b(x))} \left(f(u) \exp\left(\int_{(u,b(x)} g d\mu \right) \right) d\mu(u), \quad x \in I. \tag{7.3.22}$$

(d) Suppose that g is non-negative. If ϕ is a real-valued function on I such that ϕ is μ-integrable over $L_J(b(x))$ for every $x \in I$, and

$$\phi(x) \le f(x) + g(x) \int_{R(b(x))} \phi d\mu, \quad \mu - \ a.\,e.\ on\ I, \tag{7.3.23}$$

then $\phi(x) \le s(x)$ whenever (7.3.23) holds at $x \in I$ (so that $\phi \le s$, $\quad \mu -$ a. e. on I).

We shall illustrate by examples that Corollary 7.3.1 (so that Theorem 7.3.6) may not be true, if we use right-half closed intervals ($\{u \in I | u \leq b(x)\}$) in (7.3.21) and (7.3.22) instead of $R(b(x)) = \{u \in I | u < b(x)\}$.

Let c be a positive constant, ε_1 be the unit mass at 1 ($\varepsilon_1(A) = 1$ if $1 \in A$, and $\varepsilon_1(A) = 0$ if $1 \notin A$, for any $A \subset \mathbb{R}$), and consider the integral equation

$$y(x) = 1 + c \int_{[0,x]} y d\varepsilon_1, \quad x \in \mathbb{R}_+. \tag{7.3.24}$$

It is easy to see that (7.3.24) has a solution on $\mathbb{R}_+$ if and only if $c \neq 1$ (compare with (a) in Corollary 7.3.1 and see [238]). If $c \neq 1$, then the unique solution $s : \mathbb{R}_+ \to \mathbb{R}$ of (7.3.24) is

$$s(s) = \begin{cases} 1, & 0 \leq x \leq 1, \\ \frac{1}{1-c}, & 1 < x. \end{cases}$$

If $\phi : \mathbb{R}_+ \to \mathbb{R}, \phi(x) = 1$ and $c > 1$, then for all $x \in \mathbb{R}_+$,

$$\phi(x) \leq 1 + c \int_{[0,x]} y d\varepsilon_1,$$

but $\phi(x) > s(x)$ $(x \geq 1)$ (compare with (d) in Corollary 7.3.1). If $\frac{2}{3} < c < 1$, then for all $x \geq 1$,

$$s(x) > 1 + c \int_{[0,x]} \exp\left(\int_{[0,x]} c d\varepsilon_1\right) d\varepsilon_1(u),$$

(compare with (c) in Corollary 7.3.1).

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be the measurable spaces, and $T : (X, \mathcal{A}) \to (Y, \mathcal{B})$ be a measurable mapping. Then for every measure μ *on* $\mathcal{A}$, $\mathcal{B} \to \mu(T^{-1}(\mathcal{B}))$ defines a measure on $\mathcal{B}$. This measure is defined by $\mu(T^{-1})$.

Now we consider a result which provides a closer analogue of Theorem 7.3.6.

Theorem 7.3.8 (Györi-Horváth [245]) *Let I be an interval in $\mathbb{R}^p$, and (I, A, μ) be a measure space. Let h be a function from I to $\mathbb{R}^p$ such that $h(x_1, \cdots, x_p) = (h(x_1), \cdots, h(x_p))$, where h_i is strictly increasing and continuous ($i = 1, \cdots, p$), and $h(x) \leq x$ for every $x \in I$, and let $J = \cup\{[h(x), y] | x, y \in I\}$. Suppose f is a real-valued function on I and ϕ_0 is a real-valued function on I and ϕ_0 is a real-valued function on $J \backslash I$ such that the function $\tilde{f} \circ h$ is μ-integrable over $R_I(x)$ for every $x \in I$, where $\tilde{f} : J \to \mathbb{R}$,*

$$\tilde{f} = \begin{cases} \phi_0(x), & x \in J \backslash I, \\ f(x), & x \in I. \end{cases}$$

Suppose g is a real-valued function on I, and $g \circ h$ is μ-integrable over $R_{h^{-1}(I)}(x)$ for every $x \in I$.

Then we have the following conclusions.

(a) *The integral equation*

$$y(x) = f(x) + g(x) \int_{R_I(x)} y \circ h d\mu, \quad x \in I \tag{7.3.25}$$

has one and only one solution s on J such that the restriction of s to $J \backslash I$ is ϕ_0 ($s : J \to \mathbb{R}, s \circ h$ is μ-integrable over $R_I(h(x))$ for every $x \in I$, and $y = s$ satisfies (7.3.25) for all $x \in I$).

(b) *Introducing the function $\tilde{g} : J \to \mathbb{R}$,*

$$\tilde{g} = \begin{cases} 0, & x \in J \backslash I, \\ g(x), & x \in I, \end{cases}$$

and s can be written in the form $\sum_{n=0}^{+\infty} s_n$, where $s_0 = \tilde{f}$ and $s_{n+1}(x) = \tilde{g}(x) \int_{R_I(x)} s_n \circ h d\mu, \quad x \in J, \quad \in \mathbb{N}$.

(c) *For non-negative f, g and ϕ_0, we have for all $x \in I$,*

$$0 \le s(x) \le f(x) + g(x) \int_{R_{h^{-1}(I)}(x)} \left(f(h(u)) \exp\left(\left(\int_{(u,x)} g \circ h d\mu \right) \right) d\mu(u) \right.$$
$$\left. + g(x) \int_{R_I(x) \backslash R_{h^{-1}(I)}(x)} \phi_0 \circ h d\mu \exp\left(\int_{R_{h^{-1}(I)}(x)} g \circ h d\mu \right) \right). \tag{7.3.26}$$

(d) *Suppose g is non-negative. If ϕ is a real-valued function on J such that ϕ_0 is the restriction of ϕ to $J \backslash I$, $\phi \circ h$ is μ-integrable over $R_I(x)$ for every $x \in I$, and*

$$\phi(x) \le f(x) + g(x) \int_{R_I(x)} \phi \circ h d\mu, \quad \mu(h^{-1}) - a.\ e.\ on\ I, \tag{7.3.27}$$

then $\phi(x) \le s(x)$ whenever (7.3.27) holds at $x \in I$ (so that $\phi \le s, \quad \mu(h^{-1}) - a.\ e.\ on\ I$).

Proof If $\mathcal{B} = \{B \subset J \mid h^{-1}(B) \in \mathcal{A}\}$, then $\mathcal{B}$ is a σ-algebra in J which contains the Borel sets in J, and $h : (I, \mathcal{A}) \to (J, \mathcal{B})$ is measurable. Then the integral equation (7.3.26) is equivalent to

$$y(x) = f(x) + g(x) \int_{R_J(h(x))} y d\mu(h^{-1}),$$

and the integral inequality (7.3.27) is equivalent to

$$\phi(x) \leq f(x) + g(x) \int_{R_J(h(x))} \phi d\mu(h^{-1}), \quad \mu(h^{-1}) - a.\ e.\ on\ I,$$

thus the assertion follows from Theorem 7.3.6. □

Chapter 8
Applications of Linear Multi-Dimensional Integral and Difference Inequalities

8.1 Applications of Theorems 5.1.1 and 5.1.10, and Corollaries 5.1.2 and 5.1.4 to Nonlinear Vector Hyperbolic Partial Differential Equations

In this section, we shall use Theorems 5.1.1 and 5.1.10 and Corollaries 5.1.2 and 5.1.4 to study nonlinear vector hyperbolic partial differential equations.

Example 8.1.1 Consider the nonlinear vector hyperbolic partial differential equation

$$u_{xy} = f(x, y, u) \tag{8.1.1}$$

with boundary conditions prescribed on $x = x_0$ and $y = y_0$. We assume that f is continuous and satisfies a matrix Lipschitz condition; i.e., there is a non-negative matrix K such that

$$|f(x, y, u) - f(x, y, \overline{u})| \leq K|u - \overline{u}| \tag{8.1.2}$$

where the absolute values are taken componentwise and u and $\overline{u}$ are any vectors. If each component of f satisfies a scalar Lipschitz condition, then K reduces to a diagonal matrix and letting k be the largest of the diagonal elements we could use kI, where I is the identity matrix, for the Lipschitz constant matrix.

Using the boundary conditions the equivalent vector Volterra integral equation for (8.1.1) is

$$u(x, y) = g(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} f(s, t, u(s, t))dsdt,$$

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where g depends on the boundary values. Then for any two solutions u and $\overline{u}$ of the integral equation, we have

$$u-\overline{u}=\int_{x_0}^{x}\int_{y_0}^{y}[f(s,t,u)-f(s,t,\overline{u})]dsdt. \tag{8.1.3}$$

Hence, if $(x-x_0)(y-y_0)\geq 0$ and we take absolute values componentwise, we get

$$|u-\overline{u}|\leq\int_{x_0}^{x}\int_{y_0}^{y}|f(s,t,u)-f(s,t,\overline{u})|dsdt\leq\int_{x_0}^{x}\int_{y_0}^{y}K|u-\overline{u}|. \tag{8.1.4}$$

Applying Theorem 5.1.1 to (8.1.4), we have $|u-\overline{u}|\leq 0$ componentwise, so $u\equiv\overline{u}$; i.e., the solution is unique. □

This result can also be proved by using a norm instead of absolute values componentwise to get

$$||u-\overline{u}||\leq\int_{x_0}^{x}\int_{y_0}^{y}||f(s,t,u)-f(s,t,\overline{u})||dsdt$$

and noting that the matrix Lipschitz condition implies a norm Lipschitz condition so that

$$||u-\overline{u}||\leq\int_{x_0}^{x}\int_{y_0}^{y}\tilde{K}||u-\overline{u}||dsdt$$

where $\tilde{K}$ is a scalar. The desired conclusion now follows from the scalar two independent variable Gronwall inequality in Theorem 5.1.1 or Corollary 5.1.2.

The first method, the vector approach, may be easier to verify for some problems.

Example 8.1.2 Consider the vector characteristic initial value problem for the linear hyperbolic partial differential equation

$$u_{xy}-B(x,y)u=f(x,y), \tag{8.1.5}$$

where B is a non-negative continuous matrix and the vector u is prescribed on $x=x_0$ and $y=y_0$. Using the boundary conditions, this problem is equivalent to the vector Volterra integral equation

$$u(x,y)=F(x,y)+\int_{x_0}^{x}\int_{y_0}^{y}B(s,t)u(s,t)dsdt \tag{8.1.6}$$

where F is computed from f and the boundary conditions and we assume F is continuous. Now suppose the vectors v and w satisfy

$$v\leq F+\int_{x_0}^{x}\int_{y_0}^{y}B(s,t)v(s,t)dsdt,\quad w\geq F+\int_{x_0}^{x}\int_{y_0}^{y}B(s,t)w(s,t)dsdt. \tag{8.1.7}$$

Then applying Theorem 5.1.10 and Corollary 5.1.4 to the above inequalities shows that for any solution u to the boundary value problem, we have

$$v \leq u \leq w$$

which is a componentwise comparison theorem for the solution u.

Example 8.1.3 Consider the two vector boundary value problems

$$\begin{cases} u_{xy} = f(x, y, u), \\ u(x_0, y) = g(y), \\ u(x, y_0) = h(x), \\ g(y_0) = h(x_0) \end{cases} \tag{8.1.8}$$

and

$$\begin{cases} U_{xy} = F(x, y, U), \\ U(x_0, y) = G(y), \\ U(x, y_0) = H(x), \\ G(y_0) = H(x_0) \end{cases} \tag{8.1.9}$$

where all the functions are continuous and f satisfies a matrix Lipschitz condition. This, of course, implies a norm Lipschitz condition as in Example 8.1.1 so a norm type continuous dependence result can be obtained by using the scalar inequality in [603]. Here we shall obtain a componentwise result. We write the equivalent vector integral equations and subtract to get

$$\begin{aligned} u - U = {} & (g - G) + (h - H) - [g(y_0) - G(y_0)] \\ & + \int_{x_0}^{x} \int_{y_0}^{y} [f(s, t, u) - F(s, t, U)] ds dt. \end{aligned} \tag{8.1.10}$$

By adding and subtracting $f(s, t, U)$ in the integrand and taking absolute values componentwise, we get, for $(x - x_0)(y - y_0) \geq 0$,

$$\begin{aligned} |u - U| \leq {} & |g - G| + |h - H| - |g(y_0) - G(y_0)| \\ & + \int_{x_0}^{x} \int_{y_0}^{y} |f(s, t, u) - f(s, t, U)| ds dt \\ & + \int_{x_0}^{x} \int_{y_0}^{y} |f(s, t, U) - F(s, t, U)| ds dt. \end{aligned} \tag{8.1.11}$$

If $|g - G| \leq \epsilon$, $|h - H| \leq \epsilon$, *and* $|f(s,t,U) - F(s,t,U)| \leq \epsilon$, where ϵ is a non-negative constant, then

$$|u - U| \leq 3\epsilon + \epsilon(x - x_0)(y - y_0) + \int_{x_0}^{x}\int_{y_0}^{y} K|u - U|dsdt \tag{8.1.12}$$

where K is the non-negative Lipschitz constant matrix for f. By Theorem 5.1.10, we have

$$\begin{aligned}|u - U| &\leq 3\epsilon + \epsilon(x - x_0)(y - y_0) + \int_{x_0}^{x}\int_{y_0}^{y} V^T K[3\epsilon + \epsilon(x - x_0)(y - y_0)]dsdt \\ &= M(x,y)\epsilon\end{aligned}$$

where M is a continuous matrix function which can be computed at least in theory. On a compact domain in the xy-plane, M is bounded so the solution to the boundary value problem we started with depends continuously on f and the boundary values.

8.2 Applications of Theorem 5.1.3 to Hyperbolic Partial Integrodifferential Equations

In this section, we shall use Theorem 5.1.3 to investigate nonlinear hyperbolic partial integrodifferential equations.

Example 8.2.1 As a first application, we obtain the bound on the solution of a nonlinear hyperbolic partial integrodifferential equation

$$u_{xy} = f\left[x, y, u, \int_0^x\int_0^y k(x,y,s,t,u)dsdt\right], \tag{8.2.1}$$

with the given boundary conditions

$$u(x,0) = a(x),\ u(0,y) = b(y),\ a(0) = b(0) = 0,$$

where all the functions are continuous on their respective domains of their definitions and

$$|k(x,y,s,t,u)| \leq q(s,t)|u|, \tag{8.2.2}$$

$$|f[x,y,u,r]| \leq p(x,y)[|u| + |r|], \tag{8.2.3}$$

where p and q are as defined in Theorem 5.1.3. Equation (8.2.1) is equivalent to the Volterra integral equation

$$u(x,y) = a(x) + b(y) + \int_0^x \int_0^y f\left[s, t, u(s,t), \int_0^s \int_0^t k(s,t,\xi,\eta,u(\xi,\eta))d\xi d\eta\right] dsdt, \tag{8.2.4}$$

where $u(x, y)$ is any solution of Eq. (8.2.1). Using (8.2.2) and (8.2.3) in (8.2.4) and assuming that $|a(x)| + |b(y)| \leq M$, where $M > 0$ is a constant, and applying Theorem 5.1.3 when $g(x, y) = M$, we have

$$|u(x,y)| \leq M\left[1 + \int_0^x \int_0^y p(s,t) \exp\left(\int_0^s \int_0^t [p(\xi,\eta) + q(\xi,\eta)]d\xi d\eta\right) dsdt\right]. \tag{8.2.5}$$

Thus the right-hand side of (8.2.5) gives us the bound on the solution $u(x, y)$ of Eq. (8.2.1) in terms of the known functions. We also note that (8.2.5) implies the stability of the solution $u(x, y)$ of Eq. (8.2.1), if the bound obtained on the right side of (8.2.1) is small enough. □

Example 8.2.2 As a second application, we establish the uniqueness of solutions of Eq. (8.2.1) with the given boundary conditions. We assume that the functions k and f in Eq. (8.2.1) satisfy

$$|k(x,y,s,t,u) - k(x,y,s,t,\overline{u})| \leq q(s,t)|u - \overline{u}|, \tag{8.2.6}$$

$$|f[x,y,u,r] - f[x,y,u,\overline{r}]| \leq p(x,y)[|u - \overline{u}| + |r - \overline{r}|], \tag{8.2.7}$$

where p and q are as defined in Theorem 5.1.3. The problem is equivalent to Eq. (8.2.4). Then for any two solutions u and $\overline{u}$ of Eq. (8.2.1) with the given boundary conditions, we have

$$\begin{aligned} u - \overline{u} = g(x,y) - \overline{g}(x,y) + \int_0^x \int_0^y \Big\{ f\left[s,t,u, \int_0^s \int_0^t k(s,t,\xi,\eta,u)d\xi d\eta\right] \\ - f\left[s,t,\overline{u}, \int_0^s \int_0^t k(s,t,\xi,\eta,\overline{u})d\xi d\eta\right] \Big\} dsdt \end{aligned} \tag{8.2.8}$$

where g and $\overline{g}$ depends on the given boundary conditions. Using (8.2.6) and (8.2.7) in (8.2.8) and further assuming $|g - \overline{g}| \leq \epsilon$, for arbitrary $\epsilon > 0$, and applying

Theorem 5.1.3, we get

$$|u-\overline{u}| \leq \epsilon\left[1+\int_0^x\int_0^y p(s,t)\exp\left(\int_0^s\int_0^t [p(\xi,\eta)+q(\xi,\eta)]d\xi d\eta\right)dsdt\right]. \tag{8.2.9}$$

Since $\epsilon > 0$ is arbitrary, we have $u = \overline{u}$, i.e., there is at most one solution of Eq. (8.2.1). □

Remark 8.2.1 We note that the inequality established in Theorem 5.1.3 can be used to study the continuous dependence of the solution of Eq. (8.2.1) by following the similar argument as in [603] with suitable modifications. We omit the details.

8.3 An Application of Theorem 5.1.7 to Terminal Value Problem for the Hyperbolic Partial Differential Equations

In this section, we present some immediate applications of Theorem 5.1.7 to study certain properties of solutions of the following terminal value problem for the hyperbolic partial differential equation

$$\begin{cases} u_{xy}(x,y) = h(x,y,u(x,y)) + r(x,y), \\ u(x,\infty) = \sigma_\infty(x),\ u(\infty,y) = \tau_\infty(y), \\ u(\infty,\infty) = d, \end{cases} \tag{8.3.1}$$

where $h : \mathbb{R}_+^2 \times \mathbb{R} \to \mathbb{R}$, $r : \mathbb{R}_+^2 \to \mathbb{R}$, $\sigma_\infty, \tau_\infty : \mathbb{R}_+ \to \mathbb{R}$ are continuous functions and d is a real constant.

The following theorem deals with the estimate on the solution of problem (8.3.1)–(8.3.2).

Theorem 8.3.1 (Pachpatte [498]) *Suppose that*

$$|h(x,y,u)| \leq c(x,y)|u|, \tag{8.3.2}$$

$$\left|\sigma_\infty + \tau_\infty(y) - d + \int_x^{+\infty}\int_y^{+\infty} r(s,t)dtds\right| \leq a(x,y) \tag{8.3.3}$$

where $a(x,y), c(x,y)$ are as defined in part (a_2) of Theorem 5.1.7. Let $u(x,y)$ be a solution of problem (8.3.1)–(8.3.2) for all $x, y \in \mathbb{R}_+$, then for all $x, y \in \mathbb{R}_+$,

$$|u(x,y)| \leq a(x,y) + \bar{e}(x,y)\exp\left(\int_x^{+\infty}\int_y^{+\infty} c(s,t)dtds\right), \tag{8.3.4}$$

where

$$\bar{e}(x,y) = \int_x^{+\infty}\int_y^{+\infty} c(s,t)a(s,t)dtds.$$

Proof If $u(x,y)$ is a solution of problem (8.3.1)–(8.3.2), then it can be written as, e.g., (see [42], p. 80]) for all $x, y \in \mathbb{R}_+$,

$$u(x,y) = \sigma_\infty(x)+\tau_\infty(y)-d+\int_x^{+\infty}\int_y^{+\infty}\Big(h(s,t,u(s,t))+r(s,t)\Big)dtds. \tag{8.3.5}$$

From (8.3.6), (8.3.3), (8.3.4), it follows

$$|u(x,y)| \le a(x,y) + \int_x^{+\infty}\int_y^{+\infty} c(s,t)|u(s,t)|dtds. \tag{8.3.6}$$

Now applying part (a_2) of Theorem 5.1.7 to (8.3.6) yields the required estimate in (8.3.4). □

The next result deals with the uniqueness of the solutions of problem (8.3.1)–(8.3.2).

Theorem 8.3.2 (Pachpatte [498]) *Suppose that the function h in Eq. (8.3.1) satisfies the condition*

$$|h(x,y,u)-h(x,y,v)| \le c(x,y)|u-v|, \tag{8.3.7}$$

where $c(x,y)$ is as defined in Theorem 5.1.7. Then the problem (8.3.1)–(8.3.2) has at most one solution on $\mathbb{R}_+^2$.

Proof The problem (8.3.1)–(8.3.2) is equivalent to the integral equation (8.3.6). Let $u(x,y)$, $v(x,y)$ be two solutions of problem (8.3.1)–(8.3.2).

From (8.3.6), (8.3.8), it follows

$$|u(x,y)-v(x,y)| \le \int_x^{+\infty}\int_y^{+\infty} c(s,t)|u(s,t)-v(s,t)|dtds. \tag{8.3.8}$$

Now applying part (a_2) of Theorem 5.1.7 yields $u(x,y) = v(x,y)$, i.e., there is at most one solution to the problem (8.3.1)–(8.3.2). □

8.4 Applications of Theorem 5.1.12 and Corollaries 5.1.7–5.1.8 to Nonlinear Non-self-adjoint Vector Hyperbolic Partial Differential Equations

In this section, we shall use Theorem 5.1.12 and Corollaries 5.1.7–5.1.8 to study nonlinear, non-self-adjoint, vector hyperbolic partial differential equations.

Example 8.4.1 Let us discuss the uniqueness of the solution of the nonlinear, non-self-adjoint, vector hyperbolic partial differential equation

$$u_{xy} = \{a(x, y)u(x, y)\}_y + a(x, y)\Phi(x, y, u)$$

with the conditions prescribed on $x = x_0$, and $y = y_0$. suppose that $a(x, y)$, $\Phi(x, y, u)$ are continuous functions of their arguments, $a(x, y)$ is an $n \times n$ symmetric matrix, u and Φ are $n \times 1$ matrices, Φ satisfies a matrix Lipschitz condition, viz.,

$$|\Phi(x, y, u) - \Phi(x, y, u^*)| \le K|u - u^*|$$

for any two vectors u and u^*, where the absolute values are taken componentwise.

Let the boundary conditions be such that the given partial differential equation is equivalent to the vector Volterra integral equation given by

$$u(x, y) = g(x, y) + \int_{x_0}^{x} a(s, y)u(s, y)ds + \int_{x_0}^{x}\int_{y_0}^{y} a(s, t)\Phi(s, t, u)dsdt,$$

where $g(x, y)$ is a continuous vector function depending on boundary conditions. Then for any two solutions u and u^* of the integral equation, we have

$$u - u^* = \int_{x_0}^{x} a(s, y)\{u(s, y) - u^*(s, y)\}ds + \int_{x_0}^{x}\int_{y_0}^{y} a(s, t)\{\Phi(s, t, u) - \Phi(s, t, u^*)\}dsdt.$$

Now if $(x - x_0) \cdot (y - y_0) \ge 0$, we have

$$|u - u^*| \le K' \int_{x_0}^{x} |a| \cdot K \cdot |u - u^*|ds + \int_{x_0}^{x}\int_{y_0}^{y} |a| \cdot K \cdot |u - u^*|dsdt,$$

where $K'|a|K|u - u^*| = |a||u - u^*|$.

Now applying Corollary 5.1.8, we obtain $|u - u^*| \le 0$, componentwise, which implies $u = u^*$. Therefore there is at most one solution of the differential equation.

Example 8.4.2 Let us consider the vector characteristic initial value problem

$$u_{xy} - \{a(x, y)u(x, y)\}_y - a(x, y)u(x, y) = f(x, y),$$

where all the functions involved are continuous, and $a(x, y)$ is a non-negative matrix, and $u(x, y)$ is prescribed on $x = x_0, y = y_0$. This problem with the given conditions is equivalent to the vector Volterra integral equation

$$u(x, y) = h(x, y) + \int_{x_0}^{x} a(s, y)u(s, y)ds + \int_{x_0}^{x}\int_{y_0}^{y} a(s, t)u(s, t)dsdt,$$

where $h(x, y)$ is computed from $f(x, y)$ and the conditions at $x = x_0, y = y_0$.

Let the vectors $\bar{u}(x, y)$ and $\bar{\bar{u}}(x, y)$ satisfy

$$\bar{u}(x, y) \leq h(x, y) + \int_{x_0}^{x} a(s, y)\bar{u}(s, y)ds + \int_{x_0}^{x}\int_{y_0}^{y} a(s, t)\bar{u}(s, t)dsdt$$

and

$$\bar{\bar{u}}(x, y) \geq h(x, y) + \int_{x_0}^{x} a(s, y)\bar{\bar{u}}(s, y)ds + \int_{x_0}^{x}\int_{y_0}^{y} a(s, t)\bar{\bar{u}}(s, t)dsdt.$$

Now by (5.1.87) in Theorem 5.1.12 and Corollary 5.1.7, we find that for any solution vector u to the boundary value problem, we have

$$\bar{u} \leq u \leq \bar{\bar{u}}.$$

This is a componentwise comparison theorem for the solution vector.

Example 8.4.3 Let us consider the following pair of vector boundary value problems:

$$u_{xy} = \{a(x, y)\Phi(x, y, u)\}_y + a(x, y)u(x, y)$$

with

$$u(x_0, y) = g(y), \quad u(x, y_0) = h(x), \quad g(y_0) = h(x_0),$$
$$\Phi(x, y_0, h(x)) = f(x),$$

and

$$U_{xy} = \{a(x, y)\psi(x, y, U)\}_y + a(x, y)U(x, y)$$

with

$$U(x_0, y) = G(y), \quad U(x, y_0) = H(x), \quad G(y_0) = H(x_0),$$
$$\psi(x, y_0, H(x)) = F(x),$$

where all the functions involved are continuous and Φ satisfies the Lipschitz condition, by,

$$|\Phi(x, y, u) - \Phi(x, y, \bar{u})| \leq K \cdot |u - \bar{u}|$$

and K is the non-negative Lipschitz constant matrix for Φ for two vectors u and $\bar{u}$. Here from the equivalent vector integral equations and subtracting, it follows that

$$\begin{aligned} u - U = (g - G) + (h - H) - [g(y_0) - G(y_0)] - \int_{x_0}^{x}\int_{y_0}^{y} a(s, y_0)(f - F)ds \\ + \int_{x_0}^{x} a(s, y)[\Phi(s, y, u(s, y)) - \psi(s, y, U(s, y))]ds \\ + \int_{x_0}^{x}\int_{y_0}^{y} a(s, t)[u(s, t) - U(s, t)]dsdt. \end{aligned}$$

Adding and subtracting $\Phi(U)$ in the integrand and taking absolute values componentwise, we obtain, for $(x - x_0) \cdot (y - y_0) \geq 0$,

$$\begin{aligned} |u - U| \leq |g - G| + |h - H| + |g(y_0) - G(y_0)| + \int_{x_0}^{x} |a| \cdot |f - F|ds \\ + \int_{x_0}^{x} |a| \cdot |\Phi(u) - \Phi(U)|ds + \int_{x_0}^{x} |a| \cdot |\Phi(U) - \psi(U)|ds \\ + \int_{x_0}^{x}\int_{y_0}^{y} |a| \cdot |u - U|dsdt. \end{aligned}$$

Now if $|g - G| \leq \varepsilon$, $|h - H| \leq \varepsilon$, $|\Phi(s, t, U) - \psi(s, t, U)| \leq \varepsilon$, and $|f - F| \leq \varepsilon$, where ε is a non-negative vector, then

$$|u - U| \leq \varepsilon a(x, y) + K' \int_{x_0}^{x} A \cdot |(u - U)|ds + \int_{x_0}^{x}\int_{y_0}^{y} A \cdot |u - U|dsdt,$$

where

$$q(x, y) = 3 + 2A(x - x_0), \quad K'A|u - U| = AK|u - U|$$

and

$$A = \{|a_{ij}|\}.$$

Thus that by (5.1.87) in Theorem 5.1.12, we obtain

$$
\begin{aligned}
|u-U| \leq \varepsilon\Big[q(x,y) &+ \int_{x_0}^{x}\int_{y_0}^{y} V^T(s,t)Aq(s,t)dsdt + K\int_{x_0}^{x} A\cdot\{q(s,y) \\
&+ \int_{x_0}^{s}\int_{y_0}^{y} V^T(\theta,t)q(\theta,t)d\theta dt\}\cdot e^{K'\int_s^t A(\xi,y)d\xi}ds\Big] \\
&\leq [M(x,y)]\varepsilon,
\end{aligned}
$$

where M is a continuous matrix function and obviously bounded. If $\varepsilon \to 0$, then $u \to U$ in the domain. This means that the solution of the characteristic initial value problem depends continuously on the initial data.

8.5 Applications of Theorems 5.1.15, 5.1.18, 5.1.19 and 5.1.21 to Some Integral Inequalities in 2*D*

In this section, we shall use Theorems 5.1.15, 5.1.18, 5.1.19 and 5.1.21 to study some integral inequalities in 2D.

Example 8.5.1 Let

$$
\phi(x,y) \leq x+y+\frac{x+y}{4}\int_{x_0}^{x}\int_{y_0}^{y}\frac{\phi(x,y)}{s+t}dsdt. \tag{8.5.1}
$$

Let the domain D in Theorem 5.1.15 be conditioned in $x \geq 1, y \geq 1$; and let $\phi(x,y) \geq 0$ on D. Then, in the notation of Theorem 5.1.15,

$$
a(x,y) = x+y, \quad h(x,y) = x+y, \quad b(x,y) = 1/4(x+y).
$$

The Riemann-Green function for this problem is (Copson [149]),

$$
V(s,t,x,y) = I_0((x-s)(y-t)^{1/2}).
$$

The functions a, h, b, and ϕ in this example satisfy all the conditions of Theorems 5.1.15, 5.1.18, 5.1.19 and 5.1.21. Applications of these theorems to Eq. (8.5.1) yields the following estimates for ϕ, respectively,

$$
\begin{cases}
\phi(x,y) \leq x+y+(x+y)\int_{x_0}^{x}\int_{y_0}^{y} I_0((x-x_0)(y-t)^{1/2})dsdt, \\
\phi(x,y) \leq (x+y)^2/(x_0+y_0)I_0((x-x_0)(y-y_0)^{1/2})dsdt, \\
\phi(x,y) \leq (x+y)^2 I_0((x-x_0)(y-y_0)^{1/2}), \\
\phi(x,y) \leq (x+y)^2\exp\left((x-x_0)(y-y_0)\right).
\end{cases} \tag{8.5.2}
$$

Estimates (8.5.2) for ϕ may be compared. □

Example 8.5.2 Consider the pair of integral equations

$$\phi_1(x,y) = g_1(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} k_1(x,y,s,t,\phi_1(s,t))dsdt, \tag{8.5.3}$$

$$\phi_2(x,y) = g_2(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} k_2(x,y,s,t,\phi_2(s,t))dsdt. \tag{8.5.4}$$

Theorem 8.5.1 (Kasture-Deo [312]) *Let the solutions of problem (8.5.3) and (8.5.4) exist on a domain D. Let the kernel k_1 and k_2 satisfy on D the condition*

$$|k_1(x,y,s,t,\phi_1(s,t)) - k_2(x,y,s,t,\phi_2(s,t))| \le h(x,y)b(s,t)|\phi_1(s,t) - \phi_2(s,t)| \tag{8.5.5}$$

where $h(x,y) \ge 0$, and $b(s,t) \ge 0$ are continuous on D. Let $V(x,y,s,t)$ be the solution of the characteristic initial value problem defined by (5.1.104) and (5.1.64). Let D^+ be a sub-domain of D on which $V \ge 0$. If $(x_0,y_0),(x,y)$ with $x_0 \le x, y_0 \le y$ are points of D^+ such that

$$g_1(x,y) = g_2(x,y), \tag{8.5.6}$$

then

$$\phi_1(x,y) = \phi_2(x,y). \tag{8.5.7}$$

Proof From the integral equations (8.5.3) and (8.5.4),

$$\begin{aligned} |\phi_1(x,y) - \phi_2(x,y)| &\le |g_1(x,y) - g_2(x,y)| + \int_{x_0}^{x}\int_{y_0}^{y} |k_1 - k_2|dsdt \\ &\le \int_{x_0}^{x}\int_{y_0}^{y} h(x,y)b(s,t)|\phi_1(s,t) - \phi_2(s,t)|dsdt. \end{aligned} \tag{8.5.8}$$

Application of Theorem 5.1.15 to (8.5.8) implies (8.5.7). The proof is now complete. □

8.6 Applications of Theorem 5.1.25–5.1.30 to Some Integrodifferential Equations in 2*D*

In this section, we shall use Theorems 5.1.25–5.1.30 to investigate some integral and integrodifferential inequalities in 2*D* and prove the uniqueness and continuous dependence of the solutions of some nonlinear hyperbolic partial integrodifferential equations.

Example 8.6.1 (Uniqueness Test) As a first application, we discuss the uniqueness of solutions of the nonlinear hyperbolic partial integrodifferential equation of the form

$$u_{xy}(x, y) = f[x, y, u(x, y), \phi(x, y)], \tag{8.6.1}$$

in which

$$\phi(x, y) = \sigma(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} K(x, y, s, t, u(s, t))dsdt, \tag{8.6.2}$$

with the conditions prescribed on $x = x_0$ and $y = y_0$, where $\sigma, k,$ and f are continuous functions of their arguments and such that

$$| K(x, y, s, t, u(s, t)) - K(x, y, s, t, \bar{u}(s, t)) | \leq B | u(s, t) - \bar{u}(s, t) |, \tag{8.6.3}$$

and

$$\begin{aligned} &| f[x, y, u(x, y), \phi(x, y)] - f[x, y, \bar{u}(x, y), \bar{\phi}(x, y)] | \\ &\leq A[| u(x, y) - \bar{u}(x, y)| + |phi(x, y) - \bar{\phi}(x, y) |] \end{aligned} \tag{8.6.4}$$

for any two solutions $u(x, y)$ and $\bar{u}(x, y)$ of the given equation, where A and B are positive constants. Let the boundary conditions be such that the given equation (8.6.1)–(8.6.2) is equivalent to the Volterra integral equation given by

$$\begin{aligned} u(x, y) = g(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} & f[s, t, u(s, t), \sigma(s, t) \\ & + \int_{x_0}^{s} \int_{y_0}^{t} K(s, t, \xi, \eta, u(\xi, \eta))d\xi d\eta]dsdt, \end{aligned}$$

where $g(x, y)$ is continuous. Now if $u(x, y)$ and $\bar{u}(x, y)$ be two solutions of the given boundary value problem, then

$$\begin{aligned} u - \bar{u} = \int_{x_0}^{x} \int_{y_0}^{y} \Big\{ & f[s, t, u, \sigma + \int_{x_0}^{s} \int_{y_0}^{t} K(s, t, \xi, \eta, u)d\xi d\eta] \\ & - f[s, t, \bar{u}, \sigma + \int_{x_0}^{s} \int_{y_0}^{t} K(s, t, \xi, \eta, \bar{u})d\xi d\eta]dsdt \Big\}. \end{aligned} \tag{8.6.5}$$

Now, if $(x - x_0)(y - y_0) \geq 0$, then using (8.6.3) and (8.6.4) in (8.6.5), we have

$$|u - \bar{u}| \leq \int_{x_0}^{x} \int_{y_0}^{y} A|u - \bar{u}|dsdt + \int_{x_0}^{x} \int_{y_0}^{y} A(\int_{x_0}^{s} \int_{y_0}^{t} B|u - \bar{u}|d\xi d\eta)dsdt.$$

Now applying Theorem 5.1.25 when $a(x, y) = 0$ and $\sigma(x, y) = 0$ yields $|u-\bar{u}| \leq 0$. Therefore, $u = \bar{u}$; i.e., there is at most one solution of the problem.

Example 8.6.2 (Continuous Dependence Test) We consider continuous dependence of the solutions on the equation and boundary data of the boundary value problem

$$u_{xy}(x, y) = f[x, y, u(x, y), \phi(x, y)], \tag{8.6.6}$$

in which

$$\phi(x, y) = \sigma(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} K(x, y, s, t, u(s, t))dsdt, \tag{8.6.7}$$

with the given boundary conditions

$$u(x_0, y) = Ggy), \; u(x, y_0) = h(x), \; g(x_0) = h(x_0),$$

and

$$U_{xy}(x, y) = F[x, y, U(x, y), \Phi(x, y)], \tag{8.6.8}$$

in which

$$\Phi(x, y) = \rho(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} K(x, y, s, t, U(s, t))dsdt, \tag{8.6.9}$$

with the given boundary conditions

$$U(x_0, y) = G(y), \; U(x, y_0) = H(x), \; G(x_0) = H(x_0)$$

where all functions are continuous on their respective domains of their definitions and $|g - G| \leq \epsilon$, $|h - H| \leq \epsilon$, and

$$\begin{aligned} | f[s, t, U, \sigma + \int_{x_0}^{s} \int_{y_0}^{t} K(s, t, \xi, \eta, U)d\xi d\eta] \\ -F[s, t, U, \rho + \int_{x_0}^{s} \int_{y_0}^{t} K(s, t, \xi, \eta, U)d\xi d\eta] | \leq \epsilon, \end{aligned}$$

$$| K(s, t, \xi, \eta, u) - K(s, t, \xi, \eta, \bar{u})| \leq B|u - \bar{u}|,$$

and

$$|f[x, y, u, \phi] - f[x, y, u, \Phi] | \leq A[| u - \bar{u} | + | \phi - \Phi |],$$

where ϵ, A and B are positive constants. The equivalent integral equations of problem (8.6.6)–(8.6.7) and problem (8.6.8)–(8.6.9) are

$$u(x,y) = g(y) + h(x) - g(y_0) + \int_{x_0}^{x}\int_{y_0}^{y} f\Big[s,t,u(s,t),\sigma(s,t)$$
$$+ \int_{x_0}^{s}\int_{y_0}^{t} K(s,t,\xi,\eta,u(\xi,\eta))d\xi d\eta\Big]dsdt,$$

and

$$U(x,y) = G(y) + H(x) - G(y_0) + \int_{x_0}^{x}\int_{y_0}^{y} F\Big[s,t,U(s,t),\rho(s,t)$$
$$+ \int_{x_0}^{s}\int_{y_0}^{t} K(s,t,\xi,\eta,U(\xi,\eta))d\xi d\eta\Big]dsdt,$$

then

$$\begin{aligned} u - U &= (g-G) + (h-H) - [g(y_0) - G(y_0)] \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y}\Big\{ f\left[s,t,u,\sigma + \int_{x_0}^{s}\int_{y_0}^{t} K(s,t,\xi,\eta,u)d\xi d\eta\right] \\ &\quad - F\left[s,t,U,\rho + \int_{x_0}^{s}\int_{y_0}^{t} K(s,t,\xi,\eta,U)d\xi d\eta\right]\Big\} dsdt. \end{aligned}$$

By adding and subtracting $f[s,t,U,\sigma + \int_{x_0}^{s}\int_{y_0}^{t} K(s,t,\xi,\eta,U)d\xi d\eta]$ in the integrand, we obtain, if $(x-x_0)(y-y_0) \geq 0$,

$$\begin{aligned} |u-U| &\leq |g-G| + |h-H| - |g(y_0) - G(y_0)| \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} | f[s,t,u,\sigma + \int_{x_0}^{s}\int_{y_0}^{t} K(s,t,\xi,\eta,u)d\xi d\eta] \\ &\quad - \int_{x_0}^{x}\int_{y_0}^{y} f[s,t,U,\sigma + \int_{x_0}^{s}\int_{y_0}^{t} K(s,t,\xi,\eta,U)d\xi d\eta] \mid dsdt \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} | f[s,t,U,\sigma + \int_{x_0}^{s}\int_{y_0}^{t} K(s,t,\xi,\eta,U)d\xi d\eta] \\ &\quad - F[s,t,U,\rho + \int_{x_0}^{s}\int_{y_0}^{t} K(s,t,\xi,\eta,U)d\xi d\eta] \mid dsdt \\ &\leq \epsilon[3 + (x-x_0)(y-y_0)] + \int_{x_0}^{x}\int_{y_0}^{y} A|u-U|dsdt \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} A(\int_{x_0}^{s}\int_{y_0}^{t} B|u-U|d\xi d\eta)dsdt. \end{aligned}$$

By using Theorem 5.1.25 with $\sigma = 0$, we have

$$|u - U| \leq \epsilon\Big\{[3 + (x - x_0)(y - y_0)] + \int_{x_0}^{x}\int_{y_0}^{y}\Big[A[3 + (s - x_0)(t - y_0)] + \int_{x_0}^{s}\int_{y_0}^{t} A[3 + (\xi - x_0)(\eta - y_0)](A + B)v(\xi, \eta; s, t)d\xi d\eta\Big]\Big\}dsdt,$$

provided that $(x-x_0)(y-y_0) \geq 0$. On a compact set, the quantity in braces is bounded by some constant M^*. Therefore, $|u - U| \leq M^*\epsilon$ on this set; so the solution to such a boundary value problem depends continuously on f and the boundary values. If $\epsilon \to 0$, then $|u - U| \to 0$ on the set.

We note that Theorems 5.1.26 and 5.1.27 can be used to establish similar results as given in Examples 8.6.1 and 8.6.2 for nonlinear hyperbolic partial integro-differential equations of the forms

$$u_{xy} = f[x, y, u] + W[x, y, \int_{x_0}^{x}\int_{y_0}^{y} K(s, t, u)dsdt], \tag{8.6.10}$$

and

$$u_{xy} = f[x, y, u, \int_{x_0}^{x}\int_{y_0}^{y} K(s, t, u, \int_{x_0}^{s}\int_{y_0}^{t} e(\xi, \eta, u)d\xi d\eta)dsdt], \tag{8.6.11}$$

respectively, under some suitable condition on the functions involved in (8.6.10) and (8.6.11) and the prescribed boundary conditions. Further, we note that Theorems 5.1.28–5.1.30 can be used to study the behavior of solutions of hyperbolic integro-differential equations of the form

$$u_{xy} = f[x, y, u] + W[x, y, \int_{x_0}^{x}\int_{y_0}^{y} K(s, t, u, \int_{x_0}^{s}\int_{y_0}^{t} e(\xi, \eta, u)d\xi d\eta)dsdt], \tag{8.6.12}$$

$$u_{xy} = f(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} K(s, t, u, u_{st})dsdt, \tag{8.6.13}$$

$$u_{xy} = f(x, y) + \int_{x_0}^{x}\int_{y_0}^{y} K(s, t, u, u_{st})dsdt + \int_{x_0}^{x}\int_{y_0}^{y} w\left[s, t, \int_{x_0}^{s}\int_{y_0}^{t} e(\xi, \eta, u, u_{st})d\xi d\eta\right] dsdt, \tag{8.6.14}$$

respectively, under some suitable conditions.

We also note that the inequalities established in (5.1.177) and (5.1.178) can be used to establish similar results as given in Examples 8.6.1 and 8.6.2 for the following class of nonlinear self-adjoint hyperbolic partial integro-differential

equations of the forms

$$u_{xy} = \{a(x,y)u(x,y)\}_y + a(x,y)f[x,y,u] \\ +W[x,y,\int_{x_0}^{x}\int_{y_0}^{y} a(s,t)K(s,t,u)dsdt], \tag{8.6.15}$$

and

$$u_{xy} = \{b(x,y)u(x,y)\}_x + b(x,y)f[x,y,u] \\ +W[x,y,\int_{x_0}^{x}\int_{y_0}^{y} b(s,t)K(s,t,u)dsdt], \tag{8.6.16}$$

under some suitable conditions.

8.7 An Application of Theorem 5.1.33 to Some Nonlinear Hyperbolic Partial Integrodifferential Equations

In this section, we shall use Theorem 5.1.33 to study the boundedness and uniqueness of the solutions of some nonlinear hyperbolic partial integrodifferential equations. These applications are not stated as theorems so as to obscure the main ideas with technique details.

Example 8.7.1 As a first application, we obtain the bound on the solution of a nonlinear hyperbolic partial integrodifferential equation

$$u_{xy}(x,y) = f(x,y,u(x,y) + h\left[x,y,u(x,y),\int_{x_0}^{x}\int_{y_0}^{y} k(x,y;s,t,u(s,t))dsdt\right], \tag{8.7.1}$$

with the given boundary conditions

$$u(x,y_0) = a_1(x), \quad u(x_0,y) = a_2(y), \quad a_1(x_0) = a_2(y_0) = 0,$$

where all the functions are real-valued, continuous and defined on a domain D and such that

$$\begin{cases} |f(x,y,u)| \le c(x,y)|u|, \\ |k(x,y,s,t,u)| \le q(s,t)|u|, \\ |h[x,y,u,v]| \le p(x,y)(|u|+|v|), \end{cases} \tag{8.7.2}$$

where $c(x, y)$ $p(x, y)$ and $q(x, y)$ are as in (H_1). Equation (8.7.1) is equivalent to the Volterra integral equation

$$u(x, y) = a_1(x) + a_2(y) + \int_{x_0}^{x} \int_{y_0}^{y} f(s, t, u(s, t))dsdt$$
$$+ \int_{x_0}^{x} \int_{y_0}^{y} h\left[s, t, u(s, t), \int_{x_0}^{s} \int_{y_0}^{t} k(s, t, \xi, \eta, u(\xi, \eta))d\xi d\eta\right] dsdt, \tag{8.7.3}$$

where $u(x, y)$ is any solution of (8.7.1). Using (8.7.2) in (8.7.3) and assuming that $|a_1(x)| + |a_2(y)| \leq a(x, y)$, where $a(x, y)$ is as defined in (H_1), we have

$$|u(x, y)| \leq a(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} c(s, t)|u(s, t)|dsdt + \int_{x_0}^{x} \int_{y_0}^{y} p(s, t)\Big(|u(s, t)|$$
$$+ \int_{x_0}^{s} \int_{y_0}^{t} q(\xi, \eta)|u(\xi, \eta)|d\xi d\eta\Big)dsdt. \tag{8.7.4}$$

Now applying Theorem 5.1.33 with $b(x, y) = 1$ yields

$$|u(x, y)| \leq a(x, y) + \int_{x_0}^{x} \int_{y_0}^{y} w(s, t; x, y)\Big\{a(s, t)[c(s, t) + p(s, t)]$$
$$+p(s, t) \int_{x_0}^{s} \int_{y_0}^{t} a(\xi, \eta)[c(\xi, \eta) + p(\xi, n) + q(\xi, \eta)]$$
$$\times v(\xi, \eta; s, t)d\xi d\eta\Big\} dsdt, \tag{8.7.5}$$

where $v(s, t; x, y)$ and $w(s, t; x, y)$ are the solutions of the characteristic initial value problem (5.1.208) and (5.1.209) respectively with $b(x, y) = 1$. Thus the right-hand side in (8.7.5) gives us the bound on the solution $u(x, y)$ of Eq. (8.7.1) in terms of the known functions.

If $|a_1(x)| + |a_2(y)| \leq \epsilon$, where $\epsilon > 0$ is arbitrary, then the bound obtained in (8.7.5) reduces to

$$|u(x, y)| \leq \epsilon\Big\{1 + \int_{x_0}^{x} \int_{y_0}^{y} w(s, t; x, y)\Big[[c(s, t) + p(s, t)] + p(s, t)$$
$$\times \int_{x_0}^{s} \int_{y_0}^{t} [c(\xi, \eta) + p(\xi, \eta) + q(\xi, \eta)]v(\xi, \eta; s, t)d\xi d\eta\Big]dsdt\Big\}. \tag{8.7.6}$$

□

In this case we note that, Example 8.7.1 not only the boundedness but the stability of solution $u(x, y)$ of (8.7.1), if the bound obtained on the right-hand side in (8.7.6) is small enough.

Example 8.7.2 We discuss the uniqueness of the solution of the nonlinear hyperbolic partial integrodifferential equation (8.7.1). We assume that the functions f, k and h in Eq. (8.7.1) satisfy

$$\begin{cases} |f(x, y, u) - f(x, y, \bar{u})| \leq c(x, y)|u - \bar{u}|, \\ |k(x, y, s, t, u) - k(x, y, s, t, \bar{u})| \leq q(s, t)|u - \bar{u}|, \\ |h[x, y, u, r] - h[x, y, u, \bar{r}]| \leq p(x, y)[|u - \bar{u}| + |r - \bar{r}|], \end{cases} \tag{8.7.7}$$

where $c(x, y)$ $p(x, y)$ and $q(x, y)$ are as in (H_1). Equation (8.7.1) is equivalent to the Volterra integral equation (8.7.3). Now if $u(x, y)$ and $\bar{u}(x, y)$ be two solutions of the given boundary value problem (8.7.1) with the same boundary conditions, then we have

$$\begin{aligned} u - \bar{u} = & \int_{x_0}^{x} \int_{y_0}^{y} (f(s, t, u) - f(s, t, \bar{u}))\, dsdt \\ & + \int_{x_0}^{x} \int_{y_0}^{y} \Big\{ h\Big[s, t, u, \int_{x_0}^{s} \int_{y_0}^{t} k(s, t, \xi, \eta, u) d\xi d\eta\Big] \\ & - h\Big[s, t, \bar{u}, \int_{x_0}^{s} \int_{y_0}^{t} k(s, t, \xi, \eta, \bar{u}) d\xi d\eta\Big]\Big\} dsdt. \end{aligned} \tag{8.7.8}$$

Using (8.7.7) in (8.7.8), we have

$$\begin{aligned} |u - \bar{u}| \leq & \int_{x_0}^{x} \int_{y_0}^{y} c(s, t)|u - \bar{u}| dsdt \\ & + \int_{x_0}^{x} \int_{y_0}^{y} p(s, t) \left(|u - \bar{u}| + \int_{x_0}^{s} \int_{y_0}^{t} q(\xi, \eta)|u - \bar{u}| d\xi d\eta \right) dsdt. \end{aligned}$$

Now applying Theorem 5.1.33 yields, $|u - \bar{u}| \leq 0$. Therefore $u = \bar{u}$; i.e., there is at most one solution of the problem. □

8.8 An Application of Theorem 5.1.34 to the Nonlinear Non-self-adjoint Hyperbolic Partial Differential and Integrodifferential Equations

In this section, we shall Theorem 5.1.34 to study the behavioral relationships between the solutions of the nonlinear non-self-adjoint hyperbolic partial differential and integrodifferential equations.

Consider the nonlinear non-self-adjoint hyperbolic partial differential equation

$$u_{xy}(x, y) = \{c_0(x, y)u(x, y)\}_y + c_0(x, y)F(x, y, u(x, y)), \tag{8.8.1}$$

with the given boundary conditions

$$u(x_0, y) = g_0(y), \quad u(x, y_0) = h_0(x), \quad g_0(y_0) = h_0(x_0)$$

and the nonlinear non-self-adjoint hyperbolic partial integrodifferential equation

$$\begin{aligned} z_{xy}(x, y) = {} & [c_0(x, y)z(x, y)]_y + c_0(x, y)F(x, y, z(x, y)) \\ & + H\left[x, y, \int_{x_0}^{x}\int_{y_0}^{y} k(x, y, s, t, z(s, t))dsdt\right], \end{aligned} \tag{8.8.2}$$

with the given boundary conditions

$$z(x_0, y) = g_1(y), \quad z(x, y_0) = h_1(x), \quad g_1(y_0) = h_1(x_0),$$

where all the functions are real-valued, continuous, and defined on a domain D and are such that

$$\begin{cases} | c_0(x, y) | \le c(x, y), & (8.8.3) \\ | F(x, y, z) - F(x, y, u) | \le M_0 \mid z - u \mid, & (8.8.4) \\ | K(x, y, s, t, z) | \le M_0 c(s, t) \mid z \mid, & (8.8.5) \\ | H(x, y, \bar{z}) | \le g(x, y) \mid \bar{z} \mid, & (8.8.6) \\ | \lambda(x, y) | \le \epsilon, & (8.8.7) \end{cases}$$

where

$$\begin{aligned} \lambda(x, y) = {} & g_1(y) - g_0(y) + h_1(x) - h_0(x) - [g_1(y_0) - g_0(y_0)] \\ & - \int_{x_0}^{x} c(s, y_0)[h_1(s) - h_0(s)]ds, \end{aligned}$$

and the functions $c(x, y)$ and $g(x, y)$ are as defined in (H_1) and M_0 and ϵ are positive constants. Equations (8.8.1) and (8.8.2) are equivalent to the integral equations

$$\begin{aligned} u(x, y) = {} & g_0(y) + h_0(x) - g_0(y_0) + \int_{x_0}^{x} c_0(s, y)u(s, y)ds - \int_{x_0}^{x} c_0(s, y_0)u(s, y_0)ds \\ & + \int_{x_0}^{x}\int_{y_0}^{y} c_0(s, t)F(s, t, u(s, t))dsdt, \end{aligned} \tag{8.8.8}$$

and

$$\begin{aligned}z(x,y) &= g_1(y)+h_1(x)-g_1(y_0)+\int_{x_0}^{x} c_0(s,y)u(s,y)ds \\ &\quad -\int_{x_0}^{x} c_0(s,y_0)u(s,y_0)ds+\int_{x_0}^{x}\int_{y_0}^{y} c_0(s,t)F(s,t,u(s,t))dsdt \\ &\quad +\int_{x_0}^{x}\int_{y_0}^{y} H\left[s,t,\int_{x_0}^{s}\int_{y_0}^{t} K(s,t,\xi,\eta,z(\xi,\eta))d\xi d\eta\right] dsdt, \qquad (8.8.9)\end{aligned}$$

respectively. From (8.8.8) and (8.8.9), we derive

$$\begin{aligned}z(x,y)-u(x,y) &= \lambda(x,y)+\int_{x_0}^{x} c_0(s,y)[z(s,y)-u(s,y)]ds \\ &\quad +\int_{x_0}^{x}\int_{y_0}^{y} c_0(s,t)[F(s,t,z(s,t))-F(s,t,u(s,t))]dsdt \\ &\quad +\int_{x_0}^{x}\int_{y_0}^{y} H[s,t,\int_{x_0}^{s}\int_{y_0}^{t} k(s,t,\xi,\eta,z(\xi,\eta))d\xi\ d\eta]dsdt.\end{aligned} \tag{8.8.10}$$

Using (8.8.3)–(8.8.7) and $\mid z \mid - \mid u \mid \leq \mid z-u \mid$ in (8.8.10) and assuming that the solution $u(x,y)$ of Eq. (8.8.1) is bounded by N_0, where $N_0>0$ is a constant, we have

$$\begin{aligned}\mid z(x,y)-u(x,y)\mid &\leq a(x,y)+\int_{x_0}^{x} c(s,y)\mid z(s,y)-u(s,y)\mid ds \\ &\quad +M_0\int_{x_0}^{x}\int_{y_0}^{y} c(s,t)\mid z(s,t)-u(s,t)\mid dsdt \qquad (8.8.11) \\ &\quad +M_0\int_{x_0}^{x}\int_{y_0}^{y} g(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} c(\xi,\eta)\mid z(\xi,\eta)-u(\xi,\eta)\mid d\xi\ d\eta\right) dsdt,\end{aligned}$$

where

$$a(x,y)=\epsilon+M_0\int_{x_0}^{x}\int_{y_0}^{y} g(s,t)\left(\int_{x_0}^{s}\int_{y_0}^{t} N_0 c(\xi,\eta)d\xi\ d\eta\right) dsdt.$$

Now applying Theorem 5.1.34 yields

$$\mid z(x,y)-u(x,y)\mid \leq f_2(x,y)+\int_{x_0}^{x} c(s,y)f_2(s,y)\exp\left(\int_{s}^{x} c(\xi,y)d\xi\right) ds \tag{8.8.12}$$

where

$$f_2(x,y)=a(x,y)+M_0Q_2(x,y)+M_0\int_{x_0}^{x}\int_{y_0}^{y} g(s,t)Q_2(s,t)dsdt,$$

in which

$$Q_2(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} W(s,t;x,y)c(s,t) \times \left\{a(s,t) + M_0\int_{x_0}^{s}\int_{y_0}^{t} v(\xi,\eta;s,t)a(\xi,\eta)d\xi\, d\eta\right\} dsdt,$$

where $V(s,t;x,y)$ and $W(s,t;x,y)$ are as in Theorem 5.1.34 with suitable changes in the values of p,q,b,r and h. If the right-hand side of (8.8.12) is bounded, then we obtain the relative boundedness of the solutions $u(x,y)$ and $z(x,y)$ of (8.8.1) and (8.8.2). □

If $a(x,y)$ defined in (8.8.11) is small enough and, say, less than ϵ_0, where $\epsilon_0 > 0$ is arbitrary, then we infer from (8.8.12) that

$$\begin{aligned}| z(x,y) - u(x,y) | \le \epsilon_0\Big\{1 + M_0Q_3(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} g(s,t)Q_3(s,t)dsdt \\ + \int_{x_0}^{x} c(s,y)\exp\left(\int_{s}^{x} c(\xi,y)d\xi\right)\Big[1 + M_0Q_3(s,y) \\ +M_0\int_{x_0}^{s}\int_{y_0}^{y} g(\xi,t)Q_3(\xi,t)d\xi dt\Big]ds\Big\}.\end{aligned} \tag{8.8.13}$$

If in (8.8.13) the expression in bracks is bounded and $\epsilon_0 \to 0$, then we obtain $| z(x,y) - u(x,y) | \to 0$, which gives the equivalence between the solutions of (8.8.1) and (8.8.2). □

We note that Theorem 5.1.34 can be used to study the stability, boundedness, and continuous dependence of the solutions of (8.8.1) and (8.8.2) by following arguments similar to those in [227, 477, 603] with suitable modifications. Further we note that the integral inequality established in Theorem 5.1.34 can be used to study the similar problems for nonlinear non-self-adjoint partial differential and integrodifferential equations of the form

$$u_{xy}(x,y) = \Big(b_0(x,y)u(x,y)\Big)_x + b_0(x,y)F(x,y,u(x,y)), \tag{8.8.14}$$

and

$$\begin{aligned}z_{xy}(x,y) = \Big(b_0(x,y)z(x,y)\Big)_x + b_0(x,y)F(x,y,z(x,y)) \\ +H[x,y,\int_{x_0}^{x}\int_{y_0}^{y} k(x,y,s,t,z(s,t))dsdt]\end{aligned} \tag{8.8.15}$$

with the given boundary conditions and some suitable conditions on the functions involved in (8.8.14) and (8.8.15).

8.9 An Application of Theorem 5.1.35 to the Nonlinear Volterra Equations

In this section, we shall use Theorem 5.1.35 to study the nonlinear Volterra equations.

Let us consider the nonlinear Volterra equation

$$u(x,y) = f(x,y) + \int_0^x \int_0^y k(x,y,s,t)[u(s,t) + H(s,t,u(s,t))]dsdt, \quad x,y \geq 0, \tag{8.9.1}$$

where $f = f(x,y)$ is continuous for $x,\ y \geq 0$, $k = k(x,y,s,t)$ is continuous for $x \geq s \geq 0$, $y \geq t \geq 0$, and $H = H(x,y,u)$ is continuous in $D = \{(x,y,u) : x,\ y \geq 0,\ |u| < +\infty\}$. Suppose that for all (x,y,u) and $(x,y,\overline{u}) \in D$, we have

$$|H(x,y,u) - H(x,y,\overline{u})| \leq h_1(x,y)|u - \overline{u}|, \tag{8.9.2}$$

where $h_1 = h_1(x,y)$ is continuous for all $x,\ y \geq 0$. Assume that there exists a continuous positive function $g = g(x,y)$ such that

$$R = \sup_{x,y\geq 0}\left[\int_0^x \int_0^y |r(x,y,s,t)|(g(s,t)/g(x,y))dsdt\right] < +\infty, \tag{8.9.3}$$

and r is the resolvent kernel given by (5.1.235) in Theorem 5.1.35, and that

$$|r(x,y,s,t)| \leq M(g(x,y)/g(s,t)), \text{ for } x \geq s \geq 0,\ y \geq t \geq 0,\ M = constant \tag{8.9.4}$$

Finally, suppose that

$$\mathfrak{K} = \sup_{x,y\geq 0} \int_0^x \int_0^y a(s,t)\exp\left(\int_0^s \int_0^t a(\xi,\eta)d\xi d\eta\right)dsdt < +\infty, \tag{8.9.5}$$

where $h(x,y) = Mh_1(x,y)$ for $x,y \geq 0$. If we denote by C_g the Banach space consisting of all continuous (for all $x,y \geq 0$) functions $f = f(x,y)$, with the norm

$$\|f\|_g = \sup_{x,y\geq 0}(|f(x,y)/g(x,y)|) < +\infty, \tag{8.9.6}$$

then we can state the following stability result.

Theorem 8.9.1 (Corduneanu [152]) *Assume that the cited above conditions (8.9.2)–(8.9.5) are fulfilled. If $u_i = u_i(x,y)$ $(i = 1,2)$ are two solutions of Eq. (8.9.1) corresponding to the free terms $f_i = f_i(x,y)$ $(i = 1,2)$, then it follows*

that

$$\|u_1 - u_2\|_g \leq (1+R)(1+\mathfrak{K})\|f_1 - f_2\|_g. \tag{8.9.7}$$

Proof The existence and the uniqueness of the solution to Eq. (8.9.1) are guaranteed and we shall use the representation formula

$$u_i(x,y) = v_i(x,y) + \int_0^x \int_0^y r(x,y,s,t)H(s,t,u_i(s,t))dsdt, \quad i=1,2, \tag{8.9.8}$$

where

$$v_i(x,y) = f_i(x,y) + \int_0^x \int_0^y r(x,y,s,t)f_i(s,t)dsdt, \quad i=1,2. \tag{8.9.9}$$

From (8.9.9), we deduce that

$$\|v_1 - v_2\|_g \leq (1+R)\|f_1 - f_2\|_g. \tag{8.9.10}$$

Using the remaining conditions, and the result established in the Theorem 5.1.35, form (8.9.8), we obtain (8.9.7). □

8.10 Applications of Theorems 5.1.41–5.1.42 to Nonlinear Hyperbolic Partial Differential Equations

In this section, we shall employ Theorems 5.1.41–5.1.42 to study nonlinear hyperbolic partial differential equations.

Example 8.10.1 We shall consider the lower bound on the solution of a nonlinear hyperbolic partial differential equation of the form

$$u_{xy}(x,y) = F[x,y,u(x,y)] \tag{8.10.1}$$

with the given boundary condition

$$u(x,t) = u(s,y) = u(s,t)$$

where functions u and F are real-valued, defined, and continuous on the respective domains of their definitions and

$$|\ F[x,y,u(x,y)]\ | \leq b(x,y)W(|u(x,y)|), \tag{8.10.2}$$

where b and W are defined in Theorem 5.1.41.

Integrating (8.10.1) first with respect to y from y to t, and then with respect to x from x to s, we have

$$u(x,y) = u(s,t) + \int_x^s \int_y^t F[m,n,u(m,n)]dmdn. \tag{8.10.3}$$

Using (8.10.2) and (8.10.3), we have

$$|u(x,y)| \le |u(s,t)| + \int_x^s \int_y^t b(m,n)W(|u(m,n)|)dmdn,$$

i.e.,

$$|u(s,t)| \ge |u(x,y)| - \int_x^s \int_y^t b(m,n)W(|u(m,n)|)dmdn.$$

Now applying Theorem 5.1.41 yields

$$|u(s,t)| \ge \Omega^{-1}\left[\Omega(|u(x,y)|) - \int_x^s \int_y^t b(m,n)dmdn\right], \tag{8.10.4}$$

where Ω and Ω^{-1} are as defined in Theorem 5.1.41. Thus the right-hand side of (8.10.4) gives us the lower bound on the solution $u(s,t)$ of Eq. (8.10.1). □

Example 8.10.2 We establish the lower bound on the solution of a nonlinear hyperbolic partial integrodifferential equation of the form

$$u_{xy}(x,y) = F\left[x,\ y,\ u(x,y), \int_x^s \int_y^t k(x,\ y,\ m,\ n\ u(m,n))\ dm\ dn\right], \tag{8.10.5}$$

with the given boundary conditions $u(x,t) = u(s,y) = u(s,t)$, where u, k and F are real-valued continuous functions defined on the respective domains of their definitions and the functions k and F involved in (8.10.5) satisfy

$$|\, k(x,\ y,\ m,\ n,\ u(m,n))\,| \le c(m,\ n)\,|\, u(m,n)\,|, \tag{8.10.6}$$

$$|\, F[x,y,u(x,y),v]\,| \le b(x,y)[|\, u(x,y)\,| + |\, v\,|], \tag{8.10.7}$$

where b and c are as defined in Theorem 5.1.42. Integrating (8.10.5) as in Example 8.10.1, we have

$$u(x,y) = u(s,t) + \int_x^s \int_y^t F\left[m,n,u(m,n), \int_m^s \int_n^t k(m,n,\xi,\zeta,u(\xi,\zeta))\ d\xi\ d\zeta\right] dm\ dn. \tag{8.10.8}$$

Using (8.10.6) and (8.10.7) in (8.10.8), we have

$$| u(x, y) | \leq | u(s, t) | + \int_x^s \int_y^t b(m, n) | u(m, n) | \, dm \, dn + \int_x^s \int_y^t b(m, n) \left[\int_m^s \int_n^t c(\xi, \zeta) | u(\xi, \zeta) | \, d\xi \, d\zeta \right] dm \, dn,$$

i.e.,

$$| u(s, t) | \geq | u(x, y) | - \Big[\int_x^s \int_y^t b(m, n) | u(m, n) | \, dm \, dn + \int_x^s \int_y^t b(m, n) [\int_m^s \int_n^t c(\xi, \zeta) | u(\xi, \zeta) | \, d\xi \, d\zeta \Big] dm \, dn. \tag{8.10.9}$$

Now a suitable application of Theorem 5.1.42 yields

$$| u(s, t) | \geq | u(x, y) | \left[1 + \int_x^s \int_y^t b(m, n) \times \exp \left(\int_m^s \int_n^t [b(\xi, \zeta) + c(\xi, \zeta)] \, d\xi \, d\zeta \right) dm \, dn \right]^{-1},$$

which gives us the lower bound on the solution $u(s, t)$ of Eq. (8.10.5). □

8.11 Applications of Theorem 5.2.2 and Corollary 5.2.2 to Nonlinear Integral Equation of the Volterra-Fredholm Type

In this section, we shall use Theorem 5.2.2 and Corollary 5.2.2 to study nonlinear integral equation of the Volterra-Fredholm type. We present some applications of Theorems to study the boundedness, stability, and uniqueness of the solutions of certain integral equations, their systems, and initial boundary problems for parabolic partial differential equations.

Example 8.11.1 Consider the following nonlinear integral equation of the Volterra-Fredholm type:

$$u(x, t) = f(x, t) + \int_0^t \int_a^b K[x, t, y, s, u(y, s)] dy ds, \tag{8.11.1}$$

with assumptions:

(1) f and K are continuous in D and

$$\Theta = \{(x, t, y, s, u) : a \le x, y \le b, 0 \le s \le t < +\infty, |u| < +\infty\},$$

(2) $|K[x, t, y, s, u]| \le B(y, s)|u|$ in Ω,
(3) $|K[x_1, t_1, y_1, s_1, u_1] - K[x_2, t_2, y_2, s_2, u_2]| \le B(y, s)|u_1 - u_2|$ in Ω, where B is continuous and integrable in D.

Notice that from (8.11.1), we get the inequality

$$|u(x, t)| \le |f(x, t)| + \int_0^t \int_a^b B(y, s)|u(y, s)|dyds. \tag{8.11.2}$$

Applying Remark 5.2.1, we have

$$|u(x, t)| \le \Psi(t) \int_0^t \int_a^b B(y, s)dyds, \tag{8.11.3}$$

where

$$\Psi(t) = \sup\Big\{|f(x, t)| : a \le x \le b, 0 \le s \le t\Big\}.$$

In this way, the following result holds.

Theorem 8.11.1 (Hacia [247]) *If assumptions (1) and (2) are satisfied and $\Psi(t)$ is bounded in $I = [0, +\infty)$, then a solution of Eq. (8.11.1) is bounded in D.*

Furthermore, we can prove the stability and uniqueness of solutions to Eq. (8.11.1).

Theorem 8.11.2 (Hacia [247]) *If assumptions (1) and (3) are satisfied, then (8.11.1) has at most one solution, which is stable.*

Proof Let u_1 and u_2 be the solutions of Eq. (8.11.1) corresponding to free terms f_1,f_2, respectively, such that $|f_1(x, t) - f_2(x, t)| < \varepsilon$ for arbitrary $\varepsilon > 0$.

Then, applying assumption (3) of Example 8.11.1 to (8.11.1), we get

$$\begin{aligned}|u_1(x, t) - u_2(x, t)| &\le |f_1(x, t) - f_2(x, t)| + \int_0^t \int_a^b B(y, s)|u_1(y, s) - u_2(y, s)|dyds \\ &\le \varepsilon + \int_0^t \int_a^b B(y, s)|u_1(y, s) - u_2(y, s)|dyds.\end{aligned}$$

Using Corollary 5.2.2, we obtain the inequality

$$|u_1(x, t) - u_2(x, t)| \le \varepsilon \exp\left[\int_0^t \int_a^b B(y, s)dyds\right],$$

which gives us the stability result.

The uniqueness of solutions of Eq. (8.11.2) is proved, because if $f_1(x,t) = f_2(x,t)$, then

$$|u_1(x,t) - u_2(x,t)| \le 0,$$

i.e.,

$$u_1(x,t) = u_2(x,t), \text{ in } D.$$

□

Example 8.11.2 Now consider the following system of integral equation of Volterra-Fredholm type

$$u_i(x,t) = f_i(x,t) + \sum_{j=1}^{+\infty} \int_0^t \int_a^b k_{ij}(x,t,y,s)u_j(y,s)dyds, \tag{8.11.4}$$

where $f_i, i = 1,2,\ldots,m$ and $k_{ij}, i,j = 1,2,\ldots,m$ are continuous in D and Ω, respectively.

Introducing the following notation

$$\sum_{i=1}^{m} |u_i(x,t)| = u(x,t), \ \sum_{i=1}^{m} |f_i(x,t)| = f(x,t),$$

$$\sum_{i=1}^{m} \max_{1\le j\le m} |k_{ij}(x,t,y,s)| = B(y,s) \text{ in } \Omega,$$

we get

$$u(x,t) \le f(x,t) + \int_0^t \int_a^b B(y,s)u(y,s)dyds. \tag{8.11.5}$$

By virtue of Remark 5.2.1, we obtain

$$u(x,t) \le \Phi(t) \ \exp[\int_0^t \int_a^b B(y,s)dyds], \tag{8.11.6}$$

where

$$\Phi(t) = sup\{|f(x,s)| : a \le x \le b, 0 \le s \le t\}.$$

From the above arguments, the bounds of solutions of system (8.11.4) follow.

Theorem 8.11.3 (Hacia [247]) *Let $f_i, i = 1, 2, \ldots, m$, be continuous in D and $k_{ij}, i, j = 1, 2, \ldots, m$, be continuous in Ω, such that*

$$\sum_{j=1}^{m} \max_{1 \le j \le m} |k_{ij}(x, t, y, s)| \le B(y, s),$$

where B is continuous and integrable in D.

If $\Phi(t)$ is bounded in $I = [0, +\infty)$, then a solution $\{u_i(x, t)\}, i = 1, 2, \ldots, m$, of system (8.11.4) is bounded in D and an estimate is defined by (8.11.6).

Remark 8.11.1 If f is bounded in D, i.e., ($|f(x, t)| \le C$), then the bounded solution of system (8.11.4) is estimated by the inequality

$$\sum_{i=1}^{m} |u_i(x, t)| \le C \exp[\int_0^t \int_a^b B(y, s) dyds].$$

Theorem 8.11.4 (Hacia [247]) *If the assumptions of Theorem 8.11.3 are satisfied, then system (8.11.4) has at most one solution, which is stable.*

Proof It follows from the inequality

$$|u(x, t) - u^*(x, t)| \le \varepsilon + \int_0^t \int_a^b |u(y, s) - u^*(y, s)| dyds,$$

that

$$|u(x, t) - u^*(x, t)| \le \varepsilon \exp[\int_0^t \int_a^b B(x, t) dyds]$$

if

$$|u(x, t) - u^*(x, t)| < \varepsilon.$$

□

Example 8.11.3 Some initial-boundary-value problems for partial differential equations of the parabolic type (Fourier problems) reduces to the Volterra-Fredholm integral equation

$$u(x, t) = f(x, t) + \int_0^t \int_G k(x, t, y, s) u(y, s) dyds, \tag{8.11.7}$$

where G is a compact subset of $\mathbb{R}^n$ and f depends on the given initial and boundary conditions.

Theorem 8.11.5 (Hacia [247]) *If f and k are continuous in $G \times I$ and $(G \times I)^2$, respectively, such that*

$$|k(x,t,y,s)| \le B(y,s),$$

where B is continuous and integrable in $G \times I$, then a solution of Eq. (8.11.7) is stable.

Moreover, if f is bounded, then the solution is bounded, too.

Proof It is clear that for $|f_1(x,t) - f_2(x,t)| < \varepsilon$,

$$|u_1(x,t) - u_2(x,t)| \le \varepsilon + \int_0^t \int_G B(y,s)|u_1(y,s) - u_2(y,s)|dyds.$$

Using Corollary 5.2.2, we obtain

$$|u_1(x,t) - u_2(x,t)| \le \varepsilon \exp\Big(\int_0^t \int_G B(y,s)dyds\Big),$$

which proves the stability of the solution of Eq. (8.11.7).

The boundedness of the solution of Eq. (8.11.7) follows from the inequality

$$|u(x,t)| \le |f(x,t)| + \int_0^t \int_G B(y,s)|u(y,s)|dyds,$$

which implies

$$|u(x,t)| \le C \exp\Big(\int_0^t \int_G B(y,s)dyds\Big),$$

because f is bounded, i.e., $|f(x,t)| \le C$.

8.12 Applications of Theorem 5.3.1 to Hyperbolic Partial Delay Differential Equations

In this section, we present applications of Theorem 5.3.1 to study the boundedness, uniqueness, and continuous dependence of the solutions of the initial boundary value problem for hyperbolic partial delay differential equations of the form

$$D_2D_1z(x,y) = f(x,y,z(x,y),z(x-h_1(x),y-h_2(y))), \tag{8.12.1}$$

$$z(x,y_0) = a_1(x), \qquad z(x_0,y) = a_2(y), \qquad a_1(x_0) = a_2(y_0) = 0, \tag{8.12.2}$$

where $f \in C(\Delta \times \mathbb{R}^2, \mathbb{R}), a_1 \in C^1(J_1, \mathbb{R}), a_2 \in C^1(J_1, \mathbb{R}), h_1 \in C^1(J_1, \mathbb{R}_+), h_2 \in C^1(J_2, \mathbb{R}_+)$ such that $x - h_1(x) \geq 0, y - h_2(y) \geq 0, h_1'(x) < 1, h_2'(y) < 1$, and $h_1(x_0) = h_2(y_0) = 0$.

The first result gives us the bound on the solution of the problem (8.12.1)–(8.12.2). The notation used here is the same as in Theorem 5.3.1.

Theorem 8.12.1 (Pachpatte [501]) *Suppose that*

$$|f(x, y, u, v)| \leq a(x, y)|u| + b(x, y)|v| \tag{8.12.3}$$

and

$$|a_1(x) + a_2(y)| \leq k, \tag{8.12.4}$$

where $a, b \in C(\Delta, \mathbb{R}_+)$ and $k \geq 0$ is a constant, and let

$$M_1 = \max_{x \in J_1} \frac{1}{1 - h_1'(x)}, \qquad M_2 = \max_{y \in J_2} \frac{1}{1 - h_2'(y)}. \tag{8.12.5}$$

If $z(x, y)$ is any solution of problem (8.12.1)–(8.12.2), then

$$|z(x, y)| \leq k \exp\left(A(x, y) + \bar{B}(x, y)\right), \tag{8.12.6}$$

where $A(x, y)$ is defined by (5.3.3) in Theorem 5.3.1 and

$$\bar{B}(x, y) = M_1 M_2 \int_{\phi(x_0)}^{\phi(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{b}(\sigma, \tau) d\tau d\sigma, \tag{8.12.7}$$

in which $\phi(x) = x - h_1(x), x \in J_1, \psi(y) = y - h_2(y), y \in J_2$, and $\bar{b}(\sigma, \tau) = b(\sigma + h_1(s), \tau + h_2(t))$ for $\sigma, s \in J_1, \tau, t \in J_2$.

Proof The solution $z(x, y)$ of the problem (8.12.1)–(8.12.2) satisfies the equivalent integral equation

$$z(x, y) = a_1(x) + a_2(y) + \int_{x_0}^{x} \int_{y_0}^{y} f(s, t, z(s, t), z(s - h_1(s), t - h_2(t))) dt ds. \tag{8.12.8}$$

Using (8.12.3), (8.12.4), and (8.12.5) in (8.12.8) and making the change of variables, we have

$$|z(x, y)| \leq k + \int_{x_0}^{x} \int_{y_0}^{y} a(s, t)|z(s, t)| dt ds + M_1 M_2 \int_{\phi(x_0)}^{\phi(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{b}(\sigma, \tau)|z(\sigma, \tau)| d\tau d\sigma. \tag{8.12.9}$$

Now applying Theorem 5.3.1 to (8.12.9) yields (8.12.6). The right-hand side of (8.12.6) gives us the bound on the solution $z(x,y)$ of problem (8.12.1)–(8.12.2) in terms of the known functions. Thus if the right-hand side of (8.12.6) is bounded, then the solution of problem (8.12.1)–(8.12.2) is bounded for all $(x,y) \in \triangle$. $\square$

The next result deals with the uniqueness of the solutions of the problem (8.12.1)–(8.12.2).

Theorem 8.12.2 (Pachpatte [501]) *Suppose that the function f in (8.12.1) satisfies the condition*

$$|f(x,y,u,v) - f(x,y,\bar{u},\bar{v})| \leq a(x,y)|u-\bar{u}| + b(x,y)|v-\bar{v}|, \tag{8.12.10}$$

where $a, b \in C(\triangle, \mathbb{R}_+)$, and let $M_1, M_2, \phi, \psi, \bar{b}$ be as in Theorem 8.12.1. Then the problem (8.12.1)–(8.12.2) has at most one solution on $\triangle$.

Proof Let $z(x,y)$ and $\bar{z}(x,y)$ be two solutions of problem (8.12.1)–(8.12.2) on $\triangle$, then we have

$$\begin{aligned} z(x,y) - \bar{z}(x,y) = \int_{x_0}^{x}\int_{y_0}^{y} & [f(s,t,z(s,t),z(s-h_1(s),t-h_2(t))) \\ & -f(s,t,\bar{z}(s,t),\bar{z}(s-h_1(s),t-h_2(t)))]dtds. \end{aligned} \tag{8.12.11}$$

Using (8.12.10) in (8.12.11) and making the change of variables, we have

$$\begin{aligned} |z(x,y) - \bar{z}(x,y)| \leq & \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)|z(s,t) - \bar{z}(s,t)|dtds \\ & + M_1M_2 \int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{b}(\sigma,\tau)|z(\sigma,\tau) - \bar{z}(\sigma,\tau)|d\tau d\sigma. \end{aligned} \tag{8.12.12}$$

Now applying Theorem 5.3.1 yields

$$|z(x,y) - \bar{z}(x,y)| \leq 0.$$

Therefore $z(x,y) = \bar{z}(x,y)$; i.e., there is at most one solution of the problem (8.12.1)–(8.12.2). $\square$

The following theorem deals with the continuous dependence of solutions on the equation and the given initial boundary conditions.

Consider the problem (8.12.1)–(8.12.2) and the problem

$$\begin{cases} D_2D_1\omega(x,y) = F(x,y,\omega(x,y),\omega(x-h_1(x),y-h_2(y))), & (8.12.13) \\ \omega(x,y_0) = \bar{a}_1(x), \quad \omega(x_0,y) = \bar{a}_2(y), \quad \bar{a}_1(x_0) = \bar{a}_2(y_0) = 0, & (8.12.14) \end{cases}$$

where $F \in C(\triangle \times \mathbb{R}^2, \mathbb{R})$, $\bar{a}_1 \in C^1(J_1, \mathbb{R})$, *and* $\bar{a}_2 \in C^1(J_2, \mathbb{R})$, and h_1, h_2 are as in problem (8.12.1)–(8.12.2).

Theorem 8.12.3 (Pachpatte [501]) *Suppose that the function f in (8.12.1) satisfies the condition (8.12.10) in Theorem 8.12.2 and further assume that*

$$|a_1(x) - \bar{a}_1(x)| + |a_2(y) - \bar{a}_2(y)| \le \epsilon, \tag{8.12.15}$$

$$\begin{aligned} &\int_{x_0}^{x}\int_{y_0}^{y} |f(s,t,\omega(s,t),\omega(s-h_1(s),t-h_2(t))) \\ &\qquad -F(s,t,\omega(s,t),\omega(s-h_1(s),t-h_2(t)))|dtds \le \epsilon, \end{aligned} \tag{8.12.16}$$

where $\epsilon > 0$ is an arbitrary small constant, and let M_1, M_2, ϕ, ψ, and $\bar{b}$ be as in Theorem 8.12.1. Then the solution of problem (8.12.1)–(8.12.2) depends continuously on f and the initial boundary data.

Proof The equivalent integral equations corresponding to problem (8.12.1)–(8.12.2) and (8.12.13)–(8.12.14) are (8.12.8) and

$$\omega(x,y) = \bar{a}_1(x) + \bar{a}_2(y) + \int_{x_0}^{x}\int_{y_0}^{y} F(s,t,\omega(s,t),\omega(s-h_1(s),t-h_2(t)))dtds. \tag{8.12.17}$$

From (8.12.8) and (8.12.17) and using (8.12.10), (8.12.15), and (8.12.16), and making the change of variables, we have

$$\begin{aligned} |z(x,y) - \omega(x,y)| \le\ & |a_1(x) - \bar{a}_1(x)| + |a_2(y) - \bar{a}_2(y)| \\ & + \int_{x_0}^{x}\int_{y_0}^{y} \Big| f(s,t,z(s,t),z(s-h_1(s),t-h_2(t))) \\ & - \int_{x_0}^{x}\int_{y_0}^{y} f(s,t,\omega(s,t),\omega(s-h_1(s),t-h_2(t)))\Big| dtds, \end{aligned}$$

$$\begin{aligned} &\int_{x_0}^{x}\int_{y_0}^{y} \Big| f(s,t,\omega(s,t),\omega(s-h_1(s),t-h_2(t)))F(s,t,\omega(s,t),\omega(s-h_1(s),t-h_2(t)))\Big| dtds \\ &\le 2\epsilon + \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)|z(s,t) - \omega(s,t)|dtds \\ &+M_1M_2 \int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{b}(\sigma,\tau)|z(\sigma,\tau) - \omega(\sigma,\tau)|d\tau d\sigma. \end{aligned} \tag{8.12.18}$$

Now applying Theorem 5.3.1 to (8.12.18) yields

$$|z(x,y) - \omega(x,y)| \le 2\epsilon\left[\exp\Big(A(x,y) + \bar{B}(x,y)\Big)\right], \tag{8.12.19}$$

where $A(x,y)$ and $\bar{B}(x,y)$ are as in Theorem 8.12.1. On the compact set, the quantity in square bracket in (8.12.19) is bounded by some constant M. Therefore, $|z(x,y) - \omega(x,y)| \leq 2M\epsilon$ on the set, so the solution to boundary value problem depends continuously on f and the initial boundary values. If $\epsilon \to 0$, then $|z(x,y) - \omega(x,y)| \to 0$ on the set. $\square$

8.13 Applications of Theorems 5.3.2–5.3.3 to Retarded Non-self-adjoint Hyperbolic Partial Differential Equations

In this section, we present applications of Theorems 5.3.2–5.3.3 which display the importance to the literature. Consider the following retarded non-self-adjoint hyperbolic partial differential equation

$$z_{xy}(x,y) = D_2(a(x,y)z(x,y)) + f(x,y,z(x-h_1(x),y-g_1(y)),\cdots,z(x-h_n(x),y-g_n(y))), \tag{8.13.1}$$

with the given initial boundary conditions

$$z(x,y_0) = a_1(x),\ \ z(x_0,y) = a_2(y),\ \ a_1(x_0) = a_2(y_0) = 0, \tag{8.13.2}$$

where $f \in C(\Delta \times \mathbb{R}^n, \mathbb{R})$, $a_1 \in C^1(I_1,\mathbb{R})$, $a_2 \in C^1(I_2,\mathbb{R})$, and $a \in C(\Delta,\mathbb{R})$ is differentiable with respect to y; $h_i \in C(I_1,\mathbb{R}_+)$, $g_i \in C(I_2,\mathbb{R}_+)$ are non-increasing, and such that $x-h_i(x) \geq 0, x-h_i(x) \in C^1(I_1,I_1), y-g_i(y) \geq 0, y-g_i(y) \in C^1(I_2,I_2)$, $h_i'(x) < 1, g_i'(y) < 1, h_i(x_0) = g_i(y_0) = 0$ for $i = 1,\cdots,n$; $x \in I_1, y \in I_2$ and

$$M_i = \max_{x\in I_1} \frac{1}{1-h_i'(x)},\ N_i = \max_{y\in I_2} \frac{1}{1-g_i'(y)} \tag{8.13.3}$$

and $I_1 = [x_0,X],\ I_2 = [y_0,Y]$ and $\Delta = I_1 \times I_2$.

The first result gives us the bound on the solution of the problem (8.13.1)–(8.13.2).

Theorem 8.13.1 (Pachpatte [507]) *Suppose that*

$$\begin{cases} |f(x,y,u_1,\cdots,u_n)| \leq \displaystyle\sum_{i=1}^{n} b_i(x,y)|u_i|, & (8.13.4) \\ |e(x,y)| \leq k, & (8.13.5) \end{cases}$$

where $b_i(x,y), k$ are as in Theorem 5.3.2 and

$$e(x,y) = a_1(x) + a_2(y) - \int_{x_0}^{x} a(s,y_0)a_1(s)ds. \tag{8.13.6}$$

If $z(x, y)$ is any solution of problem (8.13.1)–(8.13.2), then for all $x \in I_1, y \in I_2$,

$$|z(x, y)| \leq k\bar{q}(x, y) \exp\left(\sum_{i=1}^{n} \int_{\phi_i(x_0)}^{\phi_i(x)} \int_{\psi_i(y_0)}^{\psi_i(y)} \bar{b}_i(\sigma, \tau)\bar{q}(\sigma, \tau)d\tau d\sigma\right), \tag{8.13.7}$$

where $\phi_i(x) = x - h_i(x),\ x \in I_1,\ \psi_i(y) = y - g_i(y),\ y \in I_2, \bar{b}_i(\sigma, \tau) = M_i N_i b_i(\sigma + h_i(s), \tau + g_i(t))$ for $\sigma, s \in I_1, \tau, t \in I_2$ and for all $x \in I_1, y \in I_2$,

$$\bar{q}(x, y) = \exp\left(\int_{x_0}^{x} |a(\xi, y)|d\xi\right). \tag{8.13.8}$$

Proof Note that the solution $z(x, y)$ of the problem (8.13.1)–(8.13.2) satisfies the equivalent integral equation

$$\begin{aligned} z(x, y) = e(x, y) + \int_{x_0}^{x} a(s, y)z(s, y)ds \\ + \int_{x_0}^{x} \int_{y_0}^{y} f(s, t, z(s - h_1(s), t - g_1(t)), \cdots, z(s - h_n(s), t - g_n(t)))dtds, \end{aligned} \tag{8.13.9}$$

where $e(x, y)$ is given by (8.13.6). From (8.13.9), (8.13.4), (8.13.5), (8.13.3) and making the change of variables, we have

$$\begin{aligned} |z(x, y)| &\leq k + \int_{x_0}^{x} |a(s, y)||z(s, y)|ds + \int_{x_0}^{x} \int_{y_0}^{y} \sum_{i=1}^{n} b_i(s, t)|z(s - h_i(s), t - g_i(t))|dtds \\ &\leq k + \int_{x_0}^{x} |a(s, y)||z(s, y)|ds + \sum_{i=1}^{n} \int_{\phi_i(x_0)}^{\phi_i(x)} \int_{\psi_i(y_0)}^{\psi_i(y)} \bar{b}_i(\sigma, \tau)|z(\sigma, \tau)|d\tau d\sigma. \end{aligned} \tag{8.13.10}$$

Now applying Theorem 5.3.2 to (8.13.10), we conclude (8.13.7). □

The next theorem deals with the uniqueness of solutions of problem (8.13.1)–(8.13.2).

Theorem 8.13.2 (Pachpatte [507]) *Suppose that the function f in Eq. (8.13.1) satisfies the condition*

$$|f(x, y, u_1, \cdots, u_n) - f(x, y, v_1, \cdots, v_n)| \leq \sum_{i=1}^{n} b_i(x, y)|u_i - v_i|, \tag{8.13.11}$$

where $b_i(x,y)$ are as in Theorem 5.3.2. Let $M_i, N_i, \phi_i, \psi_i, \bar{b}_i$ be as in Theorem 8.13.1. Then the problem (8.13.1)–(8.13.2) has at most one solution on Δ

Proof Let $u(x,y)$ and $v(x,y)$ be two solutions of problem (8.13.1)–(8.13.2) on Δ, then

$$\begin{aligned} u(x,y) - v(x,y) = &\int_{x_0}^{x} a(s,y)[u(s,y) - v(s,y)]ds \\ &+ \int_{x_0}^{x}\int_{y_0}^{y}\Big[f(s,t,u(s-h_1(s),t-g_1(t)),\cdots,u(s-h_n(s),t-g_n(t)) \\ &-f(s,t,v(s-h_1(s),t-g_1(t)),\cdots,v(s-h_n(s),t-g_n(t)))\Big]dtds. \end{aligned} \tag{8.13.12}$$

From (8.13.11)–(8.13.12), making the change of variables and in view of (8.13.3), we have

$$\begin{aligned} |u(x,y) - v(x,y)| \le &\int_{x_0}^{x} |a(s,y)||u(s,y) - v(s,y)|ds \\ &+ \int_{x_0}^{x}\int_{y_0}^{y}\sum_{i=1}^{n} b_i(s,t)|u(s-h_i(s),t-g_i(t)) - v(s-h_i(s),t-g_i(t))|dtds \\ \le &\int_{x_0}^{x} |a(s,y)||u(s,y) - v(s,y)|ds \\ &+ \sum_{i=1}^{n}\int_{\phi_i(x_0)}^{\phi_i(x)}\int_{\psi_i(y_0)}^{\psi_i(y)} \bar{b}_i(\sigma,\tau)|u(\sigma,\tau) - v(\sigma,\tau)|d\tau d\sigma. \end{aligned} \tag{8.13.13}$$

Now applying Theorem 5.3.2 to (8.13.13), we obtain

$$|u(x,y) - v(x,y)| \le 0$$

which gives us $u(x,y) = v(x,y)$, i.e., there is at most one solution of the problem (8.13.1)–(8.13.2). □

The following theorem shows the dependence of solutions of problem (8.13.1)–(8.13.2) on given initial boundary data.

Theorem 8.13.3 (Pachpatte [507]) *Let $u(x,y)$ and $v(x,y)$ be the solutions of problem (8.13.1)–(8.13.2) with the given initial boundary data*

$$u(x,y_0) = c_1(x),\ u(x_0,y) = c_2(y),\ c_1(x_0) = c_2(y_0) = 0, \tag{8.13.14}$$

and

$$v(x, y_0) = d_1(x),\ v(x_0, y) = d_2(y),\ d_1(x_0) = d_2(y_0) = 0, \tag{8.13.15}$$

respectively, where $c_1, d_1 \in C^1(I_1, \mathbb{R})$, $c_2, d_2 \in C^1(I_2, \mathbb{R})$. *Suppose that the function* f *satisfies the condition (8.13.11) in Theorem 8.13.2. Let, for all* $x \in I_1, y \in I_2$,

$$e_1(x, y) = c_1(x) + c_2(y) - \int_{x_0}^{x} a(s, y_0)c_1(s)ds, \tag{8.13.16}$$

$$e_2(x, y) = d_1(x) + d_2(y) - \int_{x_0}^{x} a(s, y_0)d_1(s)ds, \tag{8.13.17}$$

and

$$|e_1(x, y) - e_2(x, y)| \le k, \tag{8.13.18}$$

where k *is as in Theorem 5.3.2. Let* $M_i, N_i, \phi_i, \psi_i, \bar{b}_i, \bar{q}$ *be as in Theorem 8.13.1. Then for all* $x \in I_1, y \in I_2$,

$$|u(x, y) - v(x, y)| \le k\bar{q}(x, y) \exp \sum_{i=1}^{n} \left(\int_{\phi_i(x_0)}^{\phi_i(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{b}_i(\sigma, \tau)\bar{q}(\sigma, \tau)d\tau d\sigma \right). \tag{8.13.19}$$

Proof Since $u(x, y)$ and $v(x, y)$ are the solutions of problem (8.13.1)–(8.13.14) and (8.13.1)–(8.13.15) respectively, we have, for all $x \in I_1, y \in I_2$,

$$\begin{aligned} u(x, y) - v(x, y) &= e_1(x, y) - e_2(x, y) + \int_{x_0}^{x} a(s, y)\{u(s, y) - v(s, y)\}ds \\ &\quad + \int_{x_0}^{x} \int_{y_0}^{y} \Big\{ f(s, t, u(s - h_1(s), t - g_1(t)), \cdots, u(s - h_n(s), t - g_n(t))) \\ &\quad - f(s, t, v(s - h_1(s), t - g_1(t)), \cdots, v(s - h_n(s), t - g_n(t))) \Big\} dtds. \end{aligned} \tag{8.13.20}$$

From (8.13.20), (8.13.18), (8.13.11), making the change of variables and in view of (8.13.3), we have, for all $x \in I_1, y \in I_2$,

$$\begin{aligned} |u(x, y) - v(x, y)| &\le k + \int_{x_0}^{x} |a(s, y)||u(s, y) - v(s, y)|ds \\ &\quad + \sum_{i=1}^{n} \int_{\phi_i(x_0)}^{\phi_i(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{b}_i(\sigma, \tau)|u(\sigma, \tau) - v(\sigma, \tau)|d\tau d\sigma. \end{aligned} \tag{8.13.21}$$

Now an application of Theorem 5.3.2 to (8.13.21) yields the required estimate in (8.13.19), which shows the dependence of solutions of problem (8.13.1)–(8.13.2) on given initial boundary data. □

We next consider the following retarded non-self-adjoint hyperbolic partial differential equations

$$z_{xy}(x,y) = D_2(a(x,y)z(x,y)) + f(x,y,z(x-h_1(x), y-g_1(y)), \cdots, z(x-h_n(x), y-g_n(y)), \mu), \tag{8.13.22}$$

$$z_{xy}(x,y) = D_2(a(x,y)z(x,y)) + f(x,y,z(x-h_1(x), y-g_1(y)), \cdots, z(x-h_n(x), y-g_n(y)), \mu_0), \tag{8.13.23}$$

with the given initial boundary conditions (8.13.2), where $f \in C(\Delta \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R})$, and h_i, g_i are as in (8.13.1) and μ, μ_0 are real parameters.

The following theorem shows the dependence of solutions of problems (8.13.22), (8.13.2) and (8.13.23), (8.13.2) on parameters.

Theorem 8.13.4 (Pachpatte [507]) *Suppose that*

$$\begin{cases} |f(x,y,u_1,\cdots,u_n,\mu) - f(x,y,v_1,\cdots,v_n,\mu)| \le \displaystyle\sum_{i=1}^{n} b_i(x,y)|u_i - v_i|, & (8.13.24) \\ |f(x,y,u_1,\cdots,u_n,\mu) - f(x,y,u_1,\cdots,u_n,\mu_0)| \le m(x,y)|\mu - \mu_0|, & (8.13.25) \end{cases}$$

where $b_i(x,y)$ are as in Theorem 5.3.2 and $m : \Delta \to \mathbb{R}$ is a continuous function such that

$$\int_{x_0}^{x}\int_{y_0}^{y} m(s,t)\,dt\,ds \le M, \tag{8.13.26}$$

where $M \ge 0$ is a real constant. Let $M_i, N_i, \phi_i, \psi_i, \bar{b}_i$ be as in Theorem 8.13.1. If $z_1(x,y)$ and $z_2(x,y)$ are the solutions of problem (8.13.22), (8.13.2), and problem (8.13.23), (8.13.2), then for all $x \in I_1, y \in I_2$,

$$|z_1(x,y) - z_2(x,y)| \le \bar{k}\bar{q}(x,y)\exp\left(\sum_{i=1}^{n}\int_{\phi_i(x_0)}^{\phi_i(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{b}_i(\sigma,\tau)\bar{q}(\sigma,\tau)\,d\tau\,d\sigma\right) \tag{8.13.27}$$

where $\bar{k} = |\mu - \mu_0|M$ and $\bar{q}(x,y)$ is defined by (8.13.8).

Proof Let $z(x, y) = z_1(x, y) - z_2(x, y)$ for all $x \in I_1, y \in I_2$. As in the proof of Theorem 8.13.2, from the hypotheses, we have

$$\begin{aligned} z(x,y) = &\int_{x_0}^{x} a(s,y)z(s,y)ds \\ &+\int_{x_0}^{x}\int_{y_0}^{y}\Big\{f(s,t,z_1(s-h_1(s),t-g_1(t)),\cdots,z_1(s-h_n(s),t-g_n(t)),\mu) \\ &-f(s,t,z_2(s-h_1(s),t-g_1(t)),\cdots,z_2(s-h_n(s),t-g_n(t)),\mu) \\ &+f(s,t,z_2(s-h_1(s),t-g_1(t)),\cdots,z_2(s-h_n(s),t-g_n(t)),\mu) \\ &-f(s,t,z_2(s-h_1(s),t-g_1(t)),\cdots,z_2(s-h_n(s),t-g_n(t)),\mu_0)\Big\}dtds. \quad (8.13.28) \end{aligned}$$

From (8.13.28), (8.13.24)–(8.13.26), making the change of variables and in view of (8.13.3), we have

$$\begin{aligned} |z(x,y)| \le &\int_{x_0}^{x} |a(s,y)||z(s,y)|ds \\ &+\int_{x_0}^{x}\int_{y_0}^{y}\sum_{i=1}^{n} b_i(s,t)|z_1(s-h_i(s),t-g_i(t))-z_2(s-h_i(s),t-g_i(t))|dtds \\ &+\int_{x_0}^{x}\int_{y_0}^{y} m(s,t)|\mu-\mu_0|dtds \\ \le\ &\bar{k}+\int_{x_0}^{x}|a(s,y)||z(s,y)|ds+\sum_{i=1}^{n}\int_{\phi_i(x_0)}^{\phi_i(x)}\int_{\psi(y_0)}^{\psi(y)}\bar{b}_i(\sigma,\tau)|z(\sigma,\tau)|d\tau d\sigma. \end{aligned} \tag{8.13.29}$$

Applying Theorem 5.3.2 to (8.13.29) yields (8.13.27), which shows the dependence of solutions of problems (8.13.22), (8.13.2) and (8.13.23), (8.13.2) on parameters μ and μ_0. □

We note that the inequality in Theorem 5.3.2 can be used to study the similar properties as in Theorems 8.13.1–8.13.4 by replacing $D_2\ (a(x, y)z(x, y))$ by $D_1\ (a(x, y)z(x, y))$ in Eqs. (8.13.1), (8.13.22), (8.13.23) with the corresponding given initial boundary conditions under some suitable conditions on the functions involved therein.

We also note that the inequalities given in Theorem 5.3.3 can be used to establish similar results as in Theorems 8.13.1–8.13.4 by replacing $D_2(a(x, y)z(x, y))$ by

$$D_2\left(Q_1\left(x,y,z(x,y),\int_{x_0}^{x}k_1(\sigma,y,z(\sigma,y))d\sigma\right)\right)$$

or

$$D_1\left(Q_2\left(x, y, z(x, y), \int_{y_0}^{y} k_2(x, \tau, z(x, \tau\right)\right) d\tau$$

in Eqs. (8.13.1), (8.13.22), (8.13.23) with the corresponding given initial boundary conditions and under some suitable conditions on the functions involved therein. Furthermore, note that the inequalities and their applications given here can be extended very easily to functions involving many independent variables.

8.14 Applications of Theorem 5.3.6 to Retarded Volterra-Fredholm Integral Equations

In this section, we present applications of Theorem 5.3.6 to study certain properties of solutions of the retarded Volterra-Fredholm integral equation in two independent variables of the form

$$\begin{aligned} z(x, y) = f(x, y) &+ \int_{x_0}^{x}\int_{y_0}^{y} A(x, y, s, t, z(s-h_1(s), t-h_2(t)))dtds \\ &+ \int_{x_0}^{M}\int_{y_0}^{N} B(x, y, s, t, z(s-h_1(s), t-h_2(t)))dtds, \end{aligned} \tag{8.14.1}$$

where $\Delta = I_1 \times I_2$ with $I_1 = [x_0, M], I_2 = [y_0, N]$, and

$$E = \{(x, y, s, t) \in \Delta^2 : x_0 \le s \le x \le M, y_0 \le t \le y \le N\},$$

and $z, f \in C(\Delta, \mathbb{R}), A, B \in C(E \times \mathbb{R}, \mathbb{R})$ and $h_1 \in C(I_1, \mathbb{R}_+), h_2 \in C(I_2, \mathbb{R}_+)$, are non-increasing, $x - h_1(x) \ge 0, y - h_2(y) \ge 0, x - h_1(x) \in C_1(I_1, I_1), y - h_2(y) \in C^1(I_2, I_2), h_1'(x) < 1, h_2'(x) < 1, h_1(x_0) = h_2(y_0) = 0$.

The following theorem gives us the bound on the solution of Eq. (8.14.1).

Theorem 8.14.1 (Pachpatte [504]) *Suppose that the functions f, A, B in Eq. (8.14.1) satisfy the conditions*

$$|f(x, y)| \le c, \tag{8.14.2}$$

$$|A(x, y, s, t, z)| \le a(x, y, s, t)|z|, \tag{8.14.3}$$

$$|B(x, y, s, t, z)| \le b(x, y, s, t)|z|, \tag{8.14.4}$$

where $c, a(x, y, s, t), b(x, y, s, t)$ are as in Theorem 5.3.6. Let

$$M_1 = \max_{x\in I_1} \frac{1}{1 - h_1'(x)}, \quad M_2 = \max_{y\in I_2} \frac{1}{1 - h_2'(y)}, \tag{8.14.5}$$

and

$$\bar{p}(x,y) = \int_{\phi(x_0)}^{\phi(M)} \int_{\psi(y_0)}^{\psi(N)} \bar{b}(x,y,s,t) \exp\left(\int_{\phi(x_0)}^{\phi(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{a}(s,t,\sigma,\tau) d\tau d\sigma \right) dtds < 1, \tag{8.14.6}$$

where $\phi(x) = x - h_1(x), x \in I_1, \psi(y) = y - h_2(y), y \in I_2$ *and*

$$\begin{cases} \bar{a}(s,t,\sigma,\tau) = M_1 M_2 a(x,y,\sigma + h_1(s), \tau + h_2(t)), \\ \bar{b}(s,t,\sigma,\tau) = M_1 M_2 b(x,y,\sigma + h_1(s), \tau + h_2(t)). \end{cases}$$

If $z(x,y)$ *is a solution of Eq. (8.14.1) on* Δ, *then for all* $x \in I_1, y \in I_2$,

$$|z(x,y)| \leq \frac{c}{1 - \bar{p}(x,y)} \exp\left(\int_{\phi(x_0)}^{\phi(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,\sigma,\tau) d\tau d\sigma \right). \tag{8.14.7}$$

Proof Since $z(x,y)$ is a solution of Eq. (8.14.1), from (8.14.1)–(8.14.4) it follows

$$|z(x,y)| \leq c + \int_{x_0}^{x} \int_{y_0}^{y} a(x,y,s,t)|z(s - h_1(s), t - h_2(t))|dtds$$
$$+ \int_{x_0}^{M} \int_{y_0}^{N} b(x,y,s,t)|z(s - h_1(s), t - h_2(t))|dtds. \tag{8.14.8}$$

Now making the change of variables on the right-hand side of (8.14.8) and using (8.14.5), we have

$$|z(x,y)| \leq c + \int_{\phi(x_0)}^{\phi(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,\sigma,\tau)|z(\sigma,\tau)|dtds$$
$$+ \int_{\phi(x_0)}^{\phi(M)} \int_{\psi(y_0)}^{\psi(N)} \bar{b}(x,y,\sigma,\tau)|z(\sigma,\tau)|dtds. \tag{8.14.9}$$

Therefore an application of Theorem 5.3.6 to (8.14.9) yields (8.14.7). $\square$

The next result deals with the uniqueness of solutions of Eq. (8.14.1).

Theorem 8.14.2 (Pachpatte [504]) *Suppose that the functions* f, A, B *in Eq. (8.14.1) satisfy the conditions*

$$\begin{cases} |A(x,y,s,t,z) - A(x,y,s,t,\bar{z})| \leq a(x,y,s,t)|z - \bar{z}|, & (8.14.10) \\ |B(x,y,s,t,z) - B(x,y,s,t,\bar{z})| \leq b(x,y,s,t)|z - \bar{z}| & (8.14.11) \end{cases}$$

where $a(x, y, s, t), b(x, y, s, t)$ are as in Theorem 5.3.6. Let $M_1, M_2, \phi, \psi, \bar{a}, \bar{b}, \bar{p}$ be as in Theorem 8.14.1. Then Eq. (8.14.1) has at most one solution on Δ.

Proof Let $z(x, y)$ and $\bar{z}(x, y)$ be two solutions of Eq. (8.14.1) on Δ. From (8.14.1), (8.14.10), (8.14.11), it follows

$$\begin{aligned}
&|z(x, y) - \bar{z}(x, y)| \\
&\le \int_{x_0}^{x} \int_{y_0}^{y} a(x, y, s, t)|z(s - h_1(s), t - h_2(t)) - \bar{z}(s - h_1(s), t - h_2(t))|dtds \\
&\quad + \int_{x_0}^{M} \int_{y_0}^{N} b(x, y, s, t)|z(s - h_1(s), t - h_2(t)) - \bar{z}(s - h_1(s), t - h_2(t))|dtds.
\end{aligned} \tag{8.14.12}$$

Making the change of variables on the right-hand side of (8.14.12) and using (8.14.5), we have

$$\begin{aligned}
|z(x, y) - \bar{z}(x, y)| \le & \int_{\phi(x_0)}^{\phi(x)} \int_{\psi(y_0)}^{\psi(y)} \bar{a}(x, y, \sigma, \tau)|z(\sigma, \tau) - \bar{z}(\sigma, \tau)|dtds \\
& + \int_{\phi(x_0)}^{\phi(M)} \int_{\psi(y_0)}^{\psi(N)} \bar{b}(x, y, \sigma, \tau)|z(\sigma, \tau) - \bar{z}(\sigma, \tau)|dtds.
\end{aligned} \tag{8.14.13}$$

Now applying Theorem 5.3.6 to (8.14.13), we get

$$|z(x, y) - \bar{z}(x, y)| \le 0$$

which gives us $z(x, y) = \bar{z}(x, y)$, i.e., there is at most one solutions to Eq. (8.14.1). □

The following theorem deals with the continuous dependence of solutions of Eq. (8.14.1) on the right-hand side terms.

Consider Eq. (8.14.1) and the following equation

$$\begin{aligned}
w(x, y) = g(x, y) &+ \int_{x_0}^{x} \int_{y_0}^{y} F(x, y, s, t, w(s - h_1(s), t - h_2(t)))dtds \\
&+ \int_{x_0}^{M} \int_{y_0}^{N} G(x, y, s, t, w(s - h_1(s), t - h_2(t)))dtds,
\end{aligned} \tag{8.14.14}$$

where $w, g \in C(\Delta, \mathbb{R}), F, G \in C(E \times \mathbb{R}, \mathbb{R})$ and h_1, h_2 are as in Eq. (8.14.1).

Theorem 8.14.3 (Pachpatte [504]) *Suppose that the functions A, B in Eq. (8.14.1) satisfy the conditions (8.14.10), (8.14.11) in Theorem 8.14.2 and further assume*

that

$$\begin{cases} |f(x,y)-g(x,y)| \le \varepsilon, & (8.14.15) \\ \int_{x_0}^{x}\int_{y_0}^{y} |A(x,y,s,t,w(s-h_1(s),t-h_2(t))) \\ \qquad -F(x,y,s,t,w(s-h_1(s),t-h_2(t)))|dtds \le \varepsilon, & (8.14.16) \\ \int_{x_0}^{M}\int_{y_0}^{N} |B(x,y,s,t,w(s-h_1(s),t-h_2(t))) \\ \qquad -G(x,y,s,t,w(s-h_1(s),t-h_2(t)))|dtds \le \varepsilon, & (8.14.17) \end{cases}$$

where $\varepsilon > 0$ is an arbitrary small constant, and let $M_1, M_2, \phi, \psi, \bar{a}, \bar{b}, \bar{p}$ be as in Theorem 8.14.1. Then the solution of Eq. (8.14.1) depends continuously on the functions involved on the right-hand side of Eq. (8.14.1).

Proof Let $z(x,y)$ and $w(x,y)$ be the solutions of problem (8.14.1) and (8.14.14) respectively. Then we have,

$$\begin{aligned} &z(x,y)-w(x,y) = f(x,y)-g(x,y) \\ &+\int_{x_0}^{x}\int_{y_0}^{y} \{A(x,y,s,t,z(s-h_1(s),t-h_2(t))) - A(x,y,s,t,w(s-h_1(s),t-h_2(t)))\}\,dtds \\ &+\int_{x_0}^{x}\int_{y_0}^{y} \Big\{A(x,y,s,t,w(s-h_1(s),t-h_2(t))) - F(x,y,s,t,w(s-h_1(s),t-h_2(t)))\Big\}dtds \\ &+\int_{x_0}^{M}\int_{y_0}^{N} \{B(x,y,s,t,z(s-h_1(s),t-h_2(t))) - B(x,y,s,t,w(s-h_1(s),t-h_2(t)))|dtds\} \\ &+\int_{x_0}^{M}\int_{y_0}^{N} \Big\{B(x,y,s,t,w(s-h_1(s),t-h_2(t))) \\ &-G(x,y,s,t,w(s-h_1(s),t-h_2(t)))|dtds\Big\}. \end{aligned} \tag{8.14.18}$$

Using (8.14.10), (8.14.11), (8.14.15)–(8.14.17) in (8.14.18), we get

$$\begin{aligned} &|z(x,y)-w(x,y)| \\ &\le 3\varepsilon + \int_{x_0}^{x}\int_{y_0}^{y} a(x,y,s,t)|z(s-h_1(s),t-h_2(t)) - w(s-h_1(s),t-h_2(t))|dtds \\ &\quad+\int_{x_0}^{M}\int_{y_0}^{N} b(x,y,s,t)|z(s-h_1(s),t-h_2(t)) \\ &\quad-w(s-h_1(s),t-h_2(t))|dtds. \end{aligned} \tag{8.14.19}$$

Making the change of variables on the right-hand side of (8.14.19) and using (8.14.5), we infer

$$|z(x,y)-w(x,y)| \le 3\varepsilon + \int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,s,t)|z(\sigma,\tau)-w(\sigma,\tau)|dtds \\ + \int_{\phi(x_0)}^{\phi(M)}\int_{\psi(y_0)}^{\psi(N)} \bar{b}(x,y,s,t)|z(\sigma,\tau)-w(\sigma,\tau)|d\tau d\sigma. \tag{8.14.20}$$

Now applying Theorem 5.3.6 to (8.14.20) yields, for all $x \in I_1, y \in I_2$,

$$|z(x,y)-w(x,y)| \le 3\varepsilon\left[\frac{1}{1-\bar{p}(x,y)}\exp\left(\int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,\sigma,\tau)d\sigma d\tau\right)\right]. \tag{8.14.21}$$

On the compact set, the quantity in square brackets in (8.14.21) is bounded by some positive constant M. Therefore

$$|z(x,y)-w(x,y)| \le 3\overline{M}\varepsilon$$

on the set, which implies that the solution to Eq. (8.14.1) depends continuously on the functions involved on the right-hand side of Eq. (8.14.1). If $\varepsilon \to 0$, then $|z(x,y)-w(x,y)| \to 0$ on the set. □

We next consider the following retarded Volterra-Fredholm integral equations

$$\begin{cases} z(x,y) = f(x,y) + \displaystyle\int_{x_0}^{x}\int_{y_0}^{y} A(x,y,s,t,z(s-h_1(s),t-h_2(t)),\mu)dtds \\ \qquad + \displaystyle\int_{x_0}^{M}\int_{y_0}^{N} B(x,y,s,t,z(s-h_1(s),t-h_2(t)),\mu)dtds, & (8.14.22) \\ z(x,y) = f(x,y) + \displaystyle\int_{x_0}^{x}\int_{y_0}^{y} A(x,y,s,t,z(s-h_1(s),t-h_2(t)),\mu_0)dtds \\ \qquad + \displaystyle\int_{x_0}^{M}\int_{y_0}^{N} B(x,y,s,t,z(s-h_1(s),t-h_2(t)),\mu_0)dtds, & (8.14.23) \end{cases}$$

where $z, f \in C(\Delta, \mathbb{R}), A, B \in C(E \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and μ, μ_0 are real parameters.

The following theorem shows the dependence of solutions of Eqs. (8.14.22) and (8.14.23) on parameters.

Theorem 8.14.4 (Pachpatte [504]) *Suppose that*

$$\begin{cases} |A(x,y,s,t,z,\mu)-A(x,y,s,t,\bar{z},\mu)| \ \le \ A(x,y,s,t,)|z-\bar{z}|, & (8.14.24)\\ |A(x,y,s,t,\bar{z},\mu)-A(x,y,s,t,\bar{z},\mu_0)| \le r(x,y,s,t)|\mu-\mu_0|, & (8.14.25)\\ |B(x,y,s,t,z,\mu)-B(x,y,s,t,\bar{z},\mu)| \ \le b(x,y,s,t)|z-\bar{z}|, & (8.14.26)\\ |B(x,y,s,t,\bar{z},\mu)-B(x,y,s,t,\bar{z},\mu_0)| \le e(x,y,s,t)|\mu-\mu_0|, & (8.14.27) \end{cases}$$

where $a(x,y,s,t), b(x,y,s,t)$ are as in Theorem 5.3.6. and $r, e \in C(E, \mathbb{R}_+)$ are such that

$$\begin{cases} \int_{x_0}^{x}\int_{y_0}^{y} r(x,y,s,t)dtds \le k_1, & (8.14.28)\\ \int_{x_0}^{M}\int_{y_0}^{N} e(x,y,s,t)dtds \le k_2, & (8.14.29) \end{cases}$$

where k_1, k_2 are positive constants. Let $M_1, M_2, \phi, \psi, \bar{a}, \bar{b}, \bar{p}$ be as in Theorem 8.14.1. Let $z_1(x,y)$ and $z_2(x,y)$ be the solutions of Eqs. (8.14.22) and (8.14.23) respectively. Then for all $x \in I_1, y \in I_2$,

$$|z_1(x,y)-z_2(x,y)| \le \frac{(k_1+k_2)|\mu-\mu_0|}{1-\bar{p}(x,y)} \exp\left(\int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{a}(x,y,s,t)dtds\right). \tag{8.14.30}$$

Proof Let $z(x,y) = z_1(x,y) - z_2(x,y), (x,y) \in \triangle$. Then

$$\begin{aligned} z(x,y) = &\int_{x_0}^{x}\int_{y_0}^{y} \Big\{A(x,y,s,t,z_1(s-h_1(s),t-h_2(t)),\mu)\\ &-A(x,y,s,t,z_2(s-h_1(s),t-h_2(t)),\mu)\Big\}dtds\\ &+\int_{x_0}^{x}\int_{y_0}^{y} \Big\{A(x,y,s,t,z_2(s-h_1(s),t-h_2(t)),\mu)\\ &-A(x,y,s,t,z_2(s-h_1(s),t-h_2(t)),\mu_0)\Big\}dtds\\ &+\int_{x_0}^{M}\int_{y_0}^{N} \Big\{B(x,y,s,t,z_1(s-h_1(s),t-h_2(t)),\mu) \end{aligned}$$

$$-B(x, y, s, t, z_2(s-h_1(s), t-h_2(t)), \mu)\Big\}dtds$$
$$+\int_{x_0}^{M}\int_{y_0}^{N}\Big\{B(x, y, s, t, z_2(s-h_1(s), t-h_2(t)), \mu)$$
$$-G(x, y, s, t, z_2(s-h_1(s), t-h_2(t)), \mu_0)\Big\}dtds. \tag{8.14.31}$$

Inserting (8.14.24)–(8.14.29) into (8.14.31), we get

$$|z(x, y)| \le |\mu-\mu_0|k_1 + |\mu-\mu_0|k_2 + \int_{x_0}^{x}\int_{y_0}^{y} a(x, y, s, t)|z(s-h_1(s), t-h_2(t))|dtds$$
$$+\int_{x_0}^{M}\int_{y_0}^{N} b(x, y, s, t)|z(s-h_1(s), t-h_2(t))|dtds. \tag{8.14.32}$$

Making the change of variables on the right-hand side of (8.14.32) and (8.14.5), we get

$$|z(x, y)| \le |\mu-\mu_0|(k_1+k_2) + \int_{\phi(x_0)}^{\phi(x)}\int_{\psi(y_0)}^{\psi(y)} \bar{a}(x, y, \sigma, \tau)|z(\sigma, \tau)|d\sigma d\tau$$
$$+\int_{\phi(x_0)}^{\phi(M)}\int_{\psi(y_0)}^{\psi(N)} \bar{b}(x, y, \sigma, \tau)|z(\sigma, \tau)|d\sigma d\tau. \tag{8.14.33}$$

Therefore applying Theorem 5.3.6 to (8.14.33) yields (8.14.30), which shows the dependence of solutions of (8.14.22) and (8.14.23) on parameters. □

Remark 8.14.1 We note that the results in Theorems 5.3.6 and 8.14.1–8.14.4 can be extended very easily to functions involving many independent variables. Since the formulations of such results are quite straightforward in view of the results given above (see also [495]) and hence we omit the details.

Remark 8.14.2 For the study of behavior of solutions of Volterra-Fredholm integral equations involving functions of one independent variable, see [32, 406, 485].

8.15 Applications of Theorems 5.4.8 and 5.4.49 to Hyperbolic Partial Differential Equations Involving *N* Variables

In this section, we shall use Theorems 5.4.8 and 5.4.49 to study the uniqueness and continuous dependence and comparison of the solutions of hyperbolic partial differential equations involving n variables.

Example 8.15.1 (Uniqueness Test) We discuss the uniqueness of solutions of the hyperbolic partial integrodifferential equation

$$\begin{cases} \frac{\partial^n u(x)}{\partial x_1 \cdots \partial x_n} = f(x, u(x), w(x)), \\ w(x) = p(x) + \int_{x^0}^{x} K(x, s, u(s)) ds \end{cases} \tag{8.15.1}$$

with the conditions prescribed on $x = x^0$. Here p, K and f are continuous functions of their arguments and such that

$$|f(x, u(x), w(x)) - f(x, \overline{u}(x), \overline{w}(x))| \leq c_1[|u(x) - \overline{u}(x)| + |w(x) - \overline{w}(x)|],$$
$$|K(x, s, u(s)) - K(x, s, \overline{u}(s))| \leq c_2|u(s) - \overline{u}(s)|$$

for any two solutions $u(x)$ and $\overline{u}(x)$ of the given Eq. (8.15.1), where c_1 and c_2 are positive constants. Let the boundary conditions be such that the given boundary value problem (8.15.1) is equivalent to the Volterra integral equation given by

$$u(x) = n(x) + \int_{x^0}^{x} f\Big[s, u(s), p(s) + \int_{x^0}^{s} K(s, t, u(t)) dt\Big] ds,$$

where $n(x)$ is continuous. If $u(x)$ and $\overline{u}(x)$ are two solutions of the given boundary value problem, then

$$u(x) - \overline{u}(x) = \int_{x^0}^{x} \Big(f\Big[s, u(s), p(s) + \int_{x^0}^{s} K(s, t, u(t)) dt\Big]$$
$$-f\Big[s, \overline{u}(s), p(s) + \int_{x^0}^{s} K(s, t, \overline{u}(t)) dt\Big]\Big) ds.$$

If $x > x^0$, then

$$|u(x) - \overline{u}(x)| \leq \int_{x^0}^{x} c_1|u(s) - \overline{u}(s)| + \int_{x^0}^{x} c_1 \left(\int_{x^0}^{x} c_2|u(t) - \overline{u}(t)| dt \right) ds. \tag{8.15.2}$$

Applying Theorem 5.4.49 to the above inequality (8.15.2), we obtain $|u(x) - \overline{u}(x)| \leq 0$. Thus

$$u(x) = \overline{u}(x)$$

which means there is at most one solution of the problem (8.15.1). □

Example 8.15.2 (Continuous Dependence Test) Let us consider the pair of boundary value problem

$$\begin{cases} D_1 \cdots D_n u(x) &= f(x, u(x), w(x)), \\ \quad w(x) &= p(x) + \int_{x^0}^{x} K(x, s, u(s)) ds, \end{cases} \tag{8.15.3}$$

with

$$
\begin{aligned}
u(x_1^0, x_2, \cdots, x_n) &= u_{11}(x_2, \ldots, x_n), \\
u(x_1, x_2^0, x_3, \cdots, x_n) &= u_{12}(x_1, x_3, \cdots, x_n), \\
&\cdots, \\
u(x_1, \cdots, x_{n-1}, x_n^0) &= u_{1n}(x_1, \cdots, x_{n-1}); \\
u(x_1^0, x_2^0, x_3, \cdots, x_n) &= u_{21}(x_3, \cdots, u_n), \\
u(x_1^0, x_2, x_3^0, x_4, \cdots, x_n) &= u_{22}(x_2, x_4, \cdots, x_n), \\
&\cdots, \\
u(x_1, \cdots, x_{n-2}, x_{n-1}^0, x_n^0) &= u_{2,\frac{n(n-1)}{2}}(x_1, \cdots, x_{n-2}); \\
&\cdots, \\
u(x_1^0, \cdots, x_{n-1}^0, x_n) &= u_{n-1,1}(x_n),
\end{aligned}
$$

$$
\begin{aligned}
u(x_1^0, \cdots, x_{n-2}^0, x_{n-1}, x_n^0) &= u_{n-1,2}(x_{n-1}), \\
&\cdots, \\
u(x_1, x_2^0, \cdots, x_n^0) &= u_{n-1,n}(x_1); \\
u_{n-1,1}(x_n^0) = u_{n-1,2}(x_{n-1}^0) &= \cdots = u_{n-1,n}(x_1^0) = u(x_1^0, \cdots, x_n^0)
\end{aligned}
$$

and

$$
\begin{cases} D_1 \cdots D_n U(x) & = F(x, U(x), W(x)), \\ \qquad W(x) & = q(x) + \int_{x^0}^{x} K(x, s, U(s)) ds \end{cases} \tag{8.15.4}
$$

with the given boundary conditions

$$
\begin{aligned}
U(x_1^0, x_2, \cdots, x_n) &= U_{11}(x_2, \cdots, x_n), \\
U(x_1, x_2^0, x_3, \cdots, x_n) &= U_{12}(x_1, x_3, \cdots, x_n), \cdots, \\
U(x_1, \cdots, x_{n-1}, x_n^0) &= U_{1n}(x_1, \cdots, x_{n-1}), \\
U(x_1^0, x_2^0, x_3, \cdots, x_n) &= U_{21}(x_3, \cdots, x_n), \\
U(x_1^0, x_2, x_3^0, x_4, \cdots, x_n) &= U_{22}(x_2, x_4, \cdots, x_n), \cdots, \\
U(x_1, \cdots, x_{n-2}, x_{n-1}^0, x_n^0) &= U_{2,\frac{n(n-1)}{2}}(x_1, \cdots, x_{n-2}); \cdots; \\
U(x_1^0, \cdots, x_{n-1}^0, x_n) &= U_{n-1,1}(x_n),
\end{aligned}
$$

$$U(x_1^0, \cdots, x_{n-2}^0, x_{n-1}, x_n^0) = U_{n-1,2}(x_{n-1}), \cdots,$$
$$U(x_1, x_2^0, \cdots, x_n^0) = U_{n-1,n}(x_1)$$
$$U_{n-1,1}(x_n^0) = U_{n-1,2}(x_{n-1}^0) = \cdots = U_{n-1,n}(x_1^0) = U(x_1^0, \cdots, x_n^0)$$

where all functions are continuous on their respective domains of their definitions and

$$|u_{kl} - U_{kl}| \leq \epsilon \tag{8.15.5}$$

and

$$\left|f[s, u(s), p(s) + \int_{x^0}^{s} K(s, t, u(t))dt] - F[s, U(s), q(s) + \int_{x^0}^{s} K(s, t, U(t))dt]\right| \leq \epsilon, \tag{8.15.6}$$

$$\Big|K(s, t, u(t)) - K(s, t, \overline{u}(t))| \leq c_2|u(t) - \overline{u}(t)\Big|, \tag{8.15.7}$$

$$\Big|f(s, u(s), w(s)) - f(s, \overline{u}(s), W(s))\Big| \leq c_1[|u(s) - \overline{u}(s)| + |w(s) - W(s)|], \tag{8.15.8}$$

where ϵ, c_1, c_2 are positive constants. The given boundary value problems of (8.15.3) and (8.15.4) are equivalent to the Volterra integral equations respectively given by

$$u(x) = h(x) + \int_{x_0}^{x} f[s, u(s), p(s) + \int_{x^0}^{s} K(s, t, u(t))dt]ds \tag{8.15.9}$$

and

$$U(x) = H(x) + \int_{x_0}^{x} F[s, U(s), q(s) + \int_{x^0}^{s} K(s, t, U(t))dt]ds, \tag{8.15.10}$$

where $h(x)$ and $H(x)$ are functions similar to those of Theorem 5.4.8. Then

$$\begin{aligned} u(x) - U(x) = & \sum[u(x_1^0, x_2, \cdots, x_n) - U(x_1^0, x_2, \cdots, x_n)] \\ & - \sum[u(x_1^0, x_2^0, x_3, \cdots, x_n) - U(x_1^0, x_2^0, x_3, \cdots, x_n)] \\ & + \sum[u(x_1^0, x_2^0, x_3^0, x_4, \cdots, x_n) - U(x_1^{\cdot}x_2^0, x_3^0, x_4, \cdots, x_n)] \\ & - \cdots + (-1)^{n-1}[u(x_1^0, \ldots, x_n^0) - U(x_1^0, \ldots, x_n^0)] \cdots \\ & + \int_{x_0}^{x} \Big(f[s, u(s), p(s) + \int_{x^0}^{s} K(s, t, u(t))dt] \\ & - F[s, U(s), q(s) + \int_{x^0}^{s} K(s, t, U(t))]\Big)dtds. \end{aligned}$$

Adding and substituting $f[s, U(s), p(s) + \int_{x^0}^{s} K(s, y, U(t))dt]$ in the integrand, we obtain

$$
\begin{aligned}
|u(x) - U(x)| \leq & \sum |u(x_1^0, x_2, \cdots, x_n) - U(x_1^0, x_2, \cdots, x_n)| \\
& + \sum |u(x_1^0, x_2^0, x_3, \cdots, x_n) - U(x_1^0, x_2^0, x_3, \cdots, x_n)| \\
& + \cdots + |u(x_1^0, \cdots, x_n^0) - U(x_1^0, \cdots, x_n^0)| \\
& + \int_{x_0}^{x} |f[s, u(s), p(s) + \int_{x^0}^{s} K(s, t, u(t))dt] - f[s, U(s), p(s) \\
& + \int_{x^0}^{s} K(s, t, U(t))dt]|ds \\
& + \int_{x^0}^{x} |f[s, U(s), p(s) + \int_{x^0}^{s} K(s, t, U(t))dt] - F[s, U(s), q(s) \\
& + \int_{x^0}^{s} K(s, t, U(t))dt]|ds \\
\leq & \epsilon[2^n - 1 \prod_{i=1}^{n} (x_1 - x_i^0)] + \int_{x^0}^{x} c_1 |u(s) - U(s)| ds \\
& + \int_{x^0}^{x} c_1 (\int_{x^0}^{s} c_2 |u(t) - U(t)| dt) ds. \qquad (8.15.11)
\end{aligned}
$$

Using Theorem 5.4.8, we have for all $x > x^0$,

$$
\begin{aligned}
|u(x) - U(x)| \leq \epsilon \Big\{ & 2^n - 1 + \prod_{i=1}^{n} (x_i - x_i^0) + c_1 \int_{x^0}^{x} \Big([2^n - 1 + \prod_{i=1}^{n} (s_i - x_i^n)] \\
& + \int_{x^0}^{s} [2^n - 1 + \prod_{i=1}^{n} (t_i - x_i^0)](c_1 + c_2) v(t; s) dt \Big) ds \Big\}. \qquad (8.15.12)
\end{aligned}
$$

On a compact set S, the quantity in large bracks is bounded by some constant M. Therefore $|u(x) - U(x)| \leq M\epsilon$ on this set S, so that the solution of such a boundary value problem depends continuously on f and the boundary values. If $\epsilon \to 0$, then $|u(x) - U(x)| \to 0$ on this set S. □

8.16 Applications of Theorems 5.4.16–5.4.17 and Corollaries 5.4.6 and 5.4.8 to Some Partial Integrodifferential Equations

In this section, we shall use Theorems 5.4.16–5.4.17 and Corollaries 5.4.6 and 5.4.8 to study some partial integrodifferential equations.

We now consider the following example on the boundedness and continuous dependence of the solutions of some partial integrodifferential equation.

Example 8.16.1 Consider the nonlinear hyperbolic partial integrodifferential equation

$$y'(x) = p(x) + \int_0^x q(x, s, y(s), y'(s))ds, \quad x \in \mathbb{R}_+^n, \tag{8.16.1}$$

with initial conditions $y(x) = 0$ if $x_i = 0$ for some $i = 1, \dots, n$, where p, q are continuous function,

$$|p(x)| \leq M$$

and

$$|q(x, s, y(s), y'(s))| \leq w(s)(|y(s)| + |y'(s)|)$$

for any $x, s \in \mathbb{R}_+^n$, where $M > 0$ is a constant and $w \in C(\mathbb{R}_+^n, \mathbb{R}_+)$. If $y(x) \geq 0$ is a solution of Eq. (8.16.1) such that y is non-decreasing in each variable and y' is continuous, then for any $x \in \mathbb{R}_+^n$,

$$y'(t) = |y'(t)| \leq M + \int_0^x w(s)(y(s) + y'(s))ds.$$

Hence by Corollary 5.4.6, we obtain for any $x \in \mathbb{R}_+^n$,

$$y'(t) \leq M \left[1 + \int_0^x w(s) \exp\left(\int_0^s (1 + w(t))dt \right) ds \right].$$

Further, integrating both sides of the last inequality gives us an upper bound estimate for $y(x)$.

Example 8.16.2 We study the continuous dependence of the solutions on the right-hand side for the following two initial value problems

$$\begin{cases} y'(x) = f\left(x, y(x), \int_0^x h(x, s, y(s))ds\right), \\ y(x) = 0 \quad \text{if } x_i = 0 \text{ for some } i = 1, \dots, n \end{cases} \tag{8.16.2}$$

and

$$\begin{cases} y'(x) = F\left(x, Y(x), \int_0^x h(x, s, Y(s))ds\right), \\ Y(x) = 0 \quad \text{if } x_i = 0 \text{ for some } i = 1, \dots, n, \end{cases} \tag{8.16.3}$$

where all functions are supposed to be continuous, and

$$|f(x,y,z) - f(x,Y,Z)| \le M(|y-Y| + |z-Z|),$$
$$|h(x,s,y) - f(x,s,Y)| \le q(s)(|y-Y|)$$

for some constant $M > 0$ and some function $q \in C(\mathbb{R}^n_+, \mathbb{R}_+)$. Now problems (8.16.2) and (8.16.3) are equivalent to the following integral equations respectively:

$$\begin{cases} y(x) = \int_0^x f\big(s, y(s), \int_0^s h(s,t,y(t))dt\big)ds, & x \in \mathbb{R}^n_+, \\ Y(x) = \int_0^x F\big(s, Y(s), \int_0^s h(s,t,Y(t))dt\big)ds, & x \in \mathbb{R}^n_+. \end{cases}$$

Note that for any $x \in \mathbb{R}^n_+$,

$$\begin{aligned} |y(x) - Y(x)| \le& \int_0^x \left| f\left(s, y(s), \int_0^s h(s,t,y(t))dt\right) - f\left(s, Y(s), \int_0^s h(s,t,Y(t))dt\right) \right| ds \\ &+ \int_0^x \left| f\left(s, Y(s), \int_0^s h(s,t,Y(t))dt\right) - F\left(s, Y(s), \int_0^s h(s,t,Y(t))dt\right) \right| ds. \end{aligned}$$

Now if

$$\int_0^x \left| f\left(s, Y(s), \int_0^s h(s,t,Y(t))dt\right) - F\left(s, Y(s), \int_0^s h(s,t,Y(t))dt\right) \right| ds \le \varepsilon,$$

then

$$|y(x) - Y(x)| \le \int_0^x M\left[|y(s) - Y(s)| + \int_0^s q(t)|y(t) - Y(t)|dt\right] ds + \varepsilon$$

and so by Corollary 5.4.8, for any $x \in \mathbb{R}^n_+$,

$$|y(x) - Y(x)| \le \varepsilon\left[1 + \int_0^x M \exp\left(\int_0^s (M + q(t))dt\right) ds\right].$$

Therefore, the solution y depends continuously on y. Furthermore, although in order to avoid tedious manipulations, we have not made it explicit, we observe that y also depends continuously on its initial values on the hyperplanes $x_i = 0,\ i = 1, \ldots, n$, provided that these initial values are equicontinuous.

8.17 Applications of Theorem 5.4.26 to Third-Order Differential Equations

In this section, we shall use Theorem 5.4.26 to study the third-order differential equations.

We consider the third-order differential equation

$$\left[r_2(t)[r_1(t)y']'\right]' + f(t)y = g(t), \quad t \in \mathbb{R}_+ \tag{8.17.1}$$

where we assume that the functions $f(t), g(t)$ and $r_i(t)$, $(i = 1, 2)$ belong to the class $C(\mathbb{R}_+, \mathbb{R})$, and $r_i(t)$ do not change their signs on $\mathbb{R}_+$. Moreover, we suppose that $r_1'(t)$ exists on $\mathbb{R}_+$.

Obviously, Eq. (8.17.1) is equivalent to the following Volterra-type integral equation, which may be obtained by integrating from 0 to t three times and using the initial data

$$\begin{aligned} y(t) &= y(0) + r_1(0)y'(0)\int_0^t [r_1(s)]^{-1}ds + r_2(0)\left[r_1'(0)y'(0) + r_1(0)y''(0)\right] \\ &\quad\times \int_0^t [r_1(s)]^{-1}\left[\int_0^s [r_2(u)]^{-1}du\right]ds \\ &\quad+ \int_0^t [r_1(s)]^{-1}\left(\int_0^s [r_2(u)]^{-1}\left[\int_0^u g(v)dv\right]du\right)ds \\ &\quad- \int_0^t [r_1(s)]^{-1}\left[\int_0^s |r_2(u)|^{-1}\left(\int_0^u f(v)y(v)dv\right)du\right]ds, \ t \in \mathbb{R}_+. \end{aligned} \tag{8.17.2}$$

Therefore, for all $t \in \mathbb{R}_+$,

$$|y(t)| \le q(t) + \int_0^t [r_1(s)]^{-1}\left[\int_0^s |r_2(u)|^{-1}\left(\int_0^u |f(v)||y(v)|dv\right)du\right]ds, \tag{8.17.3}$$

where

$$\begin{aligned} q(t) &= |y(0)| + |r_1(0)||y'(0)|\int_0^t |r_1(s)|^{-1}ds + r_2(0)[r_1'(0)y'(0) + r_1(0)y''(0)] \\ &\quad\times \int_0^t |r_1(s)|^{-1}\left[\int_0^s |r_2(u)|^{-1}du\right]ds \\ &\quad+ \int_0^t |r_1(s)|^{-1}\left(\int_0^s |r_2(u)|^{-1}\left[\int_0^u g(v)dv\right]du\right)ds. \end{aligned} \tag{8.17.4}$$

An application of Theorem 5.4.26 to (8.17.2) gives us

$$x(t) \le q(t)U(t), \quad t \in \mathbb{R}_+ \tag{8.17.5}$$

here $U(t) = V_3(t, t)$ and $V_3(T, t)$ is defined by (5.4.161) and (5.4.162).

In the present case, we have

$$\begin{cases} V_1(T,t) = \exp \int_0^t [|r_1(s)|^{-1} + |r_2(s)|^{-1} + |f(s)|]ds, \\ V_2(T,t) = F_2(T,t)\left[1 + \int_0^t |r_2(s)|^{-1} \frac{V_1(T,s)}{F_2(T,s)} ds\right], \\ V_3(T,t) = F_1(T,t)\left[1 + \int_0^t |r_1(s)|^{-1} \frac{V_2(T,s)}{F_1(T,s)} ds\right], \end{cases} \tag{8.17.6}$$

and

$$\begin{cases} F_2(T,t) = \exp\left(\int_0^t [|r_1(s)|^{-1} - |r_2(s)|^{-1}]ds\right), \\ F_1(T,t) = \exp\left(\int_0^t -|r_1(s)|^{-1} ds\right). \end{cases}$$

Hence we obtain

$$U(t) = \exp\left(\int_0^t -|r_1(s)|^{-1} ds\right)\left\{1 + \int_0^t |r_1(s)|^{-1} \exp\left(\int_0^s |r_1(u)|^{-1} du\right) R(s)ds\right\}, \tag{8.17.7}$$

where

$$R(t) = \exp \int_0^t [|r_1(s)|^{-1} - |r_2(s)|^{-1}]ds \times \left[1 + \int_0^t |r_2(s)|^{-1} \left(\exp \int_0^s \left(2|r_2(u)|^{-1} + |f(u)|\right)du\right) ds\right].$$

We can easily observe from inequality (8.17.3) that, if the functions $|f(t)|$ and $|r_i(t)|^{-1}$, $(i = 1, 2)$, belong to the class $L^1(0, +\infty)$ and the condition

$$\int_0^t |r_1(s)|^{-1} \left(\int_0^s |r_2(u)|^{-1} \mid \int_0^u g(v)dv \mid du\right) ds < +\infty$$

holds for all $t \in \mathbb{R}_+$, then all solutions of (8.17.1) are bounded on $\mathbb{R}_+$. Furthermore, if here we have $g(t) \equiv 0$, then the trivial solution $y(t) \equiv 0$ of Eq. (8.17.1) is stable in the sense of Lyapunov. □

Now following the same argument as above and paying close attention to the structure of the function $V_n(q,t)$ given by (5.4.161), then we can easily verify the following more general result.

Theorem 8.17.1 (Yang [657]) *Consider the following nth-order differential equation*

$$[r_{n-1}(t)[\cdots[r_1(t)y']'\cdots]']' + f(t)y = g(t), \quad t \in \mathbb{R}_+ \tag{8.17.8}$$

here $f(t), g(t)$ and $r_i(t)$ belong to the class $C(\mathbb{R}_+, \mathbb{R})$, $i = 1, 2, \cdots, n-1$, $r_i(t)$ do not change their signs on $\mathbb{R}_+$ and derivatives $r_i'(t)$ exists on $\mathbb{R}_+$ and suppose that

the functions $|f(t)|$ *and* $|r_i(t)|^{-1}$ *belong to the class* $L^1(0, +\infty)$*, and the equation*

$$|\int_0^t g(u)du| < +\infty, \quad t \in \mathbb{R}_+, \tag{8.17.9}$$

holds, then all of the solutions of Eq. (8.17.1) are bounded on $\mathbb{R}_+$*. In addition, if* $g(t) \equiv 0$ *holds too, then the trivial solution* $y(t) \equiv 0$ *of Eq. (8.17.1) is stable in the sense of Lyapunov.*

8.18 Applications of Theorems 5.4.43–5.4.44 to Nonlinear Integral Equations

In this section, we shall use Theorems 5.4.43–5.4.44 to study nonlinear integral equations.

Example 8.18.1 As the first application, we obtain the lower bound on the solution of a nonlinear integral equation of the form

$$u(x) = u(s) + \int_x^s F(\xi, u(\xi))d\xi, \quad x, s \in \Omega \tag{8.18.1}$$

where all the functions in (8.18.1) are real-valued and defined on the respective domains of their definitions and it holds that

$$| F(x, u) | \leq b(x)W(|u|) \tag{8.18.2}$$

where $b(x)$ and $W(r)$ are as defined in Theorem 5.4.43. Using (8.18.2) in (8.18.1), we have

$$| u(x) | \leq | u(s) | + \int_x^s b(\xi)W(|b(\xi)|)d\xi,$$

i.e.,

$$| u(s) | \geq | u(x) | - \int_x^s b(\xi)W(|b(\xi)|)d\xi. \tag{8.18.3}$$

Now assuming that $u(x)$, $(x < s; x, s \in \Omega)$ is positive and applying Theorem 5.4.43, we have

$$| u(s) | \geq G^{-1}\Big[G(| u(x) |) - \int_x^s b(\xi)d\xi\Big] \tag{8.18.4}$$

where G and G^{-1} are as defined in Theorem 5.4.43. Thus the right-hand side of (8.18.4) gives us the lower bound on the solution $u(s)$ of Eq. (8.18.1). □

Example 8.18.2 We establish the lower bound on the solution of a nonlinear integral equation of the form

$$u(x) = u(s) + \int_x^s F\Big[\xi, u(\xi), \int_\xi^s k(\xi, \zeta, u(\xi))d\zeta\Big]d\xi, \quad x, s \in \Omega \tag{8.18.5}$$

where all the functions involved in (8.18.5) are real-valued and defined on the respective domains of their definitions and it holds that

$$| \, k(x, \xi, u) \, | \leq c(\xi)|u|, \tag{8.18.6}$$

$$| \, F(x, u, v) \, | \leq b(x)(|u| + |v|), \tag{8.18.7}$$

where $b(x)$ and $c(x)$ are defined as in Theorem 5.4.43. Using (8.18.6) and (8.18.7) in (8.18.5), we have

$$| \, u(x) \, | \leq | \, u(s) \, | + \int_x^s b(\xi) \left[u(\xi) + \int_\xi^s c(\zeta)|u(\zeta)|d\zeta \right] d\xi,$$

i.e.,

$$| \, u(s) \, | \geq | \, u(x) \, | \left[1 + \int_x^s b(\xi) \exp\left(\int_\xi^s [b(\zeta) + c(\zeta)]d\zeta \right) d\xi \right]^{-1} \tag{8.18.8}$$

which gives us the lower bound on the solution $u(s)$ of (8.18.5). □

8.19 Applications of Theorems 5.4.57–5.4.58 to Hyperbolic Differential Systems and Hyperbolic Integrodifferential Equations

In this section, we shall use Theorems 5.4.57–5.4.58 to prove the uniqueness and continuous dependence for the solutions of hyperbolic differential systems and hyperbolic integrodifferential equations of a more general type, then those given in [88, 90, 95].

We shall use Theorem 5.4.58 to provide an upper bound on the solutions of the nonlinear hyperbolic integrodifferential equation

$$u_x(x) = f\left(x, u(x), \int_y^x k(x, s, u(s)) \, ds \right) \tag{8.19.1}$$

together with the given suitable boundary conditions $u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n), \; 1 \leq i \leq n.$

The functions f and k are continuous on their respective domains of definitions and

$$\begin{cases} |f(x,u(x),v(x))| \le f_{11}(x)|u(x)| + f_{12}(x)|v(x)|, & (8.19.2)\\ |k(x,s,u(s))| \le f_{22}(s)|u(s)|, & (8.19.3)\end{cases}$$

where f_{11}, f_{12}, f_{22} are the same as those appearing in (5.4.394).

Any solution $u(x)$ of problem (8.19.1) satisfying the boundary conditions is also a solution of the Volterra integral equation

$$u(x) = a(x) + \int_y^x f(x^1, u(x^1), \int_y^{x^1} k(x^1, x^2, u(x^2))\, dx^2)\, dx^1, \tag{8.19.4}$$

where $a(x)$ takes care of the boundary conditions.

We use (8.19.2), (8.19.3) in (8.19.4) to obtain

$$|u(x)| \le |a(x)| + \int_y^x \left[f_{11}(x^1)|u(x^1)| + f_{12}(x^1) \int_y^{x^1} f_{22}(x^2)|u(x^2)|\, dx^2 \right] dx^1. \tag{8.19.5}$$

From Theorem 5.4.58, we find

$$\begin{aligned} |u(x)| \le |a(x)| + \int_y^x &\left[f_{11}(x^1)|a(x^1)| + f_{12}(x^1) \int_y^{x^1} f_{22}(x^2)|a(x^2)|\, dx^2 \right] \\ &\times \exp\left(\int_{x^1}^x \left[f_{11}(x^2) + f_{12}(x^2) \int_y^{x^2} f_{22}(x^3)\, dx^3 \right] dx^2 \right) dx^1. \end{aligned} \tag{8.19.6}$$

If, $|a(x)| \le M$, where $M > 0$ is a constant, then from (8.19.6) or (8.19.5) with Theorem 5.4.57, we get

$$|u(x)| \le M \exp\left(\int_y^x \left[f_{11}(x^1) + f_{12}(x^1) \int_y^{x^1} f_{22}(x^2)\, dx^2 \right] dx^1 \right). \tag{8.19.7}$$

Further, if $f_{11} = f_{12}$, then from (8.19.7), we obtain

$$|u(x)| \le M \exp\left(\int_y^x f_{11}(x^1) \left[1 + \int_y^{x^1} f_{22}(x^2)\, dx^2 \right] dx^1 \right). \tag{8.19.8}$$

Estimate (8.19.8) is not comparable with

$$|u(x)| \leq M\left[1+\int_y^x f_{11}(x^1)\exp\left(\int_y^{x^1}[f_{11}(x^2)+f_{22}(x^2)]\,dx^2\right)dx^1\right] \tag{8.19.9}$$

as obtained in [95] for $n = 2$.

In order for $|u(x)|$ to remain bounded in (8.19.9), it is necessary to have

$$\int_y^x [f_{11}(x^1)+f_{22}(x^1)]dx^1 < +\infty,$$

which is the same as

$$\int_y^x f_{11}(x^1)\,dx^1 < +\infty,\ \int_y^x f_{22}(x^1)\,dx^1 < +\infty. \tag{8.19.10}$$

In (8.19.8), we require

$$\int_y^x f_{11}(x^1)\left[1+\int_y^{x^1} f_{22}(x^2)\,dx^2\right] < +\infty, \tag{8.19.11}$$

which is obviously satisfied if (8.19.10) holds, but in several cases (8.19.11) more general than (8.19.10), for example, let $f_{22}(x) = \exp\left(\sum_{i=1}^n (x_i - y_i)\right)$, $f_{11}(x) = \exp\left(-2\sum_{i=1}^n (x_i - y_i)\right)$; for this (8.19.10) is not satisfied, where as (8.19.11) holds. Thus the results obtained here will be applicable to more general situations. □

8.20 An Application of Theorem 5.4.59 to Integral Equations

In this section, we shall use Theorem 5.4.59 to study integral equations.

Example 8.20.1 Suppose the following integral equation

$$v(x) = k(s) + \int_x^s A(s,t;v(t))dt + \int_x^s g(s,t)\left(\int_t^s B(s,r;v(r))dr\right)dt \tag{8.20.1}$$

holds for all $0 \leq x \leq s$, where $s \in I^n$ is a vector-valued parameter; and $k : I^n \to \mathbb{R}, g : I^n \times I^n \to \mathbb{R}$, and A and $B : I^n \times I^n \times \mathbb{R} \to \mathbb{R}$ are known as continuous functions. We assume further that the inequalities

$$\begin{cases} |A(s,t;p)| & \leq f(s,t)|p|, \\ |B(s,t;q)| & \leq h(s,t)|q|, \ \textit{for all}\ s,t \in I^n, t \leq s; p,q \in \mathbb{R}, \end{cases} \tag{8.20.2}$$

are satisfied, where f and $h : I^n \times I^n \to \mathbb{R}_+$ are known continuous functions. Then if $v(x)$ is a continuous solution of Eq. (8.20.1) on I^n, we easily obtain from (8.20.1) that for all $0 \le x \le t \le s, \quad s \in I^n$,

$$|k(s)| \ge |v(x)| - \int_x^s f(s,t)|v(t)|dt - \int_x^s |g(s,t)| \left(\int_t^s h(s,r)|v(r)|dr \right). \tag{8.20.3}$$

Setting $u(s) = |k(s)|$ and $w(x) = |v(x)|$ in (8.20.3), and applying Theorem 5.4.59 to the above inequality, then we obtain $0 \le x \le s, \quad s \in I^n$,

$$|v(x)| \le |k(s)| \exp\left(\int_x^s [f(s,t) + |g(s,t)| + h(s,t)]dt \right),$$

and

$$|v(x)| \le |k(s)| \Big\{ 1 + \int_x^s (f(s,t) + |g(s,t)|) \\ \times \left(\exp \int_t^s (f(s,r) + |g(s,r)| + h(s,r))dr \right) dt \Big\},$$

since $A_1(s,x) \le f(s,x) + |g(s,x)|$, where $A_1(s,x)$ is defined by $A_1 = \max(f(s,x), |g(s,x)|)$ for each $s \in I^n$ fixed. □

8.21 An Application of Theorem 5.4.63 to Nonlinear Hyperbolic Functional Integrodifferential Equations of the Retarded Type

In this section, we give some applications of Theorem 5.4.63 to obtain properties of solutions of a certain class of nonlinear hyperbolic functional integrodifferential equations of the retarded type. We consider the hyperbolic equation

$$\frac{\partial^n u(x)}{\partial x_1, \partial x_2, \cdots \partial x_n} = G(x, u(\sigma(x)), Tu(x)) \tag{8.21.1}$$

with the given suitable boundary conditions

$$u(x_1, x_{i-1}, x_i^0, x_{i+1}, \cdots, x_n), \quad 1 \le i \le n, \tag{8.21.2}$$

where

$$G \in C(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \quad Tu(x) = \int_{x0}^{x} k(x, y, u(\rho(y)))dy$$

with $k \in C(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $\sigma, \rho \in \mathcal{F}$.

Any solution $u(x)$ of Eq. (8.21.1) satisfying the boundary conditions is also a solution of the Volterra integral equation

$$u(x) = n(x) + \int_{x0}^{x} G(s, u(\sigma(s)), Tu(s))ds, \tag{8.21.3}$$

where $n(x)$ takes care of the boundary conditions. The following theorem provides an upper bound on the solutions of Eq. (8.21.1).

Theorem 8.21.1 (Akinyele [24]) *Assume that*

(i)

$$|k(x, y, u(\sigma(y)))| \leq g(y)|u(\sigma(y))|, \tag{8.21.4}$$

$$|G(x, u(\sigma(x)), Tu(x))| \leq f(x)[|u(\sigma(x))| + |Tu(x))|] \tag{8.21.5}$$

where f and g are continuous non-negative real-valued functions such that

$$\int_{x_1^0}^{+\infty} \cdots \int_{x_n^0}^{+\infty} g(s)ds < +\infty, \quad \int_{x_1^0}^{+\infty} \cdots \int_{x_n^0}^{+\infty} f(s)ds < +\infty \tag{8.21.6}$$

(ii) $\rho(x) \leq \sigma(x)$ *for* $x \geq x^0$,

(iii) $n(x)$ *is a non-zero, non-decreasing function such that*

$$|n(x)| \leq M \tag{8.21.7}$$

for some constant $M > 0$. *Then solutions of Eq. (8.21.1) are bounded.*

Proof Using (8.21.2), we have

$$\begin{aligned} |u(x)| &\leq |n(x)| + \int_{x0}^{x} f(s)[|u(\sigma(s))| + |Tu(s))|]ds \\ &\leq |n(x)| + \int_{x0}^{x} f(s)|u(\sigma(s))|ds + \int_{x0}^{x} f(s) \left(\int_{x0}^{s} g(t)|u(\rho(t))|dt \right) ds. \end{aligned} \tag{8.21.8}$$

Applying Theorem 5.4.63 to (8.21.8), we have

$$|u(x)| \leq |n(x)| \exp\left(\int_{x_0}^{x} \left(f(t)|\frac{n(\sigma(t))}{n(\sigma(t))}| + g(t)|\frac{n(\rho(t))}{n(\sigma(t))}|dt\right)\right). \tag{8.21.9}$$

Now (ii) implies $|\frac{n(\rho(t))}{n(\sigma(t))}| \leq 1$ and assumption on f and g imply

$$|u(x)| \leq M \exp\left(\int_{x_0}^{x} \Big(f(t) + g(t)\Big)dt\right) \leq N \tag{8.21.10}$$

where N is a positive constant, which completes the proof. □

Corollary 8.21.1 (Akinyele [24]) *Assume that hypotheses (ii) and (iii) of Theorem 8.21.1 hold. Let the conditions on G and k in Theorem 8.21.1 hold with f and g satisfying the condition*

$$\int_{x_1^0}^{+\infty} \cdots \int_{x_n^0}^{+\infty} f(s) \exp\left(\int_{x_0}^{s} \Big(f(t) + g(t)\Big)dt\right) ds < +\infty. \tag{8.21.11}$$

Then solutions of Eq. (8.21.1) are bounded.

8.22 An Application of Theorem 6.1.1 to Difference Equations

In this section, we shall employ Theorem 6.1.1 to study difference equations.

Example 8.22.1 Consider the difference equation

$$u(m,n) = a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} k(s,t,u(s,t)). \tag{8.22.1}$$

Let

$$k(s,t,u(s,t)) \leq tu(s,t), \tag{8.22.2}$$

we infer from (8.22.1)–(8.22.2)

$$u(m,n) \leq a(m,n) + \sum_{s=0}^{m-1}\sum_{t=0}^{n-1} tu(s,t). \tag{8.22.3}$$

Applying Theorem 6.1.1 to (8.22.3), it follows

$$u(m,n) \leq a(m,n) \prod_{t=0}^{n-1}(1 + mt). \tag{8.22.4}$$

8.23 An Application of Theorem 6.1.3 to Nonlinear Sum-Difference Equations

In this section, we present an application of Theorem 6.1.3 to obtain the bound on the solution of a nonlinear sum-difference equation of the form,

$$u(m,n) = F(m,n) + \sum_{s=m+1}^{+\infty} \sum_{t=n+1}^{+\infty} B(m,n,s,t,u(s,t)), \tag{8.23.1}$$

where $u, F : \mathbb{N}_0^2 \to \mathbb{R}, B : \mathbb{N}_0^2 \times \mathbb{N}_0^2 \times \mathbb{R} \to \mathbb{R}$ and

$$|F(m,n)| \le a(m,n), \tag{8.23.2}$$

$$|B(m,n,s,t,u)| \le b(s,t)|u|, \tag{8.23.3}$$

where $a(m,n)$ and $b(s,t)$ are as in Theorem 6.1.3 . Let $u(m,n)$ be a solution of Eq. (8.23.1). From (8.23.1)–(8.23.3), we derive

$$|u(m,n)| \le a(m,n) + \sum_{s=m+1}^{+\infty} \sum_{t=n+1}^{+\infty} b(s,t)|u(s,t)|. \tag{8.23.4}$$

Now applying Theorem 6.1.3 to (8.23.4), we have

$$u(m,n)| \le a(m,n) \prod_{s=m+1}^{+\infty} [1 + \sum_{t=n+1}^{+\infty} b(s,t)]. \tag{8.23.5}$$

The right-hand side of (8.23.5) gives us an upper bound on the solution $u(m,n)$ of Eq. (8.23.1) in terms of the known functions.

8.24 Applications of Theorem 6.1.5 to Nonlinear Finite Difference Equations

In this section, we present some applications of Theorem 6.1.5 to the study of boundedness, uniqueness and continuous dependence of the solutions of a few class of nonlinear finite difference equations in two independent variables. Each of these applications could be stated formally as a theorem. This has not been done so as not to obscure the essential ideas with technical details.

Example 8.24.1 As a first application, we obtain a bound on the solution of a nonlinear fourth order finite difference equation

$$\Delta_2[a_3(m,n)\Delta_2[a_2(m,n)\Delta_1[a_1(m,n)\Delta_1 u(m,n)]]] = f(m,n,u)(m,n) \quad (8.24.1)$$

with the given boundary conditions at $m = 0, n = 0$

$$\begin{cases} u(0,n) = \phi_1(n), \\ a_1(0,n)\Delta_1 u(0,n) = \phi_2(n), \\ a_2(m,0)\Delta_1[a_1(m,0)\Delta_1 u(m,0)] = \psi_1(m), \\ a_3(m,0)\Delta_2[a_2(m,0)\Delta_1[a_1(m,0)\Delta_1 u(m,0)]] = \psi_2(m). \end{cases} \quad (8.24.2)$$

Here a_1, a_2, a_3 are real-valued positive functions defined on $\mathbb{N}_0^2$, $f : \mathbb{N}_0^2 \times \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers; $\phi_1(n), \phi_2(n), \psi_1(m), \psi_2(m)$ are real-valued non-negative functions defined for all $m, n \in \mathbb{N}_0$. We assume that

$$|f(m,n,u)| \le h(m,n)|u| \quad (8.24.3)$$

where $h(m,n)$ is a real-valued non-negative function defined for all $m, n \in \mathbb{N}_0$. It is easy to observe that the problem (8.24.1)–(8.24.2) is equivalent to the equation

$$u(m,n) = b(m,n) + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)} \sum_{t=0}^{y-1} f(s,t,u(s,t)) \quad (8.24.4)$$

where

$$b(m,n) = \phi_1(n) + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \phi_2(n) + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \psi_1(s)$$
$$+ \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \psi_2(s) \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)}. \quad (8.24.5)$$

Suppose that

$$|b(m,n)| \le k \quad (8.24.6)$$

where k is a non-negative constant. Using (8.24.3), (8.24.6) in (8.24.4), we have

$$|u(m,n)| \le k + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)} \sum_{t=0}^{y-1} \frac{1}{a_3(s,n)} \sum_{t=0}^{y-1} h(s,t)|u(s,t)|.$$

Now an application of Theorem 6.1.5 yields the bound on the solution $u(m,n)$ of problem (8.24.1)–(8.24.2) in terms of the known functions.

Example 8.24.2 As a second application, we shall discuss the uniqueness of the solution of the problem (8.24.1)–(8.24.2). We assume that the function f in (8.24.1) satisfies

$$|f(m,n,u) - f(m,n,\bar{u})| \leq h(m,n)|u-\bar{u}| \tag{8.24.7}$$

where $h(m,n)$ is as in Example 8.24.1. The problem (8.24.1)–(8.24.2) is equivalent to Eq. (8.24.4). Then for any two solutions u and $\bar{u}$ of Eqs. (8.24.1)–(8.24.2), we have

$$|u(m,n) - \bar{u}(m,n)| \leq \varepsilon + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)} \sum_{t=0}^{y-1} h(s,t)|u(s,t) - \bar{u}(s,t)| \tag{8.24.8}$$

where $\varepsilon > 0$ is arbitrary constant. The assumption (8.24.7) is used to get the inequality in (8.24.8). Now an application of Theorem 6.1.5 yields

$$|u(m,n) - \bar{u}(m,n)| \leq \varepsilon \left\{ \Pi_{x=0}^{n-1} \left[1 + \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)} \sum_{t=0}^{y-1} h(s,t) \right] \right\}.$$

Since $\varepsilon > 0$ is arbitrary, we have $u = \bar{u}$, i.e., there is at most one solution of the problem (8.24.1)–(8.24.2).

Example 8.24.3 The third application is an example of continuous dependence of the solution on the equation and boundary data. Consider the problem (8.24.1)–(8.24.2) in Example 8.24.1 and the problem

$$\Delta_2[a_3(m,n)\Delta_2[a_2(m,n)\Delta_1[a_1(m,n)\Delta_1 u(m,n)]]] = F\Big(m,n,z(m,n)\Big) \tag{8.24.9}$$

with the given boundary conditions at $m=0, n=0$

$$\begin{cases} z(0,n) = \bar{\phi}_2(n), \\ a_1(0,n)\Delta_1 z(0,n) = \bar{\phi}_2(n), \\ a_2(m,0)\Delta_1[a_1(m,0)\Delta_1 z(m,0)] = \bar{\psi}_1(m), \\ a_3(m,0)\Delta_2[a_2(m,0)\Delta_1[a_1(m,0)\Delta_1 z(m,0)]] = \bar{\psi}_2(m). \end{cases} \tag{8.24.10}$$

Here a_1, a_2, a_3 are as in Example 8.24.1, $F : \mathbb{N}_0^2 \times \mathbb{R} \to \mathbb{R}, \bar{\phi}_1(n), \bar{\phi}_2(n), \bar{\psi}_1(n), \bar{\psi}_2(m)$ are real-valued non-negative functions defined for $m,n \in \mathbb{N}_0$. The equations

equivalent to (8.24.1)–(8.24.2) and (8.24.9)–(8.24.10) are (8.24.4) and

$$z(m,n) = \bar{b}(m,n) + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)} \sum_{t=0}^{y-1} F(s,t,z(s,t)) \tag{8.24.11}$$

where $\bar{b}(m,n)$ is obtained from the definition of $b(m,n)$ by replacing $\phi_1(n), \phi_2(n), \psi_1(m), \psi_2(m)$ in the right side in (8.24.5) by $\bar{\phi}_1(n), \bar{\phi}(n), \bar{\phi}(m), \bar{\phi}_2(m)$ respectively. From (8.24.4) and (8.24.11), we have

$$\begin{aligned} u(m,n) - z(m,n) &= b(m,n) - \bar{b}(m,n) \\ &+ \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)} \sum_{t=0}^{y-1} \Big(f(s,t,u(s,t)) - F(s,t,z(s,t)) \Big). \end{aligned} \tag{8.24.12}$$

Suppose that the function f in (8.24.1) satisfies the condition (8.24.7) and further we assume that

$$|b(m,n) - \bar{b}(m,n)| \le \varepsilon, \tag{8.24.13}$$

$$\sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)} \sum_{t=0}^{y-1} \Big(f(s,t,u(s,t)) - F(s,t,z(s,t)) \Big) \le \varepsilon \tag{8.24.14}$$

where ε is a arbitrary constant. Subtracting and adding $f(s,t,z(s,t))$ in the brackets on the right-hand side of Eq. (8.24.12) and using (8.24.7), (8.24.13), (8.24.14), we obtain

$$\begin{aligned} |u(m,n) - z(m,n)| \le 2\varepsilon + \sum_{x=0}^{m-1} \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)} \\ \times \sum_{t=0}^{y-1} h(s,t) |u(s,t) - z(s,t)| \le \varepsilon \end{aligned} \tag{8.24.15}$$

Now an application of Theorem 6.1.5 yields

$$|u(m,n) - z(m,n)| \le 2\varepsilon \left\{ \Pi_{x=0}^{m-1} \left[1 + \frac{1}{a_1(x,n)} \sum_{s=0}^{x-1} \frac{1}{a_2(s,n)} \sum_{y=0}^{n-1} \frac{1}{a_3(s,y)} \sum_{t=0}^{y-1} h(s,t) \right] \right\}. \tag{8.24.16}$$

If $h(m, n)$ is bounded on some compact set $0 \leq m \leq m_0, 0 \leq n \leq n_0, m, m_0, n, n_0 \in \mathbb{N}_0$, then the quantity in brackets on the right-hand of (8.24.16) is bounded by some constant M on the set $0 \leq m \leq m_0, 0 \leq n \leq n_0$. Therefore $|u(m, n) - z(m, n)| \leq 2M_\varepsilon$ on the set $0 \leq m \leq m_0, 0 \leq n \leq n_0$; so the solution $u(m, n)$ of (8.24.1)–(8.24.2) depends continuously on f and the boundary data. If $\varepsilon \to 0$, then $|u(m, n) - z(m, n)| \to 0$ on this set.

8.25 Applications of Theorem 6.2.2 to Discrete Hyperbolic Partial Differential Equations

In this section, we present some applications of Theorem 6.2.2 to the boundedness, uniqueness and continuous dependence of the solutions of discrete version of hyperbolic partial differential equations involving three variables.

Example 8.25.1 As a first application, we obtain a bound on the solution of a summary difference equation

$$\Delta^3 u_{xyz} = f\left(x, y, z, u, \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} h(x, y, z, s, t, r, u)\right), \tag{8.25.1}$$

with given boundary conditions at $x = 0,\ y = 0,\ z = 0$, where all the functions are defined on their respective domains of definitions and

$$|f[x, y, z, u, v]| \leq p(x, y, z)\,[|u| + |v|], \tag{8.25.2}$$

$$|h(x, y, z, s, t, r, u)| \leq q(s, t, r)|u|, \tag{8.25.3}$$

where p and q satisfy the hypotheses of Theorem 6.2.2. By using the given boundary conditions, Eq. (8.25.1) can be presented by the equivalent summary difference equation

$$\begin{aligned} u(x, y, z) = g(x, y, z) &+ \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} f[s, t, r, u(s, t, r), \\ &+ \sum_{k=0}^{s-1}\sum_{l=0}^{t-1}\sum_{n=0}^{r-1} h(s, t, r, k, l, n, u(k, l, n))], \end{aligned} \tag{8.25.4}$$

where $g(x, y, z)$ depends on the given boundary conditions. If $|g(x, y, z)| \leq a(x) + b(y) + c(z)$, where $a(x), b(y)$ and $c(z)$ are as defined in Theorem 6.2.2, then using (8.25.2), (8.25.3) and (8.25.4) and then applying Theorem 6.2.2, we obtain a bound on the solution $u(x, y, z)$ of Eq. (8.25.1). □

Example 8.25.2 As a second application, we shall establish the uniqueness of solutions of Eq. (8.25.1) with the given boundary conditions. We assume that the functions h and f in (8.25.1) satisfy

$$|h(x,,y,z,s,t,r,u) - h(x,y,z,s,t,r,\overline{u})| \le q(s,t,r)|u - \overline{u}|, \tag{8.25.5}$$

$$|f[x,y,z,u,v] - f[x,y,z,\overline{u},\overline{v}]| \le q(x,y,z)[|u-\overline{u}| + |v-\overline{v}|], \tag{8.25.6}$$

where p and q are as in Example 8.25.1. The problem (8.25.1) is equivalent to the Eq. (8.25.4). Then for any two solutions u and $\overline{u}$ of Eq. (8.25.1), we have

$$\begin{aligned} u(x,y,z) - \overline{u} = {} & g(x,y,z) - \overline{g}(x,y,z) \\ & + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1}\Bigg\{ f[s,t,r,u,\sum_{k=0}^{s-1}\sum_{l=0}^{t-1}\sum_{n=0}^{r-1} h(s,t,r,k,l,n,u)] \\ & - f[s,t,r,\overline{u},\sum_{k=0}^{s-1}\sum_{l=0}^{t-1}\sum_{n=0}^{r-1} h(s,t,r,k,l,n,\overline{u})]\Bigg\}, \end{aligned} \tag{8.25.7}$$

where $g(x,y,z)$ and $\overline{g}(x,y,z)$ depend on the given boundary conditions. Using (8.25.5) and (8.25.6) in (8.25.7) and further assuming $|g - \overline{g}| \le \varepsilon$, for arbitrary $\varepsilon > 0$, we have

$$|u(x,y,z) - \overline{u}(x,y,z)| \le \varepsilon + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} p(s,t,r)[|u-\overline{u}| + \sum_{k=0}^{s-1}\sum_{l=0}^{t-1}\sum_{n=0}^{r-1} q(k,l,n)|u-\overline{u}|].$$

Now a suitable application of Theorem 6.2.2 (with $a+b+c=\varepsilon$) gives us

$$|u(x,y,z) - \overline{u}(x,y,z)| \le \varepsilon + \varepsilon \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} p(s,t,r)K(s,t,r),$$

where

$$K(s,t,r) = \prod_{k=0}^{s-1}\left[1 + \sum_{l=0}^{t-1}\sum_{n=0}^{r-1}\left(p(k,l,n) + q(k,l,n)\right)\right].$$

Since $\varepsilon > 0$ is arbitrary, we have $u = \overline{u}$, i.e., there is at most one solution of Eq. (8.25.1).

We note that, here is a case where the simpler bound $|u - \overline{u}| \le R = \varepsilon k(x,y,z)$ gives us the conclusion $u \equiv \overline{u}$ more easily. □

Example 8.25.3 The third application is an example of continuous dependence of the solution on the equation and boundary data. Consider the boundary value

problem (8.25.1) given in Example 8.25.1 and

$$\Delta^3 U_{xyz} = F[x, y, z, U, \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} H(x, y, z, s, t, r, U)], \tag{8.25.8}$$

with given boundary conditions at $x = 0,\ y = 0,\ z = 0$, where all the functions are real-valued and defined on their respective domains of their definitions and

$$\Big| f[x, y, z, U, \sum_{k=0}^{s-1} \sum_{l=0}^{t-1} \sum_{n=0}^{r-1} H(s, t, r, , k, l, n, U)] \\ -F[x, y, z, U, \sum_{k=0}^{s-1} \sum_{l=0}^{t-1} \sum_{n=0}^{r-1} H(s, t, r, , k, l, n, U)] \Big| \le \varepsilon, \tag{8.25.9}$$

and suppose further that the functions h and f in (8.25.1) satisfy the condition (8.25.5) and (8.25.6) with $q(s, t, r) = M_2$ and $p(x, y, z) = M_1$, where $\epsilon,\ M_1$, and M_2 are positive constants. The equations corresponding to (8.25.1) and (8.25.8) are (8.25.4) and

$$U(x, y, z) = G(x, y, z) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} F[s, t, r, U(s, t, r), \\ + \sum_{k=0}^{s-1} \sum_{l=0}^{t-1} \sum_{n=0}^{r-1} H(s, t, r, , k, l, n, U(k, l, n))] \tag{8.25.10}$$

where $G(x, y, z)$ depends on the given boundary conditions for Eq. (8.25.8). From (8.25.4) and (8.25.10), it follows

$$u - U = (g - G) + \sum_{s=0}^{x-1} \sum_{t=0}^{y-1} \sum_{r=0}^{z-1} \Big\{ f[s, t, r, u, \sum_{k=0}^{s-1} \sum_{l=0}^{t-1} \sum_{n=0}^{r-1} h(s, t, r, k, l, n, u)] \\ -F[s, t, r, U \sum_{k=0}^{s-1} \sum_{l=0}^{t-1} \sum_{n=0}^{r-1} H(s, t, r, , k, l, n, U] \Big\}.$$

Subtracting and adding

$$f[s, t, r, U, \sum_{l=0}^{t-1} \sum_{n=0}^{r-1} k(s, t, r, , k, l, n, U)].$$

in the braces of the above equation, and further assuming $|g - G| \le \varepsilon$ and using (8.25.9), (8.25.5) and (8.25.6) as mentioned above, we conclude

$$|u - U| \le \epsilon + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1}\left\{M_1[|u - U| + \sum_{k=0}^{s-1}\sum_{l=0}^{t-1}\sum_{n=0}^{r-1} M_2|u - U|)] + \varepsilon\right\}.$$

An application of Theorem 6.2.2, on the compact set $0 \le x, y, z \le C$, yields

$$|u - U| \le M\varepsilon\left\{[1 + M_1 \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1}\prod_{k=0}^{s-1}[1 + (M_1 + M_2)tr]]\right\} \le M^*\varepsilon$$

where $M = 1 + C^3$, and M^* is obtained by replacing x, y, z by C in the expression in brackets. Thus the solution of the given boundary value problem (8.25.1) depends continuously on f and the boundary values. If $\varepsilon \longrightarrow 0$, then $|u - U| \longrightarrow 0$ on the set.

We note that the inequalities and applications presented here can be extended very easily to n independent variables. We omit the details.

8.26 An Application of Theorem 6.2.5 to Discrete Partial Integrodifferential Equations

In this section, we use Theorem 6.2.5 to obtain the bounds on the solutions of discrete versions of partial integrodifferential equations involving three independent variables. We believe that the discrete inequalities may be used in the theory of finite difference equations involving three independent variables in essentially the same capacity as the inequalities of the Gronwll and Bihari type are used in the theory of ordinary differential and integral equations.

We establish the bound on the solutions of discrete versions of partial integrodifferential equations involving three independent variables of the form

$$\Delta^3 u_{xyz} = f(x, y, z, u) + F\left(x, y, z, u, \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} h(x, y, z, s, t, r, u)\right) \tag{8.26.1}$$

with the given boundary conditions at $x = 0, y = 0, z = 0$, where all the functions are defined on their respective domains of definitions and

$$\left\{\begin{array}{ll} |\, f(x, y, z, u)\, | \le p(x, y, z)W(|u|), & (8.26.2)\\ |\, F(x, y, z, u, v)\, | \le b(x, y, z)(|u| + |v|), & (8.26.3)\\ |\, h(x, y, z, s, t, r, u)\, | \le c(s, t, r)|u| & (8.26.4) \end{array}\right.$$

for $x \geq 0, y \geq 0, z \geq 0$, where $W, b(x,y,z), c(x,y,z)$, and $p(x,y,z)$ are as defined in Theorem 6.2.5. By using the given boundary conditions, (8.26.1) can be represented by equivalent summary difference equation

$$u(x,y,z) = g(x,y,z) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} f(s,t,r,u(s,t,r)) + \sum_{s=0}^{x-1}\sum_{t=0}^{y-1}\sum_{r=0}^{z-1} \times F\left(s,t,r,u(s,t,r) + \sum_{k=0}^{s-1}\sum_{l=0}^{t-1}\sum_{n=0}^{r-1} h(s,t,r,k,l,n,u(k,l,n))\right) \tag{8.26.5}$$

where $g(x,y,z)$ depends on the given boundary conditions. If $|g(x,y,z)| \leq M$, then using (8.26.2)–(8.26.4) in (8.26.5) and then applying Theorem 6.2.5, we obtain the bound on the solution $u(x,y,z)$ of Eq. (8.26.1).

8.27 Applications of Theorems 6.3.3–6.3.5 to Difference Equations

The results in Theorems 6.3.3–6.3.5 can be directly used to prove the uniqueness and continuous dependence for the solutions of discrete versions of hyperbolic partial differential equations involving n independent variable of more general type than given in [511, 571, 595], since the arguments are similar, the details are not repeated here. We shall provide an upper bound on the solutions of difference equation of the form

$$\Delta^n u_x(x) = F(x, u(x), \sum_{s=0}^{x-1} K(x,s,u(s))) \tag{8.27.1}$$

together with the given suitable boundary conditions $u(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)$, $1 \leq i \leq n$.

The function F and K are defined on their respective domains of definitions and

$$\begin{cases} |F(x,u(x),v(x))| \leq f_{11}(x)|u(x)| + f_{12}(x)|v(x)|, & (8.27.2) \\ |K(x,s,u(s))| \leq f_{22}(s)|u(s)| & (8.27.3) \end{cases}$$

where f_{11}, f_{12}, f_{22} are same as appear in Theorem 6.3.3.

Any solution $u(x)$ of Eq. (8.27.1) satisfying the boundary conditions is also a solution of the Volterra difference equation

$$u(x) = g(x) + \sum_{x^1=0}^{x-1} F(x^1, u(x^1), \sum_{x^2=0}^{x^1-1} K(x^1, x^2, u(x^2))) \tag{8.27.4}$$

where $g(x)$ takes care of the boundary conditions.

Using (8.27.2), (8.27.3) in (8.27.4), we obtain

$$|u(x)| \le |g(x)| + \sum_{x^1=0}^{x-1} [f_{11}(x^1)|u(x^1)| + f_{12}(x^1) \sum_{x^2=0}^{x^1-1} f_{22}(x^2)|u(x^2)|].$$

If $|g(x)| \le a(x)$, where $a(x)$ the is same as in Theorem 6.3.4, we find by Theorem 6.3.4

$$|u(x)| \le a(x) \prod_{x_1^1=0}^{x_1-1} \Big[1 + \sum_{x_2^1=0}^{x_2-1} \dots \sum_{x_n^1=0}^{x_n-1} [f_{11}(x^1) + \sum_{x^2=0}^{x^1-1} f_{22}(x^2)]\Big]. \tag{8.27.5}$$

If $|g(x)| \le \sum_{i=1}^{n} a_i(x_i)$ where $a_i(x_i)$ are the same as in Theorem 6.3.3, we find by Theorem 6.3.3

$$\begin{aligned} |u(x)| \le [a_1(0) + \sum_{i=2}^{n} a_i(x_i)] \prod_{x_1^1=0}^{x_1-1} \Big[1 + \frac{\Delta a_1(x_1^1)}{a_1(x_1^1) + a_2(0) + \sum_{i=3}^{n} a_i(x_i)} \\ + \sum_{x_2^1=0}^{x_2-1} \dots \sum_{x_n^1=0}^{x_n-1} (f_{11}(x^1) + \sum_{x^2=0}^{x^1-1} f_{22}(x^2))\Big] \end{aligned} \tag{8.27.6}$$

also, in case $f_{11}(x) = f_{12}(x)$, from Theorem 6.3.5 it follows that

$$|u(x)| \le P_i(x), \quad i = 1, 2 \tag{8.27.7}$$

where

$$\begin{cases} P_1(x) = [a_1(0) + \sum_{i=2}^{n} a_i(x_i)] \prod_{x_1^1=0}^{x_1-1} \Big[1 + \frac{\Delta a_1(x_1^1)}{a_1(x_1^1)+a_2(0)+\sum_{i=3}^{n} a_i(x_i)} \\ \qquad + \sum_{x_2^1=0}^{x_2-1} \dots \sum_{x_n^1=0}^{x_n-1} (f_{11}(x^1) + \sum_{x^2=0}^{x^1-1} f_{22}(x^2))\Big], \\ P_2(x) = \sum_{i=1}^{n} a_i(x_i) + \sum_{x^1=0}^{x-1} (f_{11}(x^1) P_1(x^1). \end{cases}$$

The estimate (8.27.6) cannot be obtained from (8.27.5) except when $|g(x)| = \text{const}$. Also (8.27.7) cannot be obtained from (8.27.6). For $n = 3$, (8.27.7) is the same as obtained in [511].

8.28 Applications of Theorems 7.2.5–7.2.6 to Integro-Functional Equations

In this section, we shall employ Theorems 7.2.5–7.2.6 to study integro-functional equations.

Let $T = [t_0, t_1)$ (here $t_1 \leq +\infty$). Consider in T the usual topology (with respect to which T is connected) and the Lebesgue measure denoted by μ. Let for any $x \in T,\ T_x := [t_0, t(x)]$, where $t(\cdot)$ is such a continuous function defined in T that for every $x \in T$, the inequalities $t(t_0) \leq t(x) \leq x$ hold. By V denote the operator defined in $L^2(T)$ in the following way

$$Vf(x) \leq \int_{t_0}^{t(x)} K(x,y) f_1(\varphi(y)) d\mu(y), \tag{8.28.1}$$

where the kernel $K \in L^2(T \times T)$, φ is an invertible real function with continuous derivative for which $\varphi(x) \leq x$, while $f_1(t) = f(t)$ if $t_0 \leq t$ and $f_1(t) = 0$ if $t < t_0$.

It is easy to verify that all assumptions in Theorem 7.2.5 are satisfied. Hence, the integro-functional equations

$$\begin{aligned} h(x) &= g(x) + \int_{t_0}^{t(x)} K(x,y) h_1(\varphi(y)) d\mu(y) \\ &= g(x) + Vh(x) \quad (\epsilon + L_\epsilon(T)) \end{aligned} \tag{8.28.2}$$

possesses a unique solution. In view of Theorem 7.2.6, if the kernel K is non-negative and under the assumption that $f(y) \leq g(y) + Vf(y)$ for almost any $y \in [t_0, t(x)]$, it follows that

$$f(y) \leq h(y) \tag{8.28.3}$$

for almost all $y \in [t_0, t]$ (here $h = g + Vh$). In view of Theorem 7.2.6, if for two functions $f_i,\ f_s \in L^2(T)$, the inequality $f_i - Vf_i \leq f_s - Vf_s$ holds, then

$$f_i \leq f_s. \tag{8.28.4}$$

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